Side branch resonators modelling with Green's function methods

E. Perrey-Debain, R. Maréchal, J.-M. Ville

Laboratoire Roberval UMR 6253, Equipe Acoustique,
Université de Technologie de Compiègne, France

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Outline

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- Impedance matrix and Green's function formalism
- Simplified models
- Low-frequency applications : Helmholtz and Herschel-Quincke resonators
- Medium-frequency applications : HQ-liner system for inlet fan noise reduction
- Conclusions

Introduction

Side branch resonators are commonly used for engine exhaust noise control : (i) low-frequency applications with a single plane wave mode (automotive) (ii) medium-frequency applications and highly multimodal context (aeronautics).

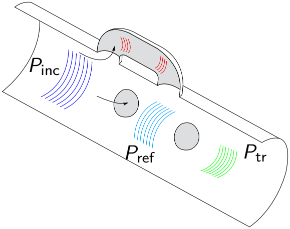


Figure: Side branch resonator (cavity Ω) with two openings Γ.

Numerical predictions : (i) 1D approximations (with length corrections) ; (ii) FEM, BEM -> computationally demanding (this includes mesh preparation etc...) especially in a highly multimodal context, lack of physical interpretation.

Impedance matrix

In the frequency domain, the acoustic pressure must obey the Helmholtz equation

$$\Delta p + k^2 p = 0, \quad k = \omega / c,$$

and $q = \partial_n p = 0$ everywhere *except* on Γ. The Green's function for the rigid-wall cavity is given by the infinite series

$$G_\Omega(\mathbf{r}, \mathbf{r}') = \sum_{n=0}^{\infty} \frac{\phi_n(\mathbf{r})\phi_n(\mathbf{r}')}{\lambda_n - \lambda}$$

where $\lambda = \omega^2$. Eigenfunctions ϕ_n are properly normalized so that application of the Green's theorem in the cavity yields

$$p(\mathbf{r}) = \int_\Gamma G_\Omega(\mathbf{r}, \mathbf{r}')\, q(\mathbf{r}')\, \mathrm{d}\Gamma(\mathbf{r}')$$

Impedance matrix

We need to precompute a finite set of eigenfunctions and estimate the truncation error...
Consider the eigenvector Φ_n, the FE discretization of nth eigenmode ϕ_n :

$$\mathbf{A}(\lambda_n)\Phi_n = 0 \quad \text{where} \quad \mathbf{A}(\lambda) = \mathbf{K} - \lambda\mathbf{M}$$

After reduction to the interfacial nodes, we obtain the impedance matrix

$$\mathbf{Z}(\lambda) = \mathbf{I}_\Gamma^{\mathrm{T}}\,\mathbf{A}^{-1}(\lambda)\,\mathbf{I}_\Gamma$$

with

$$\mathbf{A}^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Phi_n\Phi_n^{\mathrm{T}}}{\lambda_n - \lambda} = \Phi\,\mathbf{D}(\lambda)\,\Phi^{\mathrm{T}}$$

Finally,

$$\tilde{\mathbf{p}} = \mathbf{Z}(\lambda)\,\tilde{\mathbf{F}}\,\mathbf{q} = \mathbf{G}_\Omega\,\mathbf{q}$$

Truncation

Keeping the first N eigenmodes gives

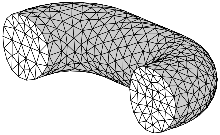
$$\mathbf{Z}(\lambda) = (\tilde{\Phi}\, \mathbf{D}(\lambda)\, \tilde{\Phi}^{\mathrm{T}})|_N + \mathbf{R}(\lambda).$$

The correction term $\mathbf{R}$ remains *weakly* dependent on the frequency, so we can take the first order Taylor expansion

$$\mathbf{R}(\lambda) \approx \mathbf{R}(\overline{\lambda}) + (\lambda - \overline{\lambda})\frac{\partial \mathbf{R}}{\partial \lambda} + \dots$$

The residual matrices are computed via

$$\begin{aligned} \mathbf{R} &= \mathbf{I}_\Gamma^{\mathrm{T}}\, \mathbf{A}^{-1}\, \mathbf{I}_\Gamma - (\tilde{\Phi}\, \mathbf{D}(\lambda)\, \tilde{\Phi}^{\mathrm{T}})|_N \\ \partial_\lambda \mathbf{R} &= \mathbf{I}_\Gamma^{\mathrm{T}}\, \mathbf{A}^{-1} \mathbf{M}\, \mathbf{A}^{-1} \mathbf{I}_\Gamma - (\tilde{\Phi}\, \partial_\lambda \mathbf{D}\, \tilde{\Phi}^{\mathrm{T}})|_N \end{aligned}$$



