Solving second order ordinary differential equations

1 Revision

Consider the equation
\[ y'' + a_1(x)y' + a_2(x)y = r(x) \quad (1) \]
satisfied by the function \( y(x) \) where a prime indicates differentiation with respect to \( x \). It is linear and so we know that its solution has the form \( y(x) = ycF(x) + yp1(x) \), where \( ycF(x) \) is the general solution of \( (1) \) with \( r = 0 \) and, as the equation is second order, contains two arbitrary constants. So, \( ycF(x) = Ay_1(x) + By_2(x) \) where both \( y_1 \) and \( y_2 \) satisfy \( (1) \) with \( R = 0 \) and are linearly independent, meaning there are no \( C_{1,2} \) such that \( C_{1,y_1} + C_{2,y_2} = 0 \) for all \( x \). The function \( yp1(x) \) is any function that satisfies \( (1) \) and is often difficult to find and up until now the method has been to use guided guesswork but there are general methods that we shall investigate shortly.

2 Linear dependence - the Wronskian

If two functions \( y_{1,2} \) are linearly dependent then \( C_1y_1(x) + C_2y_2(x) = 0 \) for all \( x \) for some \( C_{1,2} \). Differentiating this expression yields \( C_1y'_1(x) + C_2y'_2(x) = 0 \). If we treat this as two equations for the values of \( C_{1,2} \) then we have
\[ \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow W = y_1y'_2 - y'_1y_2 = 0. \quad (2) \]
The expression \( W(x) \) is called the Wronskian of the functions \( y_{1,2} \). See [http://en.wikipedia.org/wiki/Wronskian](http://en.wikipedia.org/wiki/Wronskian)

Exercises

1. Can you extend the idea to more than two functions, a situation which would be relevant to differential equations of higher order?

2. Show that the functions \( \{e^x,e^{-x}\} \) are linearly independent. How about \( \{1,x,x^2\} \) or \( \{1,x,3-x\} \)?

3 Reduction of order

It is not easy to find two linearly independent functions to form \( ycF \). The reduction of order method allows one to find a second, if you know a first. Let \( y = u(x) \) be a solution of \( (1) \) with \( r = 0 \) and let us look for a second solution of the form \( y(x) = u(x)v(x) \). Substitution gives
\[ uv'' + 2u'v' + u''v + a_1(uv' + u'v) + a_0uv = v(u'' + a_1u' + a_0u) + uv'' + (2u' + a_1u)v' = 0, \]
so that, if we introduce \( z = v' \), we get a first order equation for \( z \) and we have reduced the order of the equation.

Dividing by \( u \), we have,
\[ \frac{z'}{z} + 2\frac{u'}{u} + a_1(x) = 0, \quad \Rightarrow \quad \ln z + \ln u^2 + \int a_1(t) \, dt = \text{constant} \quad \Rightarrow \quad v'(x) = z(x) = \frac{A}{u^2} e^{-\int a_1(t) \, dt} \]
and integrating again, and recalling \( y = uv \),
\[ y(x) = Au(x) \int ^x \frac{1}{u^2(t)} e^{-\int ^{s} a_1(s) \, ds} \, dt + Bu(x) \quad \text{original solution} \quad \text{new solution} \quad (3) \]

Rather than remembering \( (3) \), it sometimes better to proceed from first principles, I suggest. See also [http://en.wikipedia.org/wiki/Reduction_of_order](http://en.wikipedia.org/wiki/Reduction_of_order)

Example

Legendre’s equation of order one is
\[ (1 - x^2)y'' - 2xy' + 2y = 0. \]
Its easy to spot that \( y = x \) is one solution. To find a second, write \( y = xv \) and substitute to get

\[
(1 - x^2)(xv'' + 2v') - 2x(xv' + v) + 2xv = (1 - x^2)(xv'' + 2v') - 2x^2v' = 0, \quad \Rightarrow \frac{v''}{v'} + \frac{2 - 4x^2}{x(1 - x^2)} = 0,
\]

and, using partial fractions, and integrating

\[
\ln v' + \int \frac{1}{t-1} + \frac{2}{t} + \frac{1}{1+t} \, dt = \ln v' + \ln(x-1) + 2\ln x + \ln(1+x) = \text{constant} \quad \Rightarrow \quad v' = \frac{B}{x^2(x^2 - 1)},
\]

for a constant \( B \). Using partial fractions to integrate again,

\[
v(x) = B \int \frac{1}{2(t-1)} - \frac{1}{t^2} - \frac{1}{2(t+1)} \, dt = \frac{1}{x} + \frac{1}{2} \ln \left( \frac{1-x}{1+x} \right) \quad \Rightarrow \quad y(x) = xv(x) = B \left( x + \frac{x}{2} \ln \left( \frac{1-x}{1+x} \right) \right).
\]

