Bloch wave excitation and the Wiener–Hopf technique

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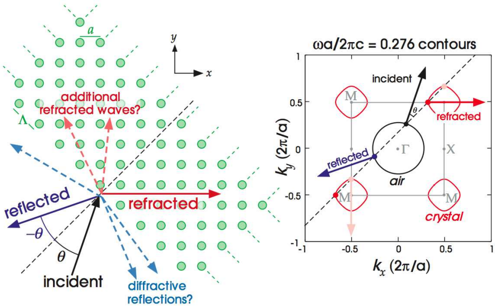
Bloch waves

- A plane wave incident from outside can excite Bloch waves in a crystalline structure that supports them.
- Joannopoulos et al. describe the process, but no calculations are given.

'The amplitudes of the refracted and reflected waves . . . require a more detailed solution of the Maxwell equations.'



'Photonic Crystals Molding the Flow of Light'
Joannopoulos et al. (2008)

The wave equation

- Consider the acoustic wave equation, with speed of sound c:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) U(\mathbf{r}, t) = 0.$$

Other physical contexts (electromagnetism, water waves, elasticity) are similar; the algebra for the electromagnetic case is a lot worse.

- Look for time-harmonic solutions:

$$\begin{aligned} U(\mathbf{r}, t) &= u_1(\mathbf{r})\cos(\omega t) - u_2(\mathbf{r})\sin(\omega t) \\ &= \mathrm{Re}\left[u(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right] \end{aligned}$$

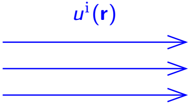
where u is a complex valued function of position.

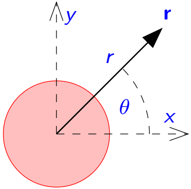
- Now we just have to solve the Helmholtz equation for u:

$$(\nabla^2 + k^2)u(\mathbf{r}) = 0, \quad k = \omega/c.$$

Single scattering

- Consider scattering by one circular cylinder (no variation in z).





- By separation of variables, the incident and scattered waves can be expanded in the form

$$u^{\mathrm{i}}(\mathbf{r}) = \sum_{n=-\infty}^{\infty} I_n \mathcal{J}_n(\mathbf{r}) \quad \text{and} \quad u^{\mathrm{s}}(\mathbf{r}) = \sum_{n=-\infty}^{\infty} A_n \mathcal{H}_n(\mathbf{r})$$

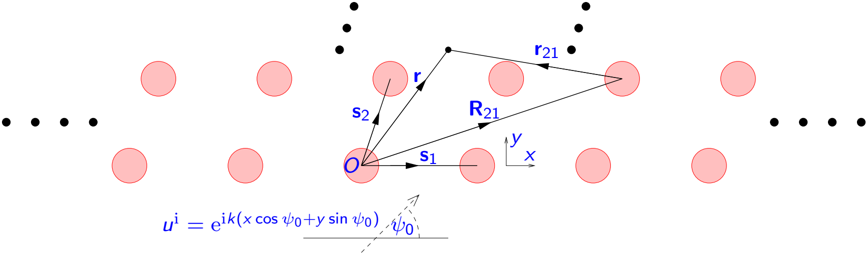
where

$$\mathcal{J}_n(\mathbf{r}) = \mathrm{J}_n(kr)\mathrm{e}^{\mathrm{i}n\theta} \qquad \text{and} \quad \mathcal{H}_n(\mathbf{r}) = \Big[\mathrm{J}_n(kr) + \mathrm{i}\,\mathrm{Y}_n(kr)\Big]\mathrm{e}^{\mathrm{i}n\theta}.$$

- I_n is known; A_n is related to I_n by the boundary conditions.
- At low to moderate frequencies, $A_n \to 0$ rapidly as $|n| \to \infty$.
- We are not treating scatterers as points!

Multiple scattering

- The same idea works for multiple bodies, but now there is a set of unknowns associated with each scatterer.



