Homogenization of ‘micro-resonances’ and localization of waves.

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(joint work with Ilia Kamotski UCL, and Shane Cooper Bath/ Cardiff)

Outline:

- **Classical homogenization** for waves (= **low frequencies**)

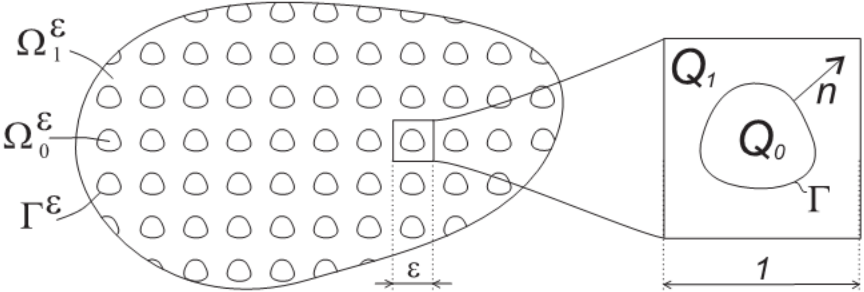
- Higher frequencies + higher contrasts ('degeneracies')
$\longrightarrow$ **'resonant' homogenization**

- **Effects**: (frequency/wavenumber) band gaps, dispersion, 'negative' materials, etc.

- **'Partial' degeneracies** and resonances (more of effects; general theory)

- Photonic Crystal Fibers as an example of **partial degeneracies**
$\longrightarrow$ Band gaps in PCFs. (Cooper, Kamotski, V.S., 2012).

Classical Periodic Homogenisation for waves (= *low frequencies*) ('anti-plane' elastodynamics/ TM electrodynamics, for simplicity)



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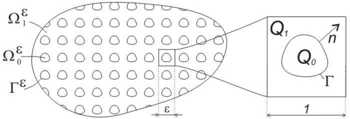

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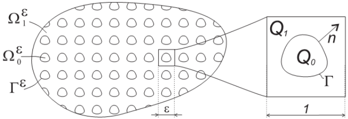

$$\rho^\varepsilon(x)u_{tt} - \mathrm{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right)\right) = f(x,t)$$

$f(x,t) \equiv 0,\ t \le 0;\quad u(x,t) \equiv 0,\ t \le 0.$ $\quad a$ 'stiffness', ρ 'density'

$a^\varepsilon(x) = a(x/\varepsilon),\ \rho^\varepsilon = \rho(x/\varepsilon)$ $\quad a(y), \rho(y)$ Q-periodic in y

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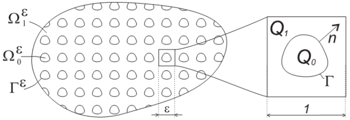

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Asymptotic expansion

$\mathrm{u}^\varepsilon(x,t) \sim$
$u^0(x, x/\varepsilon, t) + \varepsilon u^1(x, x/\varepsilon, t) + \ldots$ $\ u^0(x,y,t), u^1(x,y,t)$ Q-periodic in y

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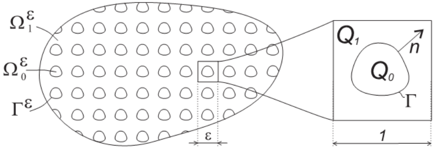
The Homogenisation Theorem

$C > 0$ independent of ε such that $\| u^\varepsilon - (u^0 + \varepsilon u^1) \|_{\mathcal{H}^1} \leq C \varepsilon^{1/2}$

High-contrast (= 'micro-resonant') homogenization and 'non-classical' two-scale limits (Zhikov 2000, 2004)



$$A^\varepsilon u = -\operatorname{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right)$$

$$a^\varepsilon(x) = \begin{cases} \varepsilon^2 & \text{on } \Omega_0^\varepsilon \text{ ('soft' phase)} \\ 1 & \text{on } \Omega_1^\varepsilon \text{ ('stiff' phase)} \end{cases}$$

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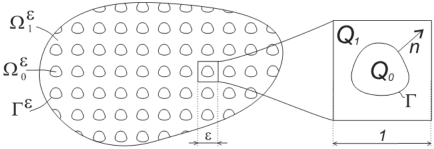

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Contrast $\delta \sim \varepsilon^2$ is a **critical scaling** giving rise to 'non-classical' effects (Khruslov 1980s; Arbogast, Douglas, Hornung 1990; Panasenko 1991; Allaire 1992; Sandrakov 1999; Brianne 2002; Bourget, Mikelic, Piatnitski 2003; Bouchitte & Felbaq 2004, ...): elliptic, spectral, parabolic, hyperbolic, nonlinear, non-periodic/ random, ...