The first term in this expression is the solution we know already and the solution to Legendre’s equation of order one is

\[
y(x) = Ax + B \frac{x}{2} \ln \left( \frac{1-x}{1+x} \right).
\]

4 The Wronskian of the functions in the complementary function

Here we consider some properties of \( W(x) = y_1y_2' - y_1'y_2 \) where \( y_{1,2} \) satisfy

\[
y_1'' + a_1y_1' + a_0y_1 = 0, \tag{4}
\]

\[
y_2'' + a_1y_2' + a_0y_2 = 0. \tag{5}
\]

Taking \( y_1 \[5\] - y_2 \[4\] \) gives \( y_1y_2'' - y_1'y_2' + a_1(y_1y_2' - y_1'y_2) = 0 \). Now \( W' = (y_1y_2' - y_1'y_2)' = y_1'y_2' + y_1y_2'' - y_1'y_2 - y_1'y_2' = y_1y_2'' - y_1'y_2' \). Hence

\[
\frac{dW}{dx} + a_1W = 0 \quad \Rightarrow \quad W = Ce^{-\int a_1(t) \, dt}. \tag{6}
\]

5 Variation of Parameters

We now return to looking for solutions to \( 1 \) with \( r(x) \) non-zero. We presume we have two solutions \( y_1(x) \) and \( y_2(x) \) which make up the complementary function and look for a solution to \( 1 \) of the form \( y(x) = A(x)y_1(x) + B(x)y_2(x) \), looking for suitable functions \( A(x) \) and \( B(x) \). There is too much freedom in this choice as we could manage with \( B = 0 \), as we do in the method of reduction of order. We can after all write any function as \( A(x)y_1(x) \) for appropriate \( A(x) \). We use this freedom to our advantage. Evaluating \( y' \) gives

\[
y' = A'y_1 + B'y_2 + Ay_1' + By_2' = Ay_1' + By_2' \quad \text{if we use our freedom to insist} \quad A'y_1 + B'y_2 = 0. \tag{7}
\]

With this choice, differentiating again, gives \( y'' = A'y_1' + B'y_2' + Ay_1'' + By_2'' \), so that we have avoided second derivatives of \( A \) and \( B \) in this expression and, substitution into \( 1 \) gives

\[
A \left[ y_1'' + a_1y_1' + a_0y_1 \right] + B \left[ y_2'' + a_1y_2' + a_0y_2 \right] + A'y_1' + B'y_2' = r(x). \tag{8}
\]

We have two linear equations now for \( A' \) and \( B' \), namely \( 7 \) and \( 8 \). Solving these leads to

\[
\frac{dA}{dx} = -\frac{ry_2}{W}, \quad \Rightarrow \quad A(x) = -\int^x r(t)y_2(t) \frac{1}{W(t)} \, dt, \quad \& \quad \frac{dB}{dx} = \frac{ry_1}{W} \quad \Rightarrow \quad B(x) = \int^x r(t)y_1(t) \frac{1}{W(t)} \, dt, \quad W = y_1'y_2 - y_1y_2'. \tag{9}
\]

Since \( y = Ay_1 + By_2 \), constants of integration, \( a \) and \( b \), say, in \( A \) and \( B \) will lead to reproduction of the complementary function \( ay_1 + by_2 \). Putting all this together leads to

\[
y(x) = ay_1 + by_2 + \int^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} R(t) \, dt. \tag{10}
\]