- We consider an array that is infinite in x and semi-infinite in y, so

$$u^{\mathrm{s}}(\mathbf{r}) = \sum_{j=-\infty}^{\infty}\sum_{p=0}^{\infty}\sum_{n=-\infty}^{\infty} A_n^{jp}\,\mathcal{H}_n(k\mathbf{r}_{jp}).$$

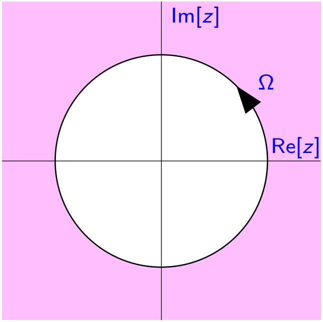
- Also $u(\mathbf{r} + j\mathbf{s}_1) = \mathrm{e}^{\mathrm{i}kjs_1\cos\psi_0}u(\mathbf{r})$, so $A_n^{jp} = \mathrm{e}^{\mathrm{i}kjs_1\cos\psi_0}A_n^{0p}$; we need 'only' determine A_n^{0p}.

Array scanning

- Introduce the z-transform by writing

$$A_n^{0p} = \frac{1}{2\pi \mathrm{i}} \int_\Omega A_n^+(z) z^{-p-1} \, \mathrm{d}z,$$

  where Ω is the unit circle (possibly with indentations).
- Dependence on row number (p) now appears in the exponent only.



- Since $A_n^{0p} = 0$ for $p < 0$, there must be no singularities inside Ω.
- Poles with $|z| > 1$: contribution to $A_n^{0p} \to 0$ as $p \to \infty$.
- Poles on $|z| = 1$: Bloch waves — $A_n^{0p} \not\to 0$ as $p \to \infty$.

Array scanning (ctd.)

- After z-transformation, we have

$$u^{\mathrm{s}}(\mathbf{r}) = \frac{1}{2\pi\mathrm{i}} \int_{\Omega} \left[\sum_{n=-\infty}^{\infty} A_n^{+}(z) \sum_{j=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kjs_1 \cos\psi_0}}{z^{p+1}} \mathcal{H}_n(k\mathbf{r}_{jp}) \right] \mathrm{d}z.$$

  - '*looks*' quasiperiodic in transform space,
  - the slowly convergent series contain no unknowns.

- There is one unknown function $A_n^{+}(z)$ for every mode included in the local expansions about the scatterers.

- Applying the boundary conditions on the cylinder surfaces leads to a matrix Wiener–Hopf equation for these.

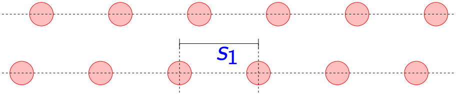
Another view — grating modes

- Look for solutions with the same quasi-periodicity as the incident field:

$$u(\mathbf{r}) = \mathrm{e}^{\mathrm{i}kx\cos\psi_j}\phi(y) \quad \Rightarrow \quad \phi(y) = \mathrm{e}^{\pm\mathrm{i}ky\sin\psi_j},$$

  where $\cos\psi_j = \cos\psi_0 + 2j\pi/(ks_1)$.

- The field between rows can be expanded using a spectral basis.



$$u(x,y) = \sum_{j=-\infty}^{\infty} \underbrace{\mathcal{A}_j^+ \mathrm{e}^{\mathrm{i}k(x\cos\psi_j + y\sin\psi_j)}}_{\text{upwards propagating}} + \underbrace{\mathcal{A}_j^- \mathrm{e}^{\mathrm{i}k(x\cos\psi_j - y\sin\psi_j)}}_{\text{downwards propagating}},$$

- A good approximation is obtained by truncating the series at $|j| = Q$, say, provided that $\cos\psi_Q > 1$.
- This simple structure (plane & evanescent modes) rules out branch points in the Wiener–Hopf equation.

Wiener–Hopf equation

- Writing the transformed equations in matrix form yields (eventually)

$$K(z)\mathbf{A}^+(z) = \mathbf{T}^+(z) + \mathbf{T}^-(z).$$

All functions are rational, $\mathbf{T}^+(z)$ is known and $\mathbf{T}^-(z) \to 0$ as $z \to \infty$.

- Poles in Ω^- (outside the contour) can be located using the LHS; $\mathbf{T}^-(z)$ is known up to a set of constants.
- At points $z_q \in \Omega^+$ at which $\det K(z) = 0$, only certain right-hand sides are permitted. In fact, if

$$K^*(z_q)\mathbf{E}_q = 0 \quad \text{with} \quad |\mathbf{E_q}| \neq 0,$$

then

$$\mathbf{E}_q^*\left(\mathbf{T}^+(z_q) + \mathbf{T}^-(z_q)\right) = 0.$$

- It can be shown that the number of points z_q is equal to the number of unknown constants in the vector $\mathbf{T}^-(z)$.