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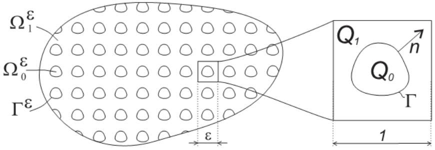

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WHY?

Two-scale formal asymptotic expansion:

$\text{div}\,(a^\varepsilon(x)\nabla u^\varepsilon) + \rho\omega^2 u^\varepsilon = 0 \;\; (timeharmonicwaves)$
$\Longleftrightarrow A^\varepsilon u^\varepsilon = \lambda u^\varepsilon, \;\; \lambda = \rho\omega^2 \;\; (spectral\; problem).$
Seek $\mathrm{u}^\varepsilon(x) \sim u^0(x, x/\varepsilon) + \varepsilon u^1(x, x/\varepsilon) + \ldots \;\; u^j(x, y) \; Q - periodic\; in\, y.$

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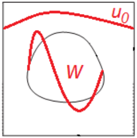
THEN:

Two-scale limit problem (Zhikov 2000, 2004)

Then $\quad u^0(x,y) =$

$$\begin{cases} u^0(x) & \text{in } Q_1 \quad \text{(still low frequency)} \\ w(x,y) & \text{in } Q_0 \quad \text{(\textit{`resonance'} frequency)} \end{cases}$$



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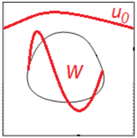

(u^0, w), $w(x,y) := u^0(x) + v(x,y)$,

solves the two-scale limit spectral problem:

$$-\mathrm{div}_x(a^{hom}\nabla_x u(x)) = \lambda u(x) + \lambda < v > (x) \quad \text{in } \Omega$$

$$-\mathrm{div}_y(a^{(0)}\nabla_y v(x,y)) = \lambda(u(x) + v(x,y)) \qquad \text{in } Q_0$$

$$v(x,y) = 0 \qquad \text{on } \partial Q_0$$

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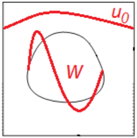

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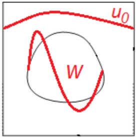

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Decouple it $\downarrow$

Two-scale limit spectral problem

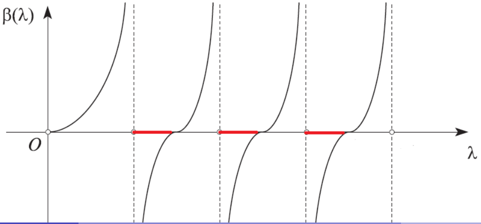
Decouple by choosing $v(x,y) = \lambda u(x) b(y)$

$$
\begin{aligned}
-\mathrm{div}_y(a^{(0)}\nabla_y b(y)) \;-\; \lambda u = 1 \qquad & \text{in } Q_0 \\
b(y) = 0 \qquad & \text{on } \partial Q_0
\end{aligned}
$$

$$-\mathrm{div}_x(a^{hom}\nabla u(x)) = \beta(\lambda) u(x), \qquad \textit{in } \Omega,$$

$$\text{where} \quad \beta(\lambda) \;=\; \lambda + \lambda^2 \sum_{j=1}^{\infty} \frac{\langle \phi_j \rangle_y^2}{\lambda_j - \lambda},$$

(λ_j, ϕ_j) Dirichlet eigenvalues/functions of inclusion Q_0 (= "micro-resonances": $\beta < 0$ "negative denstity/magnetism"' etc)



Rigorous analysis: Two-scale Convergence

Definition

1. Let $u_\varepsilon(x)$ be a bounded sequence in $L^2(\Omega)$. We say (u_ε) *weakly two-scale converges* to $u_0(x,y) \in L^2(\Omega \times Q)$, denoted by $u_\varepsilon \overset{2}{\rightharpoonup} u_0$, if for all $\phi \in C_0^\infty(\Omega)$, $\psi \in C_\#^\infty(Q)$

$$\int_\Omega u_\varepsilon(x)\phi(x)\psi\left(\frac{x}{\varepsilon}\right) \mathrm{d}x \longrightarrow \int_\Omega \int_Q u_0(x,y)\phi(x)\psi(y)\ \mathrm{d}x\mathrm{d}y$$

as $\varepsilon \to 0$.