Example

Solve \( y'' + 2y' + y = e^{-x} \ln x \). The auxiliary equation for the homogeneous version is \( \alpha^2 + 2\alpha + 1 = (\alpha + 1)^2 = 0 \) so the CF is \( y = Ae^{-x} + Bxe^{-x} \). If \( y = A(x)y_1(x) + B(x)y_2(x) \) is the solution of the forced equation then, if we choose \( A'y_1 + B'y_2 = 0 \), \( y' = Ay_1' + By_2' \) and \( y'' = A'y_1' + B'y_2' \). Since \( y_{1,2} \) satisfy the homogeneous equation substitution will lead to \( A'y_1' + B'y_2' = e^{-x} \ln x \). Solving these two equations for \( A' \) and \( B' \) gives \( A'(y_2y_1' - y_1y_2') = e^{-x} \ln x \) and \( B'(y_2y_1' - y_1y_2') = -e^{-x} \ln xy_1 \). Now \( (y_2y_1' - y_1y_2') = xe^{-x} - e^{-x} - e^{-x}(e^{-x} - xe^{-x}) = -e^{-x} \). So \( A' = e^{-x} \ln x \) and \( B' = -e^{-x} \ln x e^{-x}/(-e^{-2x}) = \ln x \).

Integrating, ignoring constants of integration, \( A = -x^2 \ln x/2 + x^2/4 \), \( B = x \ln x - x \) and \( y_{p1} = Ae^{-x} + Bxe^{-x} = x^2e^{-x}[\frac{1}{2} \ln x - \frac{3}{4}] \). Finally \( y = Ay_1 + By_2 + y_{p1} \).

6 Generalised Transforms

Recall some results involving the Laplace transform \( \tilde{y}(s) \) and Fourier transform \( \hat{y}(k) \) of the function \( y(x) \).

\[
\tilde{y}(s) = \int_0^\infty e^{-sx} y(x) \, dx, \quad \tilde{y} = s\tilde{y} - y(0), \quad \hat{y} = \tilde{y}' = -y', \quad y(x) = \frac{1}{2\pi i} \int_C e^{sx} \tilde{y}(s) \, ds,
\]

\[
\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx, \quad \hat{y} = ik\hat{y}, \quad \bar{y}(k) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) \, dx.
\]

In both cases notice that multiplying by \( x \) corresponds a differentiation of the transform, differentiation with respect to \( x \) corresponds to multiplication of the transform by the transform variable \( s \) or \( k \) and \( y \) is expressed as a contour integral in the transform variable, although the contour is the real axis in the case of the Fourier transform. Here we extend this idea and look for solutions of a differential equation

\[
(a_1x + a_0)y'' + (b_1x + b_0)y' + (c_1x + c_0)y = 0,
\]

by looking for solutions of the form

\[
y(x) = \int_C e^{xt} f(t) \, dt \quad \Rightarrow \quad y'(x) = \int_C t e^{xt} f(t) \, dt \quad \Rightarrow \quad y''(x) = \int_C t^2 e^{xt} f(t) \, dt,
\]

substituting into [13] and see what happens. We need to find an appropriate \( C \) and \( f \).

To illustrate things, we will start with the rather straightforward case of constant coefficients, so that \( a_1 = b_1 = c_1 = 0 \). Substitution gives

\[
a_0 \int_C t^2 e^{xt} f(t) \, dt + b_0 \int_C t e^{xt} f(t) \, dt + c_0 \int_C e^{xt} f(t) \, dt = \int_C [a_0 t^2 + b_0 t + c_0] e^{xt} f(t) \, dt = 0.
\]

This integral is guaranteed to be zero, as required, if \( C \) is closed and the integrand is regular inside \( C \) (http://en.wikipedia.org/wiki/Cauchy%27s_integral_theorem). Since \( e^{xt} \) is regular, we need \( (a_0 t^2 + b_0 t + c_0) f(t) \) to be regular inside our \( C \), for then Cauchy’s theorem would give \( y(x) = 0 \). Here regular means analytic or holomorphic (http://en.wikipedia.org/wiki/Holomorphic_function) However the solution, proportional to \( \int_C e^{xt} f(t) \, dt \), should not be the zero solution, so we need \( f \) to have singularities.

