Spectral theory of differential operators: what's it all about and what is its use
Part IV

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Spectral theory for type 1 systems

Looking at a self-adjoint elliptic system of m PDEs, each of even order $2n$, on a compact d-dimensional manifold M with boundary ∂M. Requires mn boundary conditions. I assume that the system is semi-bounded from below

Principal symbol is an $m \times m$ positive Hermitian matrix-function on $T^*M \setminus \{0\}$.

I assume that the eigenvalues of the principal symbol have constant multiplicity.

Extension of results from scalar case to type 1 systems is pretty straightforward.

Take eigenvalues of the principal symbol and extract $(2n)^{\mathrm{th}}$ positive root. These are the new Hamiltonians.

Reflection law: allow jumps from one Hamiltonian to another.

Formula for the first Weyl coefficient requires minor modification.

Algorithm for the evaluation of the second Weyl coefficient remains the same.

Important: the second Weyl coefficient comes from the boundary ∂M, as in the scalar case. Contributions to the second Weyl coefficient from M itself cancel out due to some symmetries.

Example of a type 1 system: linear elasticity

Equations of linear elasticity were first formulated by Baron Augustin-Louis Cauchy in 1828–29.

Variational formulation of linear elasticity

Quadratic functional

$$\mathcal{E}[\mathbf{u}] := \int_\Omega \Big(\lambda \, (\nabla_\alpha u^\alpha)^2 + \mu \, (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) \, \nabla^\alpha u^\beta \Big) \, \sqrt{\det g} \, dx \, ,$$

where λ and μ are real constants called *Lamé coefficients* which are assumed to satisfy

$$\mu > 0, \qquad d\lambda + 2\mu > 0,$$

$\mathbf{u}$ is the vector field of displacements (unknown quantity), ∇ is the Levi-Civita connection and $\sqrt{\det g}$ is the Riemannian density.

Will have to denote spectral parameter by Λ.

Variation of quadratic functional gives spectral problem

$$\mathcal{L}\mathbf{u} = \Lambda \mathbf{u},$$

where

$$(\mathcal{L}\mathbf{u})^\alpha := -\mu \left(\nabla_\beta \nabla^\beta u^\alpha + \mathrm{Ric}^\alpha{}_\beta u^\beta \right) - (\lambda + \mu) \nabla^\alpha \nabla_\beta u^\beta .$$

Principal symbol has two eigenvalues: simple eigenvalue

$$(\lambda + 2\mu)\|\xi\|^2$$

and eigenvalue

$$\mu\|\xi\|^2$$

of multiplicity $d-1$. These correspond to longitudinal and transverse elastic waves, respectively. Waves mix up when reflected from the boundary, giving us a branching Hamiltonian billiards.

Boundary conditions

Dirichlet condition

$$\mathbf{u}|_{\partial\Omega} = 0$$

or free boundary condition

$$\mathcal{T}\mathbf{u}|_{\partial\Omega} = 0$$

where $\mathcal{T}$ is the boundary traction operator defined by

$$(\mathcal{T}\mathbf{u})^\alpha := \lambda n^\alpha \nabla_\beta u^\beta + \mu \left(n^\beta \nabla_\beta u^\alpha + n_\beta \nabla^\alpha u^\beta \right).$$

Important: the free boundary condition is not the Neumann boundary condition $n^\beta \nabla_\beta u^\alpha = 0$.

In 1885 Lord Rayleigh analysed the free boundary condition and discovered *Rayleigh waves*.

Timeline of spectral analysis of linear elasticity

1912: P.Debye writes down first Weyl coefficient.

1915: H.Weyl provides rigorous proof of Debye's result.

1950: E.W.Montroll, incorrect calculation of the second Weyl coefficient.

1960: M.Dupuis, R.Mazo, and L.Onsager write down second Weyl coefficient for $d = 3$, both for Dirichlet and free boundary.

1997: I check the results of M.Dupuis, R.Mazo, and L.Onsager for $d = 3$ using my algorithm, and also deal with $d = 2$.

2022: M.Capoferri, L.Friedlander, M.Levitin, and D.Vassiliev. Second Weyl coefficient for any dimension. For odd dimensions explicit formulae in terms of Lamé parameters, for even dimensions formulae in terms of elliptic integrals.

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Two-term spectral asymptotics in linear elasticity

Matteo Capoferri, Leonid Friedlander, Michael Levitin, Dmitri Vassiliev

Motivated in part by the erroneous results in "Geometric invariants of spectrum of the Navier-Lamé operator" by Genqian Liu published in the Journal of Geometric Analysis 31 (2021), 10164--10193, we establish the two-term spectral asymptotics for boundary value problems of linear elasticity on a smooth compact Riemannian manifold of arbitrary dimension. We also present some illustrative examples and give a historical overview of the subject.

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Two-term spectral asymptotics in linear elasticity on a Riemannian manifold

Genqian Liu

In this note, by explaining two key methods that were employed in \cite{Liu-21} and by giving some remarks, we show that the proof of Theorem 1.1 in \cite{Liu-21} is a rigorous proof based on theory of strongly continuous semigroups and pseudodifferential operators. All remarks and comments to paper \cite{Liu-21}, which were given by Matteo Capoferri, Leonid Friedlander, Michael Levitin and Dmitri Vassiliev in \cite{CaFrLeVa-22}, are incorrect. The so-called "numerical counter-examples" in \cite{CaFrLeVa-22} are useless examples for the two-term asymptotics of the counting functions of the elastic eigenvalues. Clearly, the conclusion and the proof of \cite{Liu-21} are completely correct.

Elasticity operator in $\mathbb{R}^2$:

$$\mathcal{L}: \begin{pmatrix} u^1(x,y) \\ u^2(x,y) \end{pmatrix} \mapsto -\begin{pmatrix} (\lambda+2\mu)\partial^2_{xx} + \mu\partial^2_{yy} & (\lambda+\mu)\partial^2_{xy} \\ (\lambda+\mu)\partial^2_{xy} & \mu\partial^2_{xx} + (\lambda+2\mu)\partial^2_{yy} \end{pmatrix} \begin{pmatrix} u^1(x,y) \\ u^2(x,y) \end{pmatrix}$$

Reflection about x-axis:

$$\mathcal{R}: \begin{pmatrix} u^1(x,y) \\ u^2(x,y) \end{pmatrix} \mapsto \begin{pmatrix} u^1(x,-y) \\ u^2(x,-y) \end{pmatrix}.$$

Chain Rule tells us that

$$[\mathcal{L}, \mathcal{R}] \neq 0.$$