2. We say (u_ε) *strongly* two-scale converges to $u_0 \in L^2(\Omega \times Q)$, denoted by $u_\varepsilon \overset{2}{\to} u_0$, if for all $v_\varepsilon \overset{2}{\rightharpoonup} v_0(x,y)$,

$$\int_\Omega u_\varepsilon(x) v_\varepsilon(x)\mathrm{d}x \longrightarrow \int_\Omega \int_Q u_0(x,y)v_0(x,y)\ \mathrm{d}x\mathrm{d}y$$

as $\varepsilon \to 0$. (implies convergence of norms upon sufficient regularity)

Rigorous analysis (Zhikov 2000, 2004):

1. Two-scale resolvent convergence:

$\alpha > 0, \quad A^\varepsilon u^\varepsilon + \alpha u^\varepsilon = f^\varepsilon \in L^2(\Omega); \quad u^\varepsilon \in H^1_0(\Omega)$.
If $f^\varepsilon \overset{2}{\rightharpoonup} f_0(x,y)$ then $u^\varepsilon \overset{2}{\rightharpoonup} u_0(x,y)$. (If $f^\varepsilon \overset{2}{\to} f_0(x,y)$ then $u^\varepsilon \overset{2}{\to} u_0(x,y)$.)
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2. Spectral band gaps: (Let $\Omega = \mathbb{R}^d$.)

A_0 self-adjoint in $H \subset L^2(\mathbb{R} \times Q_0)$, with a band-gap spectrum $\sigma(A_0)$.

$\sigma(A^\varepsilon) \to \sigma(A_0)$ in the sense of Hausdorff. (Hence a Band-gap effect: For small enough ε waves of certain frequencies do not propagate, in any direction). The proof is based on a "two-scale spectral compactness".

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Limit band gaps $=\{\lambda : \beta(\lambda) < 0\}$. Interpretation : $\beta(\lambda) = \mu(\omega) < 0 \leftrightarrow$ negative magnetism/density (Bouchitte & Felbacq, 2004).

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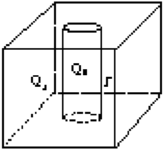
$\beta(\lambda) > 0$ - propagation with (high) dispersion $(\lambda \to \lambda_j -)$, due to "coupled resonances"' (V.S. & P. Kuchment, 2007).

‘Frequency’ vs “directional” gaps and ‘partial’ degeneracies

Cherednichenko, V.S., Zhikov (2006): spatial nonlocality for homogenised limit with highly anisotropic fibers.



$$a^{\varepsilon}(x) = \begin{cases} \sim \quad 1 & \text{in } Q_1 (matrix) \\ \sim \quad \varepsilon^2 & \text{in } Q_0 \text{ “across” fibers} \\ \sim \quad 1 & \text{in } Q_0 \text{ “along” fibers} \end{cases}$$

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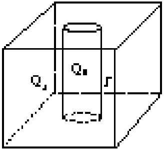

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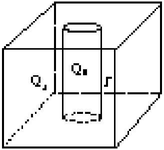