Let \( \alpha \) and \( \beta \) be the roots of the auxiliary equation, then we look for \( f \) such that \( (t - \alpha)(t - \beta)f(t) \) is free of singularities in some region, which we encircle with \( C \), but that \( f(t) \) has singularities. We choose \( f(t) = A(t - \alpha)^{-1} + B(t - \beta)^{-1} \), so that \( (a_0 t^2 + b_0 t + c_0) f(t) = a_0[A(t - \beta) + B(t - \alpha)] \) which is regular as required. With this choice

\[
y(x) = \int_C e^{xt} \left( \frac{A}{t - \alpha} + \frac{B}{t - \beta} \right) \, dt.
\]
The contour $C = C_α$ gives one solution, $y_1(x)$, $C = C_β$ gives a second independent solution ($y_2(x)$), since it is not possible to push $C_β$ to $C_α$ without crossing a singularity in $f$. The contour $C_0$ gives the zero solution as $e^{xt}f(t)$ is regular inside $C_0$. We can find explicit forms for these solutions here, using Cauchy’s integral formula

$$\int_C \frac{g(t)}{(t-t_0)}\,dt = 2\pi ig(t_0) \quad \text{if } g(t) \text{ is regular inside } C.$$

With $C = C_α, t_0 = α, g(t) = e^{xt}(A+B(t-α)/(t-β))$, gives $y_1(x) = 2\pi ig(α) = e^{αx}(A+0) = ˜Ae^{αx}$. Similarly, $C_β$ generates $y_2(x) = ˜Be^{dx}$. This of course is the solution we would have found by elementary methods. A sum of these two solutions, giving the general solution would be generated with the contour encircling both singularities $C_α+β$

The method works too if the auxiliary equation has a double root at $a_0t^2 + b_0t + c_0 = a_0(t-α)^2$. We choose $f(t) = A(t-α)^2 + B(t-α)^{-1}$ and $C_α$ gives the solution

$$y(x) = \int_{C_α} e^{xt} \left[ \frac{A}{(t-α)^2} + \frac{B}{(t-α)} \right] \,dt$$

This can be evaluated using the residue theorem [http://en.wikipedia.org/wiki/Residue_Theorem](http://en.wikipedia.org/wiki/Residue_Theorem) You will recall that the residue of $g(t)$, say, at a pole, here $t = α$, is the coefficient of $(t-α)^{-1}$ in the Laurent expansion of $g(t)$ about $t = α$ [http://en.wikipedia.org/wiki/Residue%28complex_analysis%29](http://en.wikipedia.org/wiki/Residue%28complex_analysis%29). The easiest way to find that here is to write

$$g(t) = e^{xt} \left[ \frac{A}{(t-α)^2} + \frac{B}{(t-α)} \right] = e^{αx}e^{x(t-α)} \left[ \frac{A}{(t-α)^2} + \frac{B}{(t-α)} \right] \approx e^{αx}(1 + x(t-α) + \cdots) \left[ \frac{A}{(t-α)^2} + \frac{B}{(t-α)} \right],$$

$$\approx Ae^{αx} \frac{1}{(t-α)^2} + (Ax + B)e^{αx} \frac{1}{(t-α)} + \cdots \quad t \to α.$$

The residue is therefore $(Ax + B)e^{αx}$ and the residue theorem gives $y(x)$ as $2πi$ times this, i.e. $y(x) = e^{αx}(A + ˘Bx)$, as with elementary methods.

Consider now the general case. Substitution of (14) into (13) requires

$$\int_C \{x[a_1t^2 + b_1t + c_1] + [a_0t^2 + b_0t + c_0] \} e^{xt} f(t) \,dt = 0. \quad (15)$$

There are two ways of proceeding from here. The first is to try and find a function $g(t)$ such that the integrand is $d[e^{xt}g(t)]/dt$ so that we are left with the requirement that $[e^{xt}g(t)]_C = 0$. The second is to use integration by parts on the $xe^{xt}$ term in (15). We will use illustrate the first method here and the second in later examples.

