Hybrid numerical-asymptotic methods for high frequency scattering

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Solving scattering problems

Model problem - Helmholtz equation:

$$(\Delta + k^2)u = 0, \quad k = \text{wavenumber} = \frac{\text{frequency}}{\text{wavespeed}} = \frac{2\pi}{\text{wavelength}}$$
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Solution methods:

- Increasing frequency
  - Numerical methods (FEM, BEM, ...)
  - Asymptotic methods (Geometrical Optics, ray tracing, GTD, ...)

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- Controllably accurate
- Computationally infeasible at high frequencies
- Computational cost independent of frequency
- Accurate only at high frequencies
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What to do here??

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  - Computationally infeasible at high frequencies

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  - Accurate only at high frequencies
Hybrid Numerical-Asymptotic (HNA) methods

Fuse conventional numerical methods with high frequency asymptotics to create algorithms that are controllably accurate and computationally feasible over the whole frequency range.
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Key idea: enrich the FEM/BEM approximation space with **oscillatory functions**
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Key idea: enrich the FEM/BEM approximation space with oscillatory functions

\[ v(x, k) \approx v_0(x, k) + \sum_{m=1}^{M} v_m(x, k) e^{ik\psi_m(x)}, \]

- \( v_0 \) is some known leading order asymptotic behaviour
- \( \psi_m, m = 1, \ldots, M \) are specified phase functions, from asymptotics
- \( v_m, m = 1, \ldots, M \) are unknown amplitude functions, found numerically
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Expectation: If \( v_0 \) and \( \psi_m \) are chosen appropriately, \( v_m, m = 1, \ldots, M \), will be slowly varying, and less expensive to approximate than \( v \)
Why do mathematicians like FEM/BEM?

FEM = Finite Element Method, BEM = Boundary Element Method ("Method of Moments")

- General
- Systematic
- Flexible
- Controllably accurate
- Established frameworks for error analysis
  
...
Starting point: **Partial Differential Equation** (PDE) written in “weak form”:

Given \( l \in V^* \), find \( u \in V \) such that 

\[
a(u, v) = l(v), \quad \forall v \in V
\]
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**FEM example:**

\[-(\Delta + k^2)u = f \text{ in } \Omega \quad \text{with } u = 0 \text{ on } \Gamma\]

\[a(u, v) := \int_D (\nabla u \cdot \nabla v - k^2 u v) \, dx, \quad V = H^1_0(D)\]

\[l(v) = \int_D f v \, dx, \quad V^* = H^{-1}(D)\]
Basics of BEM

Starting point: Boundary Integral Equation (BIE) written in “weak form”:

Given \( l \in V^* \), find \( \phi \in V \) such that

\[
a(\phi, \psi) = l(\psi), \quad \forall \psi \in V
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Given \( l \in V^* \), find \( \phi \in V \) such that

\[
a(\phi, \psi) = l(\psi), \quad \forall \psi \in V
\]

**BEM example:**

\[
(\Delta + k^2)u = 0
\]

\( D \)

\[
u^i = e^{ikd \cdot x}
\]

\[
|d| = 1
\]

\[
u^s := u - u^i \text{ outgoing at infinity}
\]

\[
u(x) = u^i(x) - \int_{\Gamma} \Phi(x, y) \frac{\partial u}{\partial n}(y) \, ds(y), \quad x \in D
\]

\[
S \frac{\partial u}{\partial n} = u^i \text{ on } \Gamma, \quad S\phi(x) := \int_{\Gamma} \Phi(x, y) \phi(y) \, ds(y), \quad x \in \Gamma
\]

\[
a(\phi, \psi) := \langle S\phi, \psi \rangle = \int_{\Gamma} (S\phi)(y) \overline{\psi(y)} \, ds(y), \quad V = H^{-1/2}(\Gamma)
\]

\[
l(\psi) = \int_{\Gamma} u^i(y) \overline{\psi(y)} \, ds(y), \quad V^* = H^{1/2}(\Gamma)
\]
“Continuous” problem:

Find $u \in V$ such that $a(u, v) = l(v)$, $\forall v \in V$
“Continuous” problem:

\[
\text{Find } u \in V \text{ such that } a(u, v) = l(v), \quad \forall v \in V
\]

To approximate this numerically, choose a finite dimensional subspace \( V_N \subset V \) and consider the “discrete” problem:

\[
\text{Find } u^N \in V_N \text{ such that } a(u^N, v^N) = l(v^N), \quad \forall v^N \in V_N
\]

Well-posedness and quasi-optimality

If \( a(\cdot, \cdot) \) is “nice” (continuous and coercive) then the continuous and discrete problems both have unique solutions satisfying

\[
\|u - u^N\|_V \leq C \min_{v^N \in V_N} \|u - v^N\|_V
\]