Scattering matrix

The theory starts by introducing the hard-walled duct Green's function

$$G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty} \frac{\psi_l(x,y)\psi_l^*(x',y')}{2\mathrm{i}\beta_l} \, \mathrm{e}^{\mathrm{i}\beta_l|z-z'|}$$

The transverse eigenmodes ψ_l are solution of the boundary value problem

$$(\partial_{xx}^2 + \partial_{yy}^2)\psi_l + k^2\psi_l = \beta_l^2\psi_l$$

with $\partial_n\psi_l = 0$ on the boundary line ∂S. These modes are normalized as

$$\int_S |\psi_l|^2 \, \mathrm{d}S = 1$$

in particular $\psi_0 = 1/\sqrt{A_d}$ where A_d is the cross section area of the main duct. For circular ducts :

$$\psi_l = N_{m,n} J_m(\alpha_{m,n} r)\mathrm{e}^{\mathrm{i}m\theta}, \quad \beta_l = \sqrt{k^2 - \alpha_{m,n}^2}$$

Scattering matrix

The finite element discretization of the integral equation

$$p(\mathbf{r}) = \int_\Gamma G(\mathbf{r}, \mathbf{r}')\, q(\mathbf{r}')\, \mathrm{d}\Gamma(\mathbf{r}') + p^I(\mathbf{r})$$

gives ($\mathbf{r}_i$ is the FE node location)

$$\tilde{p}_i = \sum_{j=1}^{\tilde{N}} G_{ij} q_j + p_i^I \quad \text{with} \quad G_{ij} = \int_\Gamma G(\mathbf{r}_i, \mathbf{r}')\, \tilde{\phi}_j(\mathbf{r}')\, \mathrm{d}\Gamma(\mathbf{r}')$$

The acoustic velocity is deduced from

$$\mathbf{q} = (\mathbf{G}_\Omega - \mathbf{G})^{-1}\, \mathbf{p}^I$$

Note :
i. The computation is not trivial as $\partial_{z'} G$ is discontinuous at $z' = z_i$.

ii. The matrix $\mathbf{G}_\Omega - \mathbf{G}$ is of *small* size.

Simplified models : one opening

Starting with one opening only, the impedance matrix relation can be averaged to give

$$\bar{p} = \bar{Z}(\lambda)\bar{q} \quad \text{where} \quad \bar{Z}(\lambda) = \frac{1}{\tilde{N}} \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} (\mathbf{Z}(\lambda)\,\tilde{\mathbf{F}})_{ij}$$

By the same token,

$$\bar{p} = \frac{1}{W}\bar{q} + \bar{p}^I, \quad \text{where} \quad W = \frac{2\mathrm{i}kA_d}{A}$$

and A denotes the area of the interface. An incident plane wave $p^I = \mathrm{e}^{\mathrm{i}kz}$ produces a transmitted pressure field

$$p = T\mathrm{e}^{\mathrm{i}kz} \quad \text{with} \quad T = 1 + \frac{1}{W\bar{Z}(\lambda) - 1}$$

Helmholtz resonators

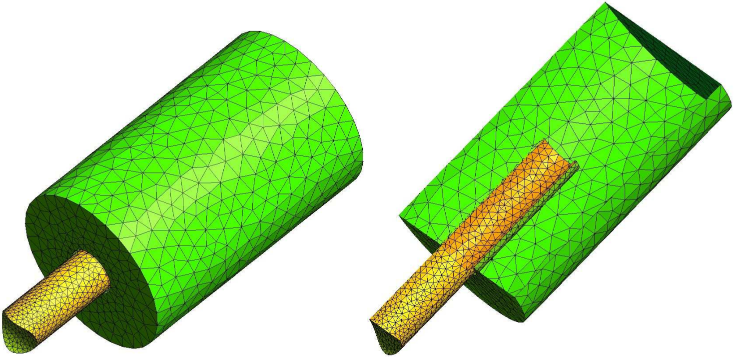
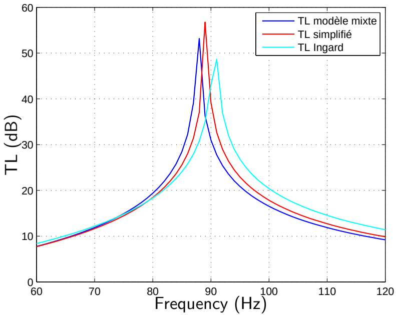
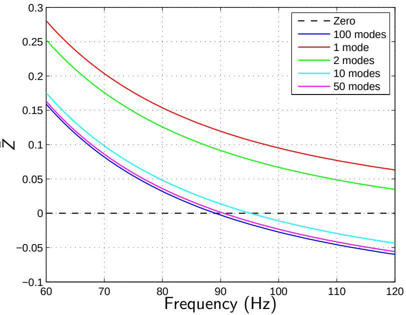