Since $d[e^{xt}g(t)]/dt = e^{xt}(g’ + xy)$, comparing with (15) we identify, comparing coefficients of $xe^{xt}$,

$$g(t) = (a_1t^2 + b_1t + c_1)f(t), \quad g’(t) = (a_1t^2 + b_1t + c_1)f(t), \quad \Rightarrow \frac{g’}{g} = \frac{a_0t^2 + b_0t + c_0}{a_1t^2 + b_1t + c_1}, \quad \Rightarrow \ln g = \int \frac{a_0t^2 + b_0t + c_0}{a_1t^2 + b_1t + c_1} \,dt.$$

Once $g(t)$ has been calculated, $f(t)$ can be found through $f(t) = g(t)/(a_1t^2 + b_1t + c_1)$ or $f(t) = g’(t)/(a_0t^2 + b_0t + c_0)$. The solution is, recall

$$\int_C e^{xt} f(t) \,dt, \quad \text{with } C \text{ chosen so that } [e^{xt}g(t)]_C = 0.$$ 

**Example**

Solve

$$xy'' + 4y’ - xy = 0, \quad x > 0, \quad \text{looking for a solution of the form } \quad y(x) = \int_C e^{xt} f(t) \,dt.$$

We will see the relevance of the condition $x ≥ 0$ as we proceed. Equation (14) reminds us of the form of the derivatives of $y$ and substitution implies we need

$$\int_C (xt^2 + 4t - x)e^{xt}f(t) \,dt = \int_C \frac{d}{dt}[e^{xt}g(t)] \,dt = [e^{xt}g(t)]_C = 0, \quad (16)$$
The solution therefore is

$$ (xt^2 + 4t - x)e^{xt}f(t) = \frac{d}{dt}[e^{xt}g(t)] = e^{xt}g'(t) + xe^{xt}g(t), \quad g' = 4tf, \quad g = (t^2 - 1)f. $$

So

$$ \frac{g'}{g} = \frac{4t}{t^2 - 1}, \quad \Rightarrow \ln g = 2\ln(t^2 - 1), \quad \Rightarrow g(t) = (t^2 - 1)^2 \quad \text{and} \quad f(t) = g(t)/(t^2 - 1) = t^2 - 1. $$

The solution therefore is

$$ \int_C e^{xt}(t^2 - 1) \, dt, \quad \text{where} \quad C \text{ is chosen so that} \quad [e^{xt}(t^2 - 1)]_C = 0. $$

There are zeros in $e^{xt}(t^2 - 1)$ at 1, $-1$ and $-\infty$ and joining any two of these zeros with a contour $C$ will make $[e^{xt}(t^2 - 1)]_C = 0$. This is where the condition $x > 0$ comes into play. There are two ways of doing this join to get two independent solutions, $C_1$ joining $-1$ and 1, generating the solution $y_1(x)$ and $C_2$, joining $-\infty$ and $-1$, generating $y_2(x)$. A contour joining $-\infty$ and 1 would generate $y_1 + y_2$ which is not independent of $y_1$ and $y_2$. Similarly a contour from 1 to $-1$ will generate $-y_1$. Contours $C_1$ and $C_2$ can run along the real axis, so that

$$ y_1(x) = \int_{-1}^1 e^{xt}(t^2 - 1) \, dt, \quad y_2(x) = \int_{-\infty}^1 e^{xt}(t^2 - 1) \, dt, \quad y(x) = Ay_1(x) + By_2(x). $$

We now turn to looking at the properties of these solutions.

Putting $x = 0$ gives $e^{xt} = 1$ and $y_1(0) = \int_{-1}^1 1(t^2 - 1) \, dt$ which is a finite number. In contrast $y_2(0) = \int_{-\infty}^1 1(t^2 - 1) \, dt$ which is an improper divergent integral, diverging at its lower limit. However for positive $x$, however small, $e^{xt} \to 0$ as $t \to -\infty$ sufficiently quickly for the integral $y_2(x)$ to converge, giving a finite result. We conclude that $y_2(x)$ has a singularity at $x = 0$.

Examining the limit $x \to \infty$, shows that $y_1(x) \to \infty$ as it has an interval of integration in which $t > 0$ and so $e^{xt}$ in the integrand gets arbitrarily large. In contrast the interval in $y_2(x)$, $(1, -1)$ has $t < 0$, so that as $x \to \infty$ the factor $e^{xt} \to 0$ and the integral $y_2(x) \to 0$. We will see more of this type of argument later in the course where we will cover techniques for extracting more information about the behaviour of these integral solutions.