← Best approx. error in \( V_N \)
Basics of FEM/BEM

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Let \( \{\phi_j\}_{j=1}^N \) be a basis for \( V_N \). Write \( u^N = \sum_{j=1}^N u_j \phi_j \), then

\[
Au = l, \quad A_{ij} = a(\phi_j, \phi_i), \quad \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \quad \mathbf{l} = \begin{pmatrix} l(\phi_1) \\ \vdots \\ l(\phi_N) \end{pmatrix}
\]
Basics of FEM/BEM

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If $a(\cdot, \cdot)$ is “nice” (continuous and coercive) then the continuous and discrete problems both have unique solutions satisfying

$$\| u - u^N \|_V \leq C \min_{v^N \in V_N} \| u - v^N \|_V \quad \leftarrow \text{Best approx. error in } V_N$$
How to choose $V_N$ for wave problems?

$$
\|u - u^N\|_V \leq C \min_{v^N \in V_N} \|u - v^N\|_V
$$

This holds for any finite-dimensional $V_N \subset V$. 

Conventional choice: $V_N = \{ \text{piecewise polynomials on a triangulation of } \Omega \text{ (or } \Gamma) \}$

Problem: requires $N = O(k^d)$ (FEM) or $N = O(k^d - 1)$ (BEM) to keep $\min_{v^N \in V_N} \|u - v^N\|_V$ fixed as $k \to \infty$.

Alternative choice: $V_N = \{ \text{piecewise polynomials } \times \text{oscillatory functions} \}$

Attraction: if chosen correctly, oscillatory functions should approximate the solution more efficiently (i.e. with smaller $N$) than piecewise polynomials alone.
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Choose oscillations based on high frequency asymptotics of solution

- FEM e.g. Giladi and Keller (2001).
- BEM e.g. Chandler-Wilde, Langdon, Hewett, Groth, Gibbs, Melenk, Graham, Dominguez, Smyshlyaev, Bruno, Huybrechs, Vandewalle, Ganesh, Hawkins...
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Many mathematical challenges:

- high frequency behaviour of solution
- estimation of \( \min_{v^N \in V_N} \| u - v^N \|_V \)
- find a “nice” (continuous and coercive) formulation, for error analysis
- evaluation of \( A_{ij} = a(\phi_j, \phi_i) \) (highly oscillatory integrals)
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HNA methodology well-understood for BEM for 2D convex scatterers.
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HNA methodology well-understood for BEM for 2D convex scatterers.
Current work: generalise to 3D, penetrable and nonconvex scatterers.
High frequency asymptotics - convex polygons

\[ u(x) = u^i(x) - \int_{\Gamma} \Phi(x, y) \frac{\partial u}{\partial n}(y) \, ds(y), \quad x \in D \]

According to Geometrical Optics/Geometrical Theory of Diffraction, on a “lit” side

\[ \frac{\partial u}{\partial n} \sim 2 \frac{\partial u^i}{\partial n} \]

incident + reflected

\[ + A + e^{ik \cdot x} \]

diffracted

\[ k \to \infty \]

where \( s \) is arc length along the side, measured from \( P_j \).
According to Geometrical Optics/Geometrical Theory of Diffraction, on a “lit” side

\[
\frac{\partial u}{\partial n} \sim 2 \frac{\partial u^i}{\partial n} + A^+ e^{iks} + A^- e^{-iks}, \quad k \to \infty
\]

where \( s \) is arc length along the side, measured from \( P_j \)
High frequency asymptotics - convex polygons

\[ u(x) = u^i(x) - \int_{\Gamma} \Phi(x, y) \frac{\partial u}{\partial n}(y) \, ds(y), \quad x \in D \]

On an “unlit” (or “shadow”) side

\[ \frac{\partial u}{\partial n} \sim A^+ e^{iks} + A^- e^{-iks}, \quad k \to \infty \]

\[ |d| = 1 \]
Regularity results - convex polygons

Theorem (Hewett, Langdon, Melenk (2013))

Let $\Omega$ be a convex polygon. Then on any side $\Gamma_j$

$$\frac{\partial u}{\partial n}(\mathbf{x}(s)) = \Psi(\mathbf{x}(s)) + v_j^+(s)e^{iks} + v_j^-(L_j - s)e^{-iks}, \quad 0 < s < L_j,$$

where

(i) $\Psi := 2\frac{\partial u^i}{\partial n}$ if $\Gamma_j$ is lit and $\Psi := 0$ otherwise,

(ii) $v_j^\pm(s)$ are analytic in $\text{Re}[s] > 0$, with

$$|v_j^+(s)| \leq Ck^2 \begin{cases} |ks|^{\pi/\Omega_j - 1}, & 0 < |s| \leq 1/k, \\ |ks|^{-1/2}, & |s| > 1/k, \end{cases}$$

where $\Omega_j$ is the exterior angle at the vertex $P_j$. 
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where $\Omega_{j+1}$ is the exterior angle at the vertex $P_{j+1}$. 
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\]

where \( \Omega_{j+1} \) is the exterior angle at the vertex \( P_{j+1} \).