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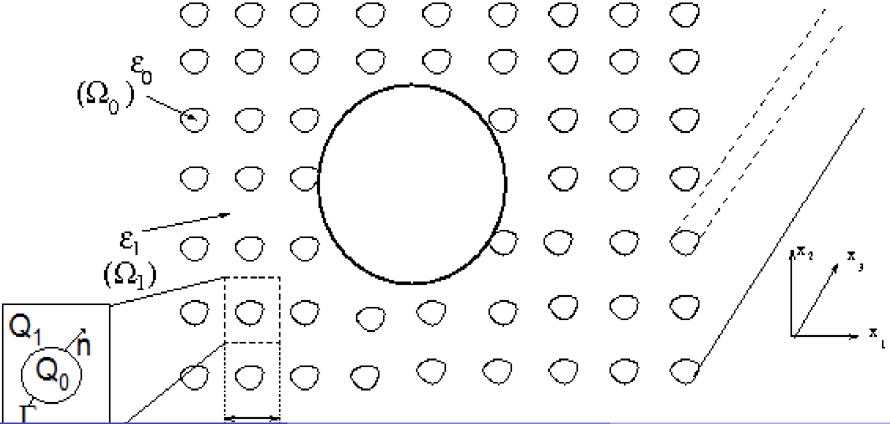
Notice, in the above fibres, $a^\varepsilon(x) = a^{(1)}(x/\varepsilon) + \varepsilon^2 a^{(0)}(x/\varepsilon)$, *where*

$$a^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad i.e. \textit{ is partially degenerate.}$$

Photonic crystal fibers: A partially degenerate problem (Cooper, I. Kamotski, V.S. 2012); cf scalar prototype problem I.Kamotski V.S. 2006; M. Cherdantsev 2009 ('full' contrast)



Photonic crystal fibers: Problem Formulation

$\nabla \times E = i\omega\mu H,$
$\nabla \times H = -i\omega\epsilon E$
$\epsilon = \epsilon_0\chi_0(x/\varepsilon) + \epsilon_1\chi_1(x/\varepsilon)$
$\epsilon_0 > \epsilon_1, \quad \mu$ constant
E=$\exp(ikx_3)E(x_1, x_2),$
H=$\exp(ikx_3)H(x_1, x_2)$

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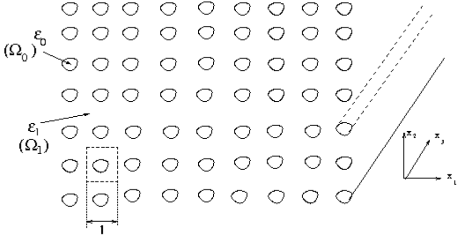

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$$\epsilon = \epsilon_0\chi_0(x/\varepsilon) + \epsilon_1\chi_1(x/\varepsilon)$$
$\epsilon_0 > \epsilon_1, \quad \mu$ constant

E=$\exp(ikx_3)E(x_1, x_2)$,

H=$\exp(ikx_3)H(x_1, x_2)$

In each phase E_3 and H_3 satisfy the following equations

$$\Delta E_3 + a^\varepsilon E_3 = 0, \qquad \Delta H_3 + a^\varepsilon H_3 = 0$$

where $a^\varepsilon = \omega^2\mu\epsilon(x/\varepsilon)$ - k^2. E$_3$ and H_3 coupled across interface Γ^ε:

$$\omega\left[\frac{\epsilon}{a^\varepsilon}\nabla E_3 \cdot n\right] = -k\left[\frac{1}{a^\varepsilon}\nabla H_3 \cdot n^\perp\right], \quad k\left[\frac{1}{a^\varepsilon}\nabla E_3 \cdot n^\perp\right] = \omega\left[\frac{\mu}{a^\varepsilon}\nabla H_3 \cdot n\right]$$

Photonic crystal fibers: A partially degenerate problem

Consider the problem in its weak form:

$$\partial_1\left(\frac{\omega\epsilon}{a^\varepsilon}E_{3,1}\right)+\partial_2\left(\frac{\omega\epsilon}{a^\varepsilon}E_{3,2}\right)+\partial_1\left(\frac{k}{a^\varepsilon}H_{3,2}\right)-\partial_2\left(\frac{k}{a^\varepsilon}H_{3,1}\right)=-\omega\epsilon E_3$$

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For $u=(E_3,H_3)$, find u such that

$$\int_{\mathbb{R}^2}\frac{\omega}{a^\varepsilon}\left(\epsilon\nabla u_1\cdot\overline{\nabla\phi_1}+\mu\nabla u_2\cdot\overline{\nabla\phi_2}\right)+\frac{k}{a^\varepsilon}\left(\left\{\overline{\phi_1},u_2\right\}+\left\{u_1,\overline{\phi_2}\right\}\right)\,\mathrm{d}x$$
$$=\int_{\mathbb{R}^2}\omega\rho(x/\varepsilon)u\cdot\overline{\phi}\quad\forall\phi\in C_0^\infty(\mathbb{R}^2)$$

$\{u,v\}:=u_{,1}v_{,2}-v_{,1}u_{,2}$ (Poisson bracket); $\quad \rho(y)=\begin{pmatrix}\epsilon(y) & 0\\ 0 & \mu\end{pmatrix}$

Form positive if $k^2<\omega^2\mu\epsilon_1$.