An alternative method of solving this type of second order equations is to look for a power series solution about $x = 0$, as discussed in Methods 4. This gives a lot of information about the behaviour of the solution as $x \to 0$ but very little about the behaviour away from $x = 0$ and, although it is possible to extract the behaviour as $x \to \infty$ from the series solution as $x \to 0$, it is not straightforward. The current method gives a form of solution which gives an explicit connection between the solution at $x = 0$ and the solution as $x \to \infty$. In the current example we see $y_1$ is finite at $x = 0$, but grows exponentially as $x \to \infty$ In contrast $y_2$ is finite as $x \to \infty$ but grows as $x \to 0$. Hence there is no solution finite as both $x \to 0$ and $x \to \infty$.

It is possible to get explicit forms for the solutions as these integrals can be evaluated. Note that if this is not the case, as explained above, the integral form of the solution is still very valuable. Splitting the range of integration in $y_1$ and making a change of variable $t \to -t$ in one range gives,

$$ y_1(x) = \int_{-1}^0 e^{xt}(t^2 - 1) \, dt + \int_0^1 e^{xt}(t^2 - 1) \, dt = \int_0^1 e^{-xt}(t^2 - 1) \, dt + \int_0^1 e^{xt}(t^2 - 1) \, dt = 2 \int_0^1 \cosh(zt)(t^2 - 1) \, dt = \frac{4}{x^3} [\sinh x - x \cosh x], $$

integrating by parts twice. The growth at infinity is clear and l’Hospital’s rule shows that it is finite at $x = 0$. In $y_2(x)$, make the substitution $t \to -t$ and integrate by parts twice

$$ y_2(x) = \int_1^\infty e^{-xt}(t^2 - 1) \, dt = \frac{2e^{-xt}}{x^3} (1 + x). $$

We will now solve the same equation using the second method, direct substitution and then integration by parts. From (16) we need

$$ 0 = \int_C [x(t^2 - 1) + 4t]e^{xt}f(t) \, dt = [e^{xt}(t^2 - 1)f(t)]_C + \int_C e^{xt} \left( 4tf(t) - \frac{d}{dt}[f(t^2 - 1)] \right) \, dt, $$

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using parts on the $xe^{xt}.f(t)(t^2-1)$ part of the integral. So we need

$$0 = \left[e^{xt}(t^2-1)f(t)\right]_C + \int_C e^{xt}(4ft-2ft-(t^2-1)f')\ dt.$$

We ensure this by ensuring $f$ satisfies $(t^2-1)f' = 2ft$ and then choosing $C$ so that $\left[e^{xt}(t^2-1)f(t)\right]_C = 0$, just as before.

We can now see explicitly that this method only works if the equation is linear and when the degree of the polynomial coefficients in the differential equation is less than the order of the equation. If the coefficients are quadratic in a second order equation, then two integration by parts are needed to get rid of the $x^2f$ which would lead to terms in $f''$ and a second order equation would need to be solved for $f$ and we are likely be no better off.

We also see the connection with Laplace and Fourier transforms in that differentiation of the function $y$ leads to a multiplication of the generalised transform $f$ by $t$ and that multiplication of $y$ by $x$ leads to differentiation of the transform through the integration by parts.

**Another example**

$$xy'' + (3x-1)y' - 9y = 0, \quad x > 0 \quad y(x) = \int_C e^{xt}f(t)\ dt.$$

Substitution and gathering together terms that give $xe^{xt}$

$$\int_C [t^2f + 3tf]xe^{xt}\ dt - \int_C [tf + 9f]e^{xt}\ dt = 0.$$

Integration by parts in the first integral, changes the $x$ into a $-d/dt$ and giving

$$\left[(t^2f + 3tf)e^{xt}\right]_C - \int_C \frac{d}{dt} [t^2f + 3tf] + (t + 9)f\ dt = 0,$$

which we ensure by insisting that the integrand in the integral is identically zero and then choosing $C$ appropriately.

$$0 = [t^2f + 3tf]' + (t + 9)f = f'(t^2+3t) + f[2t+3+t+9] \quad \Rightarrow \quad \frac{f'}{f} = -\frac{3t+12}{t(t+3)} = \frac{1}{t+3} - \frac{4}{t},$$

using partial fractions. Integration gives $f = (t+3)/t^4$ and we have a solution

$$y(x) = \int_C \frac{e^{xt}(t+3)}{t^4} \ dt \text{ with } C \text{ chosen so that } \left[t^2f + 3tf\right]_C = \left[\frac{(t+3)^2e^{xt}}{t^3}\right]_C = 0.$$

