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TWO-TERM ASYMPTOTICS OF THE SPECTRUM OF A BOUNDARY VALUE PROBLEM IN THE CASE OF A PIECEWISE SMOOTH BOUNDARY

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1. Let G_0 be a compact n-dimensional manifold with an $(n-1)$-dimensional piecewise smooth boundary G_1. By this it is meant that in a neighborhood of any point M the manifold G_0 is diffeomorphic to a neighborhood of the vertex of an l-faced solid angle in $\mathbf{R}^n$, $0 \le l \le n$; we call all of $\mathbf{R}^n$ a 0-faced angle and the half-space $\langle x, \xi^{(1)} \rangle \ge 0$, where $\xi^{(1)} \ne 0$ is some fixed element of $\mathbf{R}^n$, a 1-faced angle, while an l-faced angle for $2 \le l \le n$ we define by induction as the union or intersection of an $(l-1)$-faced angle with the half-space $\langle x, \xi^{(l)} \rangle \ge 0$, where $\xi^{(l)}$ is linearly independent of $\xi^{(1)}, \dots, \xi^{(l-1)}$. The closure of the set of points $M \in G_0$ with the same index l we denote by G_l; thus, $G_n \subseteq G_{n-1} \subseteq \cdots \subseteq G_1 \subset G_0$.

The set $G_1 \setminus G_2$ has a finite number $q \ge 0$ of connected components. We call the closures of these connected components *faces* and write them G_{1p}, $p = 1, \dots, q$.

In this note we investigate the eigenvalue problem for the system of equations

$$(Au(x) - \lambda u(x))_{G_0 \setminus G_2} = 0 \tag{1}$$

with boundary conditions

$$(B_{jp}u(x))_{G_{1p} \setminus G_2} = 0, \qquad j = 1, \dots, sm, \; p = 1, \dots, q; \tag{2}$$

here $\mathbf{u}$ is an s-dimensional column, A is a matrix-valued differential operator of order $2m$ and size $s \times s$, and the B_{jp} are "boundary" differential operators of orders m_{jp} and size $1 \times s$. Problem (1), (2) is assumed to be formally selfadjoint on functions of the class

$$\mathbf{u}(x) \in C^\infty(G_0), \qquad \operatorname{supp} \mathbf{u}(x) \cap G_2 = \varnothing, \tag{3}$$

and regularly elliptic; the concept of regular ellipticity is formulated in a manner analogous to the case of smooth boundary ($G_2 = \varnothing$); see [1], Chapter 2, §1.4, and [4], §6; we mention only that the orders of the boundary conditions m_{jp} on neighboring faces may be distinct, and the coefficients in A and B_{jp} are infinitely smooth up to G_2.

In the case $G_2 = \varnothing$ the conditions enumerated above suffice for the problem to be well-posed; however, since in this note, generally speaking $G_2 \ne \varnothing$, we must additionally require that problem (1), (2) possesses an energy functional $F(\mathbf{u}, \mathbf{v})$ which "retains" the Sobolev norm $\| \cdot \|_{H^m(G_0)}$. By this we mean that for any $\mathbf{u}$ and $\mathbf{v}$ satisfying (2) and (3) the expression $(A\mathbf{u}, \mathbf{v})$ can be transformed to the form

$$(A\mathbf{u}, \mathbf{v}) = \int_{G_0} \left(\sum_{|\alpha| \le m, |\beta| \le m} (\partial_x^\alpha \mathbf{u})^{\mathrm{T}} e_{\alpha\beta} \overline{(\partial_x^\beta \mathbf{v})} \right) dx \overset{\mathrm{def}}{=} F(\mathbf{u}, \mathbf{v})$$

($e_{\alpha\beta}(x) \in C^\infty(G_0)$, $e_{\alpha\beta} = e^*_{\beta\alpha}$), and

$$F(\mathbf{u}, \mathbf{u}) \ge c \|\mathbf{u}\|^2_{H^m(G_0)} - c^{-1} \|\mathbf{u}\|^2, \qquad c > 0,$$

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uniformly with respect to all $\mathbf{u}(x) \in \mathfrak{D}(F)$, where $\mathfrak{D}(F)$ is the linear space of column-functions of $H^m(G_0)$ satisfying the boundary conditions (2) with orders $m_{jp} \le m - 1$. We note immediately that, although the presence of an energy functional $F(\mathbf{u}, \mathbf{v})$ is a very nontrivial condition, in practically all physically realizable problems (in the theory of membranes, acoustics, the theory of bending and tangential oscillations of plates [6], [7], and the three-dimensional theory of elasticity [8]) it exists, and $F(\mathbf{u}, \mathbf{u})/2$ has the meaning of the potential energy.