To form HNA approximation space \( V_N \), replace \( v^\pm_j \) by piecewise polynomials.
Best approximation error - convex polygons

**Theorem (Hewett, Langdon, Melenk (2013))**

Under appropriate assumptions on the piecewise polynomial approximation, there exist constants $C, \tau > 0$, independent of $k$, such that

$$\min_{v^N \in V_N} \left\| \frac{\partial u}{\partial n} - v^N \right\|_{L^2(\Gamma)} \leq C k^2 e^{-\tau \sqrt{N}}.$$
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Result

We can provably achieve any required approximation accuracy with $N$ growing only like $\log^2 k$ as $k \to \infty$, rather than like $k$, as for a conventional BEM.
Numerical results - convex polygon

Plot the field arising from the numerical boundary solution:

\[ u^N(x) := u^i(x) - \int_{\Gamma} \Phi(x, y) \left( \frac{\partial u}{\partial n} \right)^N(y) \, ds(y), \quad x \in D \]
Numerical results - convex polygon

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\[ k = 10 \]

\[ k = 160 \]
Numerical results - convergence of $u^N$

**Theorem**

$$(\text{Relative maximum error in } D) \leq Ck^2 e^{-\tau \sqrt{N}}$$

(Here $p \propto \sqrt{N}$ is the maximum polynomial degree used)

Accuracy actually improves as $k$ gets larger!
Nonconvex polygons

High frequency asymptotic behaviour on $\Gamma$ is more complicated:

Multiple reflections

Partial illumination

Theorem (Chandler-Wilde, Hewett, Langdon, Twigger (2012))
For a class of nonconvex polygons we can achieve any required accuracy of approximation with $N$ growing only like $\log k$ as $k \to \infty$. 

Nonconvex polygons

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Multiple reflections

Partial illumination

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For a class of nonconvex polygons we can achieve any required accuracy of approximation with $N$ growing only like $\log^2 k$ as $k \to \infty$. 
Transmission problems - penetrable scatterers

Joint work with S. Groth and S. Langdon
(EPSRC CASE award with Met Office, Industrial supervisor A. Baran)

Motivating application: scattering by ice crystals in cirrus clouds

- First steps: 2D acoustic case, convex polygon
- High frequency asymptotic solution involves infinitely many refractions/reflections/diffractions
- Infinitely many phases to consider, even for a convex scatterer

\[
\begin{align*}
    u_1(x) &= u^i(x) + \int_{\Gamma} \left( u_1(y) \frac{\partial \Phi_1(x,y)}{\partial n(y)} - \Phi_1(x,y) \frac{\partial u_1(y)}{\partial n(y)} \right) \, ds(y), \quad x \in \Omega_1, \\
    u_2(x) &= \int_{\Gamma} \left( \Phi_2(x,y) \frac{\partial u_2(x)}{\partial n(y)} - u_2(y) \frac{\partial \Phi_2(x,y)}{\partial n(y)} \right) \, ds(y), \quad x \in \Omega_2,
\end{align*}
\]

Here \( \Omega_1 \) is exterior (\( k_1 \)), \( \Omega_2 \) is interior (\( k_2 \)).
HNA approximation space - GO terms

Compute Geometrical Optics (GO) approximation using a beam tracing algorithm:

(a) Primary beams from \( \Gamma_1 \)

(b) Primary beams from \( \Gamma_2 \)

(c) Primary beams from \( \Gamma_3 \)

(d) Secondary beams arising from transmitted beam in (a)

(e) Secondary beams arising from transmitted beam in (b)

(f) Secondary beams arising from transmitted beam in (c)
HNA approximation space - GO terms

Compute Geometrical Optics (GO) approximation using a beam tracing algorithm:

Using this alone in integral representation corresponds to Physical-Geometrical Optics Hybrid (PGOH) method of Bi et al ('11), see also Yang and Liou ('95,'96,'97), Muinonen ('89). We want to include diffracted field.
HNA approximation space - diffraction terms

Problem! No closed form solution yet known for canonical diffraction problem (transmission wedge), cf. Rawlins ’99

- Use “heuristic” choice of phases for diffracted field
- Need to include oscillations at both interior and exterior wavenumbers
- Compare GO alone with (1) adding diffraction from adjacent corners and (2) adding diffraction from opposite corners too

Compute “numerical best approximation errors” by comparison with a reference solution computed using a standard BEM