The function $(t+3)^3x^{3t}/t^3$ is zero only in two places, $t = -3$ and $t \to -\infty$ and although one suitable $C$ can be found by joining these two, with $C_2$, a second solution needs a new idea. The function is single valued (the exponents are integers and not fractions) so that any closed contour $C$ will give zero. Most closed contours will give a zero solution for $y(x)$ except those which encircle a singularity in $e^{xt}f(t) = e^{xt}(t+3)/t^4$ which would give a non-zero $y(x)$. Such a contour is $C_1$ which encircles the singularity at the origin. We have

$$y_1(x) = \frac{1}{2\pi i} \int_{C_1} \frac{e^{xt}(t+3)}{t^4} \ dt, \quad y_2(x) = \int_{-\infty}^{-3} \frac{e^{xt}(t+3)}{t^4} \ dt = \int_{3}^{\infty} \frac{e^{xt}(3-t)}{t^4} \ dt.$$

The $2\pi i$ factor ensures that $y_1$ is real. An explicit form for $y_1$ can be found using Cauchy’s integral formula for derivatives (http://en.wikipedia.org/wiki/Cauchy%27s_integral_formula),

$$f^n(t_0) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-t_0)^{n+1}} \ dt, \quad y_1(x) = \left. \frac{1}{3!} \frac{d^3}{dt^3} e^{xt}(t+3) \right|_{t=0} = \frac{1}{6}(x^2e^{xt}(t+3) + 3xe^{xt}.1 + 0 + 0)_{t=0} = x^2(x+1)/2,$$

where we have used Leibnitz’ theorem to quickly evaluate the third derivative. This solution is therefore a simple polynomial in $x$. The second solution $y_2(x)$ has an integrand that decays as $x \to 0$ and so $y_2(x) \to 0$ as $x \to \infty$.  

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The function \( y_2(x) \) has a singularity at \( x = 0 \) as its second derivative there is infinite. Differentiating under the integral sign gives

\[
y_2(x) = \int_3^\infty \frac{e^{-xt/(3-t)}}{t^4} \, dt, \quad y_2'(x) = \int_3^\infty \frac{-te^{-xt/(3-t)}}{t^4} \, dt = -\int_3^\infty \frac{e^{-xt}(3-t)}{t^3} \, dt, \quad y_2''(x) = \int_3^\infty \frac{e^{-xt}(3-t)}{t^2} \, dt.
\]

For \( x > 0 \) all these integrals converge due to the exponential decay in the integrand as \( t \to 0 \). However for \( x = 0 \) they yield

\[
y_2(0) = \int_3^\infty \frac{(3-t)}{t^4} \, dt, \quad y_2'(0) = -\int_3^\infty \frac{(3-t)}{t^3} \, dt, \quad y_2''(0) = \int_3^\infty \frac{(3-t)}{t^2} \, dt.
\]

The first two integrals converge giving a finite value for \( y_2(0) \) and \( y_2'(0) \) but the third diverges due to the \( 1/t \) behaviour in the integrand as \( t \to \infty \). Hence \( y_2''(0) \) does not exist.

**Example - Airy’s Equation**

Airy’s equation is \( y'' - xy = 0 \). It is important as it bridges the exponential behaviour of \( y'' - y = 0 \) and the oscillatory behaviour of \( y'' + y = 0 \). Its solutions are oscillatory in \( x < 0 \) and exponentially growing/decaying in \( x > 0 \). The solution that contains no component of the growing solution and so decays is denoted \( \text{Ai}(x) \). The second tabulated solution is written \( \text{Bi}(x) \). [http://en.wikipedia.org/wiki/Airy_function](http://en.wikipedia.org/wiki/Airy_function) We look for a solution \( y(x) = \int_C f(t)e^{xt} \, dt \).