Residues

- There are poles at points $z_q \in \Omega^-$ (outside Ω) where $\det K(z) = 0$.
- Write $\mathbf{A}^+(z) = \dfrac{\mathbf{B}}{z - z_q} + \hat{\mathbf{A}}^+(z)$, where $\hat{\mathbf{A}}^+$ is regular at $z = z_q$.
- Use in Wiener–Hopf equation:

$$K(z)\big[\mathbf{B} + (z - z_q)\hat{\mathbf{A}}^+(z)\big] = (z - z_q)\big[\mathbf{T}^+(z) + \mathbf{T}^-(z)\big];$$

  hence $K(z_q)\mathbf{B} = \mathbf{0} \quad (*)$.
- Also,

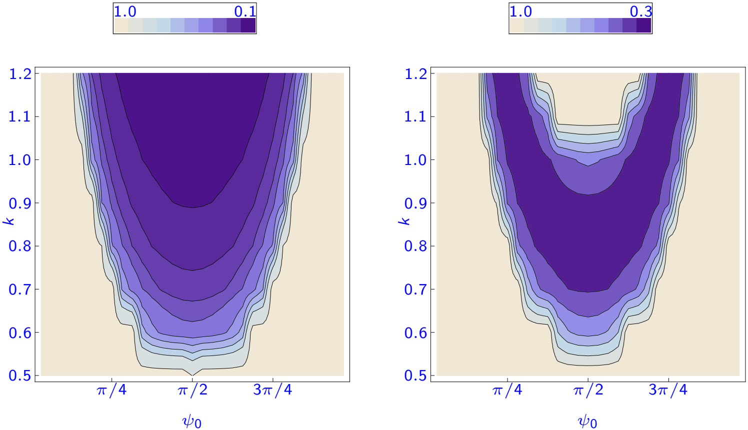
$$K(z)\hat{\mathbf{A}}^+(z) = \mathbf{T}^+(z) + \mathbf{T}^-(z) + \frac{K(z)}{z - z_q}\mathbf{B},$$

  so

$$\mathbf{E}_q^*\left(\mathbf{T}^+(z_q) + \mathbf{T}^-(z_q) + \lim_{z \to z_q}\frac{K(z)}{z - z_q}\mathbf{B}\right) = 0. \quad (\dagger).$$

- $\mathbf{B}$ is determined by $(*)$ and $(\dagger)$.

Amplitude of reflection

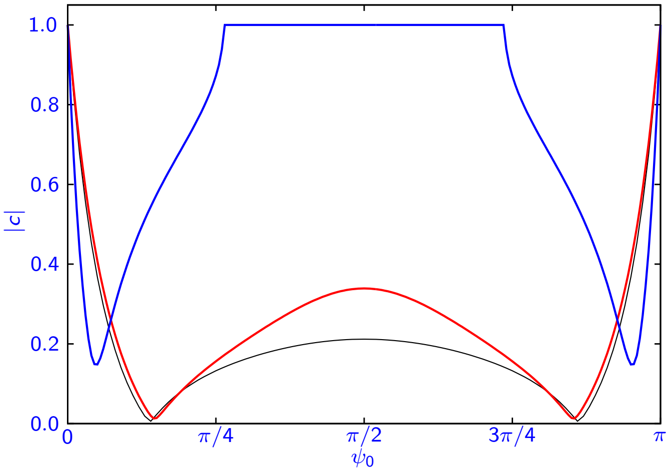


$a = 0.001$, $\mathbf{s}_1 = [2, 0]$, $\mathbf{s}_2 = [0, 2]$ $a = 0.001$, $\mathbf{s}_1 = [1, 0]$, $\mathbf{s}_2 = [0, 3]$

(Dirichlet boundary conditions.)

A new result

$\mathbf{s}_1 = [1, 0]$, $\mathbf{s}_2 = [0, 1]$, Neumann boundary conditions.
$k = 1.4$, $a = 0.25$; $k = 2.0$, $a = 0.25$; $k = 2.0$, $a = 0.45$



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  - More general work to follow.