(Full HNA BEM currently being implemented)

In our experiments we use fix $N = 168$ and vary $k = 5, 10, 20, 40, 80, 160$
Best approx. errors on the boundary

| $k_1$ | $\xi$  | $\frac{||u-u_{go}||}{||u||}$ | $\frac{||u-U_1||}{||u||}$ | $\frac{||u-U_2||}{||u||}$ |
|-------|--------|-----------------------------|-----------------------------|-----------------------------|
| 5     | 0.05   | $1.88 \times 10^{-1}$       | $1.66 \times 10^{-2}$       | $2.57 \times 10^{-3}$       |
| 10    | 0.05   | $1.37 \times 10^{-1}$       | $1.03 \times 10^{-2}$       | $1.35 \times 10^{-3}$       |
| 20    | 0.05   | $1.00 \times 10^{-1}$       | $8.41 \times 10^{-4}$       | $3.72 \times 10^{-4}$       |
| 40    | 0.05   | $7.25 \times 10^{-2}$       | $2.23 \times 10^{-4}$       | $2.20 \times 10^{-4}$       |
| 80    | 0.05   | $5.19 \times 10^{-2}$       | $2.58 \times 10^{-4}$       | $2.58 \times 10^{-4}$       |
| 160   | 0.05   | $3.69 \times 10^{-2}$       | $2.31 \times 10^{-4}$       | $2.31 \times 10^{-4}$       |
| 5     | 0.0125 | $2.48 \times 10^{-1}$       | $4.05 \times 10^{-2}$       | $8.02 \times 10^{-3}$       |
| 10    | 0.0125 | $1.84 \times 10^{-1}$       | $7.88 \times 10^{-2}$       | $9.46 \times 10^{-3}$       |
| 20    | 0.0125 | $1.28 \times 10^{-1}$       | $4.53 \times 10^{-2}$       | $9.42 \times 10^{-3}$       |
| 40    | 0.0125 | $9.13 \times 10^{-2}$       | $1.05 \times 10^{-2}$       | $2.66 \times 10^{-3}$       |
| 80    | 0.0125 | $6.69 \times 10^{-2}$       | $1.87 \times 10^{-3}$       | $1.79 \times 10^{-3}$       |
| 160   | 0.0125 | $4.84 \times 10^{-2}$       | $7.52 \times 10^{-4}$       | $7.52 \times 10^{-4}$       |
| 5     | 0      | $2.57 \times 10^{-1}$       | $5.30 \times 10^{-2}$       | $1.16 \times 10^{-2}$       |
| 10    | 0      | $2.15 \times 10^{-1}$       | $1.43 \times 10^{-1}$       | $1.95 \times 10^{-2}$       |
| 20    | 0      | $1.79 \times 10^{-1}$       | $1.48 \times 10^{-1}$       | $2.82 \times 10^{-2}$       |
| 40    | 0      | $1.50 \times 10^{-1}$       | $1.34 \times 10^{-1}$       | $3.07 \times 10^{-2}$       |
| 80    | 0      | $1.25 \times 10^{-1}$       | $1.17 \times 10^{-1}$       | $3.17 \times 10^{-2}$       |
| 160   | 0      | $1.04 \times 10^{-1}$       | $1.00 \times 10^{-1}$       | $2.81 \times 10^{-2}$       |

Refractive index is $k_2/k_1 = 1.31 + \xi i$

Smaller $\xi$ (less absorption) ⇒ need to include more diffracted terms
Smaller $k$ (lower frequency) ⇒ need to include more diffracted terms
Best approx. errors: far-field pattern

Refractive index is $\frac{k_2}{k_1} = 1.31 + 0.05i$
3D problems

Scattering by a planar screen in 3D

Complexity of high frequency asymptotics similar to that of the 2D transmission problem

- Numerical best approximation results are promising
- Currently implementing a BEM (with J. Hargreaves, Salford)
- Analysis would have to be in $\tilde{H}^{-1/2}(\Gamma)$. Already have:
  - full NA for 2D problem of multiple collinear screens (with S. Langdon and S. Chandler-Wilde)
  - $k$-explicit continuity and coercivity results for 2D and 3D case (with S. Chandler-Wilde)
Conclusions and outlook

- High frequency scattering problems are numerically challenging
- FEM/BEM offers a flexible approximation strategy but conventional approximation spaces are computationally expensive
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- Possible approach for attacking “real-world” problems: try a combination of conventional and HNA methods (Gibbs, Langdon, Chandler-Wilde)
References


- D. P. Hewett, S. Langdon, S. N. Chandler-Wilde, *A frequency-independent boundary element method for scattering by two-dimensional screens and apertures*, under review

Preprints available at www.maths.ox.ac.uk/~hewett

For a more general review: