A new rank parity computing machine

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Main results

Theorem (Constantinou, Dokchitser, Green, Morgan)

Assume $\#\text{Ш}$ is finite. For all smooth, projective curves over number fields X/K

$$\mathrm{rank}(\mathrm{Jac}_X/K) \equiv \sum_v \Lambda(X/K_v) \mod 2$$

where $\Lambda \in \{0,1\}$ is an explicit invariant computed from curves over local fields.

Work in progress theorem (Dokchitser, Green, Morgan)

Assume $\#\text{Ш}$ is finite. The Birch and Swinnerton-Dyer conjecture correctly predicts the parity of $\mathrm{rank}(\mathrm{Jac}_X/K)$ for all nice hyperelliptic curves over number fields X/K.

Theorem (Green, Maistret: $p = 2$ and E has CM)

The p-parity conjecture holds for elliptic curves over totally real number fields.

BSD and the parity conjecture

Birch and Swinnerton-Dyer conjecture

$$\mathrm{rank}(\mathrm{Jac}_X) = \mathrm{ord}_{s=1} L(\mathrm{Jac}_X, s)$$

$+$

Conjectural functional equation

$$L^*(\mathrm{Jac}_X, s) = w(\mathrm{Jac}_X) L^*(\mathrm{Jac}_X, 2-s)$$

$\Downarrow$

The Parity Conjecture

Let K be a number field and X/K a curve. Then

$$(-1)^{\mathrm{rank}(\mathrm{Jac}_X/K)} = w(\mathrm{Jac}_X/K) = \prod_v w(\mathrm{Jac}_X/K_v).$$

Assuming finiteness of Ш, this is known for:

- Elliptic curves
- Jacobians of semistable genus 2 curves $(+\ldots)$
- Jacobians of semistable hyperelliptic curves $(+\ldots)$ over quadratic extensions

Goal

Develop an arithmetic analogue of the local root number $w(\mathrm{Jac}_X/K_v)$.

Applications of local formulae

Let E/K be a semistable elliptic curve. Assuming BSD, or finiteness of Ш,

$$\text{rank}(E/K) \equiv \#\{v|\infty\} + \#\{v\nmid\infty,\ E/K_v \text{ split multiplicative}\} \mod 2.$$

- $E/\mathbb{Q} : y^2 + y = x^3 + x^2$ has split multiplicative reduction nowhere $\Rightarrow$ rank$(E/\mathbb{Q})$ is odd. Therefore E has a $\mathbb{Q}$-point of infinite order.
- If $E/\mathbb{Q}$ is semistable with split multiplicative reduction at 2 then rank$(E/\mathbb{Q}(\zeta_8))$ is odd.

Suppose a local formula exists for X/K, i.e. $\text{rank}(\text{Jac}_X/K) \equiv \sum_v \Lambda(X/K_v) \mod 2$.

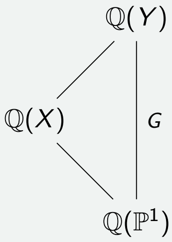
- rank$(\text{Jac}_X/\mathbb{Q}(i,\sqrt{17}))$ is even for any curve $X/\mathbb{Q}$.

Goal

Develop an arithmetic analogue of the parity conjecture which holds independently of BSD.

Ingredient 1 for the parity computing machine: field diagrams

Let $X/\mathbb{Q}$ be a curve and $\pi : X \to \mathbb{P}^1$.

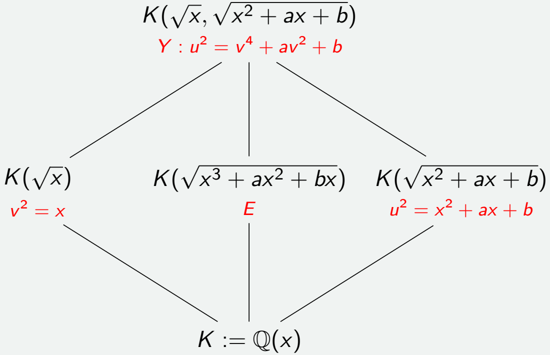


- $\mathbb{Q}(Y)^H = \mathbb{Q}(Y/H)$
- $(\mathrm{Jac}_Y(\mathbb{Q}) \otimes \mathbb{Q})^H = \mathrm{Jac}_{Y/H}(\mathbb{Q}) \otimes \mathbb{Q}$
- Tate modules
- Selmer groups
- Height pairings

Example

Let $E : y^2 = x^3 + ax^2 + bx$ and $\pi : E \to \mathbb{P}^1$,

$$(x, y) \mapsto x.$$



Let G be a finite group.