Direct substitution implies that we need

\[
0 = \int_C (t^2 - x) f e^{xt} \, dt - [f(t)e^{xt} f(t)]_C + \int_C (t^2 f + f') e^{xt} \, dt,
\]

so we choose \( f' = -t^2 f \), solve for \( f = e^{-t^3/3} \) and then choose \( C \) so that \( [e^{xt} f]_C = [e^{xt-t^3/3}]_C = 0 \). There are no zeros in \( e^{xt-t^3/3} \) except possibly as \( |t| \to \infty \). As \( |t| \to \infty \) the behaviour of \( e^{xt-t^3/3} = e^{xt}e^{-t^3/3} \) is determined by the sign of the real part of \( t^3 \). If it is positive then it is exponentially small, but it grows to be exponentially large where the sign is negative. To investigate the sign put \( t = R(\cos \theta + i \sin \theta) \) so that \( t^3 = R^3(\cos 3\theta + i \sin 3\theta) \), with real part \( R^3 \cos 3\theta \). This is positive where \(-5\pi/6 < \theta < -\pi/2, -\pi/2 < \theta < \pi/2, 3\pi/2 < 3\theta < 5\pi/2 \), i.e. \(-5\pi/6 < \theta < -\pi/2, -\pi/6 < \theta < \pi/6, \pi/2 < \theta < 5\pi/6 \). Hence any contour \( C \) that originates at infinity in one of these sectors and then exits to infinity in one will ensure \( [e^{xt-t^3/3}]_C = 0 \) and the solution will be \( \int_C e^{xt-t^3/3} \, dt \). Alternatively any closed \( C \) will give a solution, but this will be the zero solution as the exponential function is free of singularities. Similarly any contour that starts at infinity in one sector and then exits by the same sector will also give the zero solution. This is because the contour can be “closed at infinity” within the sector, where the integrand can be made arbitrarily small by making \(|t| \) large.

The closed contour contains no singularities and so the integral for \( y(x) \) gives the zero solution. To be more explicit, consider the integral along the contour \( C_1 \) shown extending to infinity and generating the solution \( y \). The contour made up of the union of the parts of \( C_1 \) with modulus less than \( R \) and the arc \( C_1(R) \) on which \( |t| = R \) is a closed contour giving a zero value for \( y \). The difference between the integral giving \( y \) and this zero integral consists of contributions along \( C_1(R) \) and having \(|t| > R \) and so can be made as small as we please as \( R \to \infty \). Hence the integral along \( C_1 \), i.e. \( y \) must be as close as we please to zero and so must be zero. The similar argument applied to \( C_2 \) which enters and exits into different sectors fails as the integrand is not exponentially small along \( C_2(R) \).

Hence we identify three contours giving three non-zero solutions for \( y, y_i(x) = \int_{C_i} e^{xt-t^3/3} \, dt \). However only two of
these are independent as \(y_1(x) + y_2(x) + y_3(x) = 0\) as the contours \(C_1, C_2\) and \(C_3\) can be "joined at infinity" to give a closed contour containing no singularities. The tabulated solutions \(Ai(x)\) and \(Bi(x)\) are given by

\[
Ai(x) = \frac{1}{2\pi i} y_1(x) = \frac{1}{2\pi i} \int_{C_1} e^{xt-t^3/3} \, dt, \quad Bi(x) = \frac{1}{2\pi} (y_2(x) - y_3(x)).
\] (17)

We will consider \(Ai(x)\) further and make a specific choice for \(C_1\) in (17), choosing \(C_1 = C(\epsilon) = -\epsilon + is, \, -\infty < s < \infty\). Remember we will get the same function \(Ai(x)\) whatever \(C_1\) we choose as long as it enters and exits the complex \(t\)-plane in the right sectors so the result will be independent of \(\epsilon\) as long as \(\epsilon > 0\). We can take the limiting case \(\epsilon \to 0\) giving a limiting contour running on the imaginary axis. The integrand in this case does not become small as \(|s| \to \infty\) but does not grow and does oscillate increasingly rapidly. The integral is not absolutely integrable but does converge.

\[
Ai(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{isx-3s^3/3} \, ids = \frac{1}{2\pi} \int_{-\infty}^{0} e^{i(sx+s^3/3)} \, ds + \frac{1}{2\pi} \int_{0}^{\infty} e^{i(sx+s^3/3)} \, ds = \frac{1}{2\pi} \int_{0}^{\infty} e^{-i(sx+s^3/3)} + e^{i(sx+s^3/3)} \, ds
\]

put \(s \rightarrow -s\)

\[
= \frac{1}{\pi} \int_{0}^{\infty} \cos(xs + s^3/3) \, ds.
\] (18)