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Form positive if $k^2<\omega^2\mu\epsilon_1$. Consider a 'near critical' k: $k^2=\omega^2\mu(\epsilon_1-\varepsilon^2)$, $\omega^2\mu=\lambda$

Photonic crystal fibers: **a partially degenerate** problem

An 'emergent' high contrast:

$$\int_{\mathbb{R}^2} A_\varepsilon(x)\nabla u \cdot \overline{\nabla\phi}\,\mathrm{d}x = \lambda \int_{\mathbb{R}^2} \rho(x/\varepsilon) u \cdot \overline{\phi}\mathrm{d}x \quad \forall \phi \in C_0^\infty(\mathbb{R}^2)$$

Here $A_\varepsilon(x) = A(x/\varepsilon)$ where $A(y) = a^{(1)}(y) + \varepsilon^2 a^{(0)}(y) \ + O(\varepsilon^4)$.

$$\mathrm{a}^{(1)} \geq 0 \text{ BUT } \mathrm{a}^{(1)}(y) + a^{(0)}(y) > \nu I, \ \nu > 0$$

$a^{(1)}(y)\nabla u \cdot \nabla u = \chi_1(y)\left(|u_{1,1} + u_{2,2}|^2 + |u_{1,2} - u_{2,1}|^2\right)$ (partially) DEGENERATES for u s.t. RHS zero. i.e. satisfies Cauchy-Riemann type equations in matrix phase.

Photonic crystal fibers: **a partially degenerate** problem

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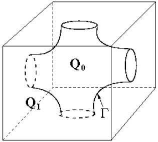
Moral: This appears a particular case of homogenization for partially degenerating PDE systems $\downarrow$

General 'Partial' Degeneracies (I. Kamotski and V.S. 2012)



Consider a 'resolvent' problem:

$\Omega \in \mathbb{R}^d, \quad \alpha > 0,$

$-\ \mathrm{div}\left(a^\varepsilon(x)\nabla u^\varepsilon\right) + \alpha\rho^\varepsilon u^\varepsilon = f^\varepsilon \in L^2(\Omega),$

$u^\varepsilon \in \left(H_0^1(\Omega)\right)^n, \ n \geq 1.$

A general degeneracy

$$a^\varepsilon(x) \ = \ a^{(1)}\left(\tfrac{x}{\varepsilon}\right) \ + \ \varepsilon^2\, a^{(0)}\left(\tfrac{x}{\varepsilon}\right),$$

$$a^{(l)} \in \left(L^\infty_\#(Q)\right)^{n\times d\times n\times d}, \quad a^{(1)} \geq 0, \ a^{(1)} + a^{(0)} > 0,$$

Weak formulation:

$$\int_\Omega \left[a^{(1)}\left(\frac{x}{\varepsilon}\right)\nabla u\cdot\nabla\phi(x) + \varepsilon^2\, a^{(0)}\left(\frac{x}{\varepsilon}\right)\nabla u\cdot\nabla\phi(x) + \alpha\,\rho^\varepsilon(x)\,u\cdot\phi(x)\right]dx$$

$$= \int_\Omega f^\varepsilon(x)\cdot\phi(x)\,dx, \ \ \forall\phi \in \left(H_0^1(\Omega)\right)^d.$$

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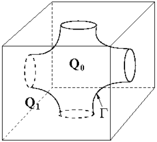

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A priori estimates:

Weak two-scale limits. Key assumption on the degeneracy

Introduce

$$V := \left\{ v \in \left(H^1_{\#}(Q)\right)^n \,\middle|\, a^{(1)}(y)\nabla_y v = 0 \right\}.$$