The variational formulation of the problem

$$\lambda_k = \inf_{U_k \subset \mathfrak{D}(F)} \left(\sup_{\mathbf{u} \in U_k, \|\mathbf{u}\| = 1} F(\mathbf{u}, \mathbf{u}) \right), \tag{4}$$

where $U_k \subset \mathfrak{D}(F)$ is a k-dimensional linear subspace, is well-posed (cf. [7] and [9]). It can be shown that problem (4) had discrete spectrum $\lambda_1 \le \lambda_2 \le \cdots$ accumulating at $+\infty$, and the column eigenfunctions $\mathbf{u}_k(x)$ on which the inf sup is achieved belong to $H^m(G_0) \cap C^\infty(G_0 \setminus G_2)$ and satisfy the system of equations (1) ($\lambda = \lambda_k$) and all the boundary conditions (2).

2. We denote by $\sigma_{2m}(x, \xi)$ the principal symbol of the operator A, by $\Lambda_k(x, \xi)$, $k = 1, \dots, r$, the distinct eigenvalues of $\sigma_{2m}(x, \xi)$, and by s_k their multiplicities (of course, $s_1 + \cdots + s_r = s$). We denote by $n_A(x, \xi, \lambda)$ and $N(\lambda)$ the distribution functions of the eigenvalues (the number of eigenvalues, counting multiplicity, less than a given λ) of the matrix $\sigma_{2m}(x, \xi)$ and of problem (4) respectively.

Repeating the arguments of [2]–[4] with minor changes, we arrive at the following result (cf. also [5]).

THEOREM 1. *Suppose the multiplicities s_k of the eigenvalues of the principal symbol are constant on $T^*G_0 \setminus 0$. Then as $\lambda \to +\infty$*

$$N(\lambda) = a\lambda^{n/2m} + O(\lambda^{(n-1)/2m}). \tag{5}$$

If, moreover, the problem is not absolutely periodic and is not of deadend type, then

$$N(\lambda) = a\lambda^{n/2m} + b\lambda^{(n-1)/2m} + o(\lambda^{(n-1)/2m}). \tag{6}$$

The constants a and b in (5) and (6) are defined by

$$a = (2\pi)^{-n} \int_{T^*G_0} n_A(x, \xi, 1)\, dx\, d\xi,$$

and

$$b = (2\pi)^{1-n} \sum_{p=1}^{q} \int_{T^*G_{1p}} \left(n_B(x_\Gamma, \xi_\Gamma, 1) + \frac{\varphi_B(x_\Gamma, \xi_\Gamma, 1)}{2\pi} \right) dx_\Gamma\, d\xi_\Gamma,$$

where $n_B(x_\Gamma, \xi_\Gamma, \lambda)$ and $\varphi_B(x_\Gamma, \xi_\Gamma, \lambda)$ for fixed x_Γ and ξ_Γ are, respectively, the distribution function of the eigenvalues and the scattering phase of a one-dimensional problem with constant coefficients on the semiaxis $0 \le x_n < +\infty$ which is obtained from (1) and (2) (p is fixed) if only terms with leading differentiations (of orders $2m$ and m_{jp}) are retained in the operators, all differentiations $\partial/\partial x_k$, $k = 1, \dots, n-1$, along the face G_{1p} are replaced by $i\xi_k$, and in the coefficients we set $x = (x_\Gamma, 0)$; see [2]–[4]. Here $x = (x_1, \dots, x_n)$ are local coordinates on G_0, and $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$. It is assumed that in a neighborhood of G_{1p} local coordinates are introduced so that $G_{1p} = \{x_n = 0\}$ and $x_n > 0$ on $G_0 \setminus G_{1p}$; $x_\Gamma = (x_1, \dots, x_{n-1})$ and $\xi_\Gamma = (\xi_1, \dots, \xi_{n-1})$.

The concepts of absolute periodicity and the deadend property are associated with the behavior of trajectories of the Hamiltonian systems

$$\dot{x} = \partial h_k / \partial \xi, \qquad \dot{\xi} = \partial h_k / \partial x \tag{7}$$

on $T^*G_0 \setminus 0$, $h_k(x,\xi) = (\Lambda_k(x,\xi))^{1/2m} > 0$, $k = 1,\dots,r$, and are defined as follows. We consider a trajectory $x(x_0,\xi_0,t)$, $\xi(x_0,\xi_0,t)$ of one ($k = k_0$) of the systems (7); $(x_0,\xi_0) \in T^*(G_0 \setminus G_1) \setminus 0$ is the initial point of the trajectory for $t = 0$. In the case of intersection of the trajectory with the boundary we reflect the trajectory inside $T^*G_0 \setminus 0$, preserving the value of the Hamiltonian and the continuity of $x(x_0,\xi_0,t)$, $\xi_\Gamma(x_0,\xi_0,t)$ at $t = \tau$ (τ is the time of reflection) and allowing the trajectory to "jump" from one Hamiltonian system (7) to another; here we agree to consider only regular reflections, i.e., those such that $\dot{x}_n(x_0,\xi_0,\tau-0) < 0$, $\dot{x}_n(x_0,\xi_0,\tau+0) > 0$, and $x(x_0,\xi_0,\tau) \notin G_2$. As in [3] and [4], we observe that the reflected trajectory, generally speaking, is not uniquely determined: to one incident trajectory there may correspond several (at most sm) reflected trajectories; this phenomenon we call *reproduction of trajectories*, and in using the notation $x(x_0,\xi_0,t)$, $\xi(x_0,\xi_0,t)$, we imply that one of the trajectories of the systems (7) is meant specifically a trajectory issuing from the point (x_0,ξ_0) and depending in an infinitely differentiable manner on x_0, ξ_0, and t for $x(x_0,\xi_0,t) \notin G_1$.