$\sum_i H_i - \sum_j H'_j$ is a *Brauer relation for* G if

$$\sum_i \mathrm{Ind}_{H_i}^{G} \mathbb{1} = \sum_j \mathrm{Ind}_{H'_j}^{G} \mathbb{1}.$$

Theorem (Kani, Rosen)

Let $Y/\mathbb{Q}$ be a curve, $G \leq \mathrm{Aut}_{\mathbb{Q}}(Y)$. If $\sum_i H_i - \sum_j H'_j$ is a Brauer relation for G then there's an isogeny

$$\prod_i \mathrm{Jac}_{Y/H_i} \to \prod_j \mathrm{Jac}_{Y/H'_j}.$$

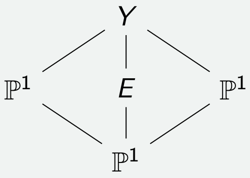
Example

A Brauer relation for $G = \mathrm{C}_2 \times \mathrm{C}_2$ is

$$\mathrm{C}_2 + \mathrm{C}'_2 + \mathrm{C}''_2 - \{1\} - G - G.$$

By Kani & Rosen,

$$\mathrm{Jac}_E \cong E \sim \mathrm{Jac}_Y.$$



By Cassels & Tate,

$$\frac{\#E(\mathbb{Q})^2_{\mathrm{tors}}}{\#J_Y(\mathbb{Q})^2_{\mathrm{tors}}} \cdot \frac{\#\text{Ш}_{J_Y}}{\#\text{Ш}_E} \cdot \frac{C_{J_Y}}{C_E} = \frac{\mathrm{Reg}_E}{\mathrm{Reg}_{J_Y}}$$

$$= \square \cdot 2^{\mathrm{rank}(E)}.$$

The parity computing machine

Theorem (Constantinou, Dokchitser, Green, Morgan)

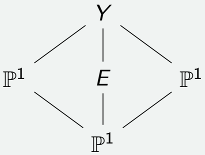
Let $Y/\mathbb{Q}$ be smooth, projective such that $\#Ш_{\mathrm{Jac}_Y}[\ell^\infty]$ is finite. Assume $Y \to \mathbb{P}^1$ is a Galois cover and let $\Theta = \sum_i H_i - \sum_j H'_j$ be a Brauer relation for its Galois group. Then

$$\mathrm{ord}_\ell \left(\frac{\prod_i \mathrm{Reg}_{\mathrm{Jac}_{Y/H_i}}}{\prod_j \mathrm{Reg}_{\mathrm{Jac}_{Y/H'_j}}} \right) \equiv \sum_v \Lambda_\Theta(Y/\mathbb{Q}_v) \mod 2$$

where Λ_Θ is an expression in ℓ and local data for $Y/\mathbb{Q}_v$.

Example

Let $E : y^2 = x^3 + ax^2 + bx$, $Y : u^2 = v^4 + av^2 + b$. Let $\Theta = C_2 + C'_2 + C''_2 - \{1\} - 2(C_2 \times C_2)$.



$$\mathrm{rank}(E) \equiv \mathrm{ord}_2\left(\frac{\mathrm{Reg}_E}{\mathrm{Reg}_{\mathrm{Jac}_Y}}\right) \equiv \underbrace{\mathrm{ord}_2\left(\frac{\Omega(\mathrm{Jac}_Y)}{\Omega(E)}\right)}_{= \Lambda_\Theta(Y/\mathbb{R})} + \sum_p \underbrace{\mathrm{ord}_2\left(\frac{c_p(\mathrm{Jac}_Y)}{c_p(E)}\right)}_{= \Lambda_\Theta(Y/\mathbb{Q}_p)}.$$

The parity computing machine

We recover local formulae for:

- E admitting a cyclic ℓ-isogeny (Cassels), if $E(K)[\ell] \neq \{O\}$ then $D_{2\ell}$
- Jac_X for X hyperelliptic over quadratic extensions (Kramer, Tunnell; Morgan), $C_2 \times C_2$
- Jac_X for X of genus 2 with a Richelot isogeny (Dokchitser, Maistret), D_8
- Jac_X for X of genus 3 such that G_K acts on $\mathrm{Jac}_X[2]$ by a 2-group (Docking). S_4

Theorem (Constantinou, Dokchitser, Green, Morgan)

Assume $\#Ш$ is finite. Let K be a number field and X/K a smooth, projective curve. There is a finite collection of Brauer relations Br *such that*

$$\mathrm{rank}(\mathrm{Jac}_X/K) \equiv \sum_{\Theta \in \mathrm{Br}} \sum_{v} \Lambda_\Theta(X/K_v) \mod 2.$$

Comparison with BSD

Theorem (Constantinou, Dokchitser, Green, Morgan)

Assume $\#Ш$ is finite. Let X/K be a smooth, projective curve over a number field. Then

$$\mathrm{rank}(\mathrm{Jac}_X/K) \equiv \sum_v \Lambda(X/K_v) \mod 2.$$

The parity conjecture predicts that

$$(-1)^{\mathrm{rank}(\mathrm{Jac}_X/K)} = \prod_v w(\mathrm{Jac}_X/K_v) \Rightarrow \mathrm{rank}(\mathrm{Jac}_X/K) \equiv \sum_v \eta(X/K_v) \mod 2.$$

Theorem (Green, Maistret)

- *The 2-parity conjecture holds for $\mathrm{Jac}_X \cong E_1 \times E_2$ where $E_1[2] \cong E_2[2]$.*
- *The p-parity conjecture holds for elliptic curves over totally real fields (we complete $p = 2$).*

Work in progress theorem (Dokchitser, Green, Morgan)

Assume $\#Ш$ is finite. The parity conjecture holds for all semistable hyperelliptic curves over number fields with good ordinary reduction at places $v \mid 2$.

Thank you for your attention!