(subspace of "microscopic oscillations"), and

$$W := \left\{ \psi \in \left(L^2_{\#}(Q)\right)^{n\times d} \,\middle|\, \mathrm{div}_y \left(\left(a^{(1)}(y)\right)^{1/2} \psi(y) \right) = 0 \text{ in } \left(H^{-1}_{\#}(Q)\right)^n \right\}$$

("microscopic fluxes")

Then, up to a subsequence, $u^\varepsilon \overset{2}{\rightharpoonup} u_0(x,y) \in L^2(\Omega; V)$

$$\varepsilon \nabla u^\varepsilon \overset{2}{\rightharpoonup} \nabla_y u_0(x,y)$$

$$\xi^\varepsilon(x) := \left(a^{(1)}(x/\varepsilon)\right)^{1/2} \nabla u^\varepsilon \overset{2}{\rightharpoonup} \xi_0(x,y) \in L^2(\Omega; W).$$

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Key assumption

There exists a constant $C > 0$ such that for all $v \in \left(H^1_\#(Q)\right)^n$

$$\|P_{V^\perp} v\|_{\left(H^1_\#(Q)\right)^n} \le C \left\|a^{(1)}(y)\nabla_y v\right\|_{L^2}$$

The two-scale Limit Operator

Let Ω be bounded Lipschitz, or $\Omega = \mathbb{R}^d$. Introduce $U \subset L^2(\Omega; V)$:

$$U := \left\{ u(x,y) \in L^2(\Omega; V) \middle| \exists\, \xi(x,y) \in L^2(\Omega; W) \text{ s.t., } \forall\, \Psi(x,y) \in C^\infty \right.$$

$$\int_\Omega \int_Q \xi(x,y) \cdot \Psi(x,y) dx dy = -\int_\Omega \int_Q u(x,y) \cdot \nabla_x \cdot \left(\left(a^{(1)}(y) \right)^{1/2} \Psi(x,y) \right)$$

Define $T : U \to L^2$ by $Tu = \xi$.

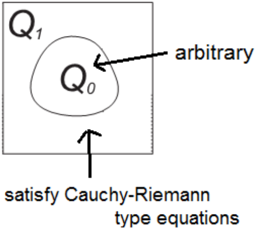
Strong two-scale resolvent convergence: Let $f^\varepsilon \xrightarrow{2} f_0(x,y)$.
Then $u^\varepsilon \xrightarrow{2} u_0(x,y)$ solving:

Find $u_0 \in U$ such that $\forall \phi \in U$

$$\int_\Omega \int_Q \Big\{ Tu_0(x,y) \cdot T\phi_0(x,y) + a^{(0)}(y) \nabla_y u_0(x,y) \cdot \nabla_y \phi_0(x,y) +$$

$$+ \alpha\, \rho(y) u_0(x,y) \cdot \phi_0(x,y) \Big\} dy\, dx = \int_\Omega \int_Q f_0(x,y) \cdot \phi_0(x,y)\, dy\, dx.$$

Back to Photonic Crystals

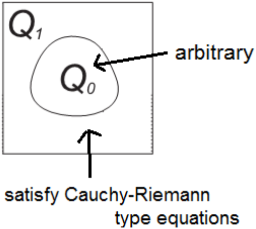


$$\mathrm{V}_\theta := \left\{ v \in \left(H^1_\theta(Q)\right)^n \,\middle|\, a^{(1)}(y)\nabla_y v = 0 \right\}.$$

$\mathrm{v} \in V_\theta$ iff v is θ-quasi-periodic, and

$\mathrm{v}_{1,1} + v_{2,2} = 0, \;\; v_{1,2} - v_{2,1} = 0$ in Q_1.

Back to Photonic Crystals



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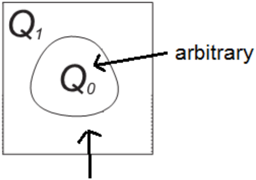
Theorem 1 (Key assumption holds)

There exists a constant $c > 0$ such that for any $u \in H^1_\theta(Q)$

$$\|P_{V_\theta^\perp} u\|_{H^1(Q)} \le c\left(\|u_{1,1} + u_{2,2}\|_{L^2(Q_0)} + \|u_{1,2} - u_{2,1}\|_{L^2(Q_0)}\right).$$