We call a point $(\bar{x}_0,\bar{\xi}_0)$ *absolutely periodic* if there exist $T > 0$ and a trajectory $x(x_0,\xi_0,t)$, $\xi(x_0,\xi_0,t)$ such that after a finite number $l \geq 0$ of regular reflections $x(\bar{x}_0,\bar{\xi}_0,T) = \bar{x}_0$, $\xi(\bar{x}_0,\bar{\xi}_0,T) = \bar{\xi}_0$ and, moreover, the diffeomorphism $(x_0,\xi_0) \to (x(x_0,\xi_0,T),\xi(x_0,\xi_0,T))$ of a ρ-neighborhood of $(\bar{x}_0,\bar{\xi}_0)$ differs from the identity mapping by $O(\rho^{+\infty})$. We call a point (x_0,ξ_0) a *deadend* point if there exists a trajectory $x(x_0,\xi_0,t)$, $\xi(x_0,\xi_0,t)$ experiencing an infinite number of regular reflections in finite time. We call the problem *not absolutely periodic* (respectively, *not of deadend type*) if the natural $(2n-1)$-dimensional Lebesgue measure of absolutely periodic (respectively deadend) points normalized by the condition $h_{k_0}(x,\xi) = 1$ is equal to zero.

3. In connection with Theorem 1 there arises the question of finding effective sufficient conditions under which problem (1), (2) is not absolutely periodic and is not of deadend type. One such set of sufficient conditions is presented in [2] (see also [4], §1): $G_2 = \varnothing$, $r = 1$, the condition of simple reflection is satisfied, G_0, G_1, and $h_1(x,\xi)$ are analytic, and G_0 is Hamiltonian-convex. Another set of sufficient conditions is formulated in Theorem 2 below. We emphasize that Theorem 2 admits $r \neq 1$ (r is the number of distinct eigenvalues of $\sigma_{2m}(x,\xi)$), i.e., it is also applicable in the case of certain reproduction of trajectories; this is important both in a theoretical and applied respect, since such a situation is observed in a number of problems of mechanics, in particular, in the theory of tangential oscillations of plates and in the three-dimensional theory of elasticity (here $r = 2$).

THEOREM 2. *Suppose the manifold G_0 is imbedded in $\mathbf{R}^n$ and its boundary has nonpositive curvature, i.e., for any point $M \in G_1 \setminus G_2$ the hyperplane tangent to G_1 passing through M belongs to G_0 in some neighborhood of M. Suppose the Hamiltonian systems (7) are isotropic, i.e., $h_k = c_k|\xi|$, $k = 1,\dots,r$, where $c_k > 0$ are constants not depending on x or ξ. Then problem (1), (2) is not absolutely periodic and is not of deadend type.*

The proof of Theorem 2 is based on the following considerations. It is easy to see that under the assumptions of the theorem any diverging pencil of trajectories on reflection from the boundary goes over into a diverging pencil, and hence for any periodic point $(\bar{x}_0,\bar{\xi}_0)$ we have $\partial x(\bar{x}_0,\xi_0,T)/\partial\xi_0|_{\xi_0=\bar{\xi}_0} \neq 0$, i.e., under the assumptions of Theorem 2 no absolutely periodic points exist. Turning to deadend points, we note that in the case $r = 1$ the measure of the set of such points is equal to zero by Lemma 2 of [10], Chapter 6, §1.2. In the case $r \neq 1$ the arguments of [10] do not go through, since they make essential use of the preservation of the elementary phase volume $dx\,d\xi$ under reflection, while in our case this volume may increase on reflection κ times, where $1 \leq \kappa \leq r$, κ a natural number, due to reproduction of trajectories. However, it is not hard to show

that under the conditions of Theorem 2 any deadend trajectory after some l_0th reflection ceases to "jump" from one Hamiltonian system (7) to another (the number l_0 depends on the trajectory, generally speaking). We obtain the required result by classifying all deadend trajectories $x(x_0,\xi_0,t)$, $\xi(x_0,\xi_0,t)$ according to the indices of the Hamiltonian systems (7) corresponding to their first l_0+1 links and noting that for each class the measure of the points (x_0,ξ_0) is equal to zero (this follows from Lemma 2 in [10], loc. cit., and the fact that after a finite number l_0 of reflections the measure cannot increase more than a finite number r^{l_0} of times) by the countable additivity of Lebesgue measure.

An important special case of the manifolds G_0 satisfying the conditions of Theorem 2 are polyhedra (the curvature of the boundary is identically zero). It is curious that the spectrum of the Laplace operator on a polyhedron was specially investigated in [11] and [12], but the second term of the asymptotics of $N(\lambda)$ was not derived.

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