Back to Photonic Crystals



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Second Result

The generalised flux $\xi =: Tu$ is zero. i.e. $Tu = 0 \; \forall u \in U$. (Two-scale limit solution u_0 determined by miscroscopic behaviour only)

Limit Spectral problem:

Find $u \in V_\theta$ such that

$$\int_Q a^{(0)}(y)\nabla_y u(y) \cdot \overline{\nabla_y(\phi(y))}\,\mathrm{d}y = \lambda \int_Q \rho(y)u(y) \cdot \overline{\phi(y)}\,\mathrm{d}y \quad \forall \phi \in V_\theta.$$

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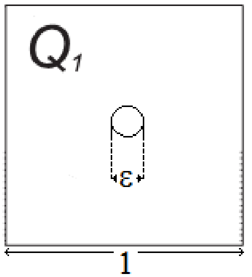
Theorem 2 (Spectral compactness: not trivial)

Let $\lambda_\varepsilon \in \sigma(A^\varepsilon)$ (Bloch spectrum). Let u^ε be associated normalized Bloch's waves: $A^\varepsilon u^\varepsilon = \lambda^\varepsilon u^\varepsilon$,
$\mathrm{u}^\varepsilon(x) = e^{i\theta^\varepsilon \cdot x/\varepsilon} v^\varepsilon(x/\varepsilon), \quad \|v^\varepsilon(y)\|_{L^2(Q)} = 1, \ \theta^\varepsilon \in (-\pi, \pi]^d$.
Let $\lambda \to \lambda_0$ and $\theta^\varepsilon \to \theta_0$. Then $\lambda_0 \in \sigma_0(A_0, \theta_0)$ (spectrum of the Limit operator), and $u^\varepsilon \to u_0(y)$, eigenfunction of $A_0(\theta)$. Hence the spectra converge.

Implication: If the limit problem displays a band-gap, the original problem must also have a gap for small enough ε.

Example: Band gaps in 'ARROW' fibres

An extreme problem: Q_0 a circle of small radius δ



Theorem:

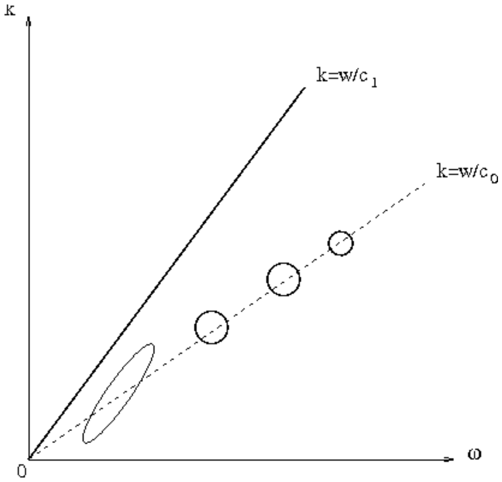
There exist constants c_1, $c_2 > 0$ independent of δ, such that, for all quasimomenta θ_0,

$$\lambda_2(\theta_0) \le -\frac{c_1}{\delta^2 \ln \delta}, \qquad \lambda_3(\theta_0) \ge c_2 \delta^{-2}.$$

This implies that, for small enough δ, there is a wide spectral gap in the limit spectrum, and therefore also for small enough ε for the original problem by the spectral compactness.

$+$ Higher gaps: $\lambda \sim \lambda_j^D(Q_0^\delta) \sim \delta^{-2}$ (re 'micro-resonances')

The band gaps in a Photonic Crystal Fiber:



Summary:

- Homogenization for a critical high contrast scaling $\delta \sim \varepsilon^2$ gives rise to numerous "non-classical" effects described by two-scale limit problems due to **"micro-resonances"**.

- **'Partial' degeneracies** often happen in physical problems, and give rise to more of such effects (e.g. **Band Gaps in Photonic Crystal Fibers**). These however have to be analysed in a new way.

- **A general two-scale homogenization theory** can be constructed for such **partial degeneracies**, under a generically held decomposition condition. Resulting limit (homogenized) operator is generically two-scale (and 'non-local').

- Associated two-scale operator and spectral convergence and compactness are held generally or for particular physical examples.

- In principle, some of this applies also to nonlinear (cf Cherednichenko & Cherdantsev 2011), discrete-to-continuous, as well as non-periodic problems.