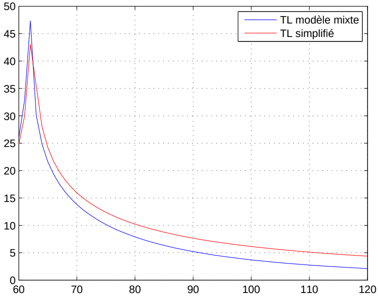
Figure: Classical (left) ; with extended neck (right).

Helmholtz resonator, classical





Helmholtz resonator, with extended neck



Simplified models : two openings

We consider a symmetric resonator connected to the main duct via two openings located at $z = z_1$ and $z = z_2$.

$$\begin{pmatrix} \bar{p}_1 \\ \bar{p}_2 \end{pmatrix} = \begin{pmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{12} & \bar{Z}_{11} \end{pmatrix} \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix}$$

Moreover,

$$\begin{pmatrix} \bar{p}_1 \\ \bar{p}_2 \end{pmatrix} = \frac{1}{W} \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}kL} \\ \mathrm{e}^{\mathrm{i}kL} & 1 \end{pmatrix} \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} + \begin{pmatrix} \bar{p}_1^I \\ \bar{p}_2^I \end{pmatrix}$$

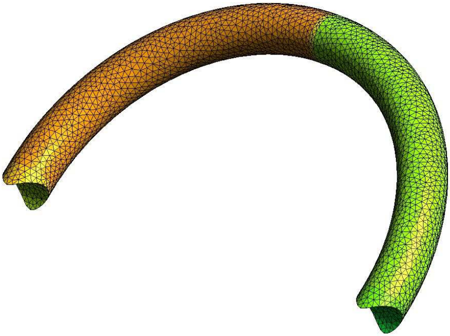
where $L = |z_2 - z_1|$. This gives

$$T = 1 + \frac{2W\bar{Z}_{11} - 2W\bar{Z}_{12}\cos(kL) + (\mathrm{e}^{2\mathrm{i}kL} - 1)}{(W\bar{Z}_{11} - 1)^2 - (W\bar{Z}_{12} - \mathrm{e}^{\mathrm{i}kL})^2}$$

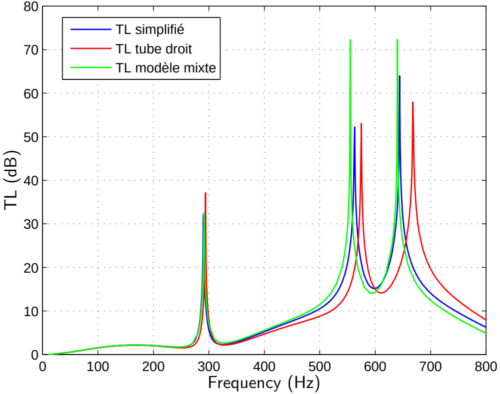
Thus, no acoustic energy is transmitted if

$$\boxed{\bar{Z}_{12}^2 - \bar{Z}_{11}^2 - \frac{A\sin(kL)}{kA_d}\bar{Z}_{12} = 0}$$

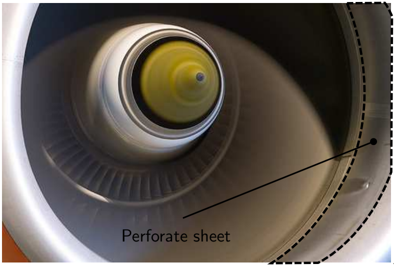
Herschel-Quincke resonator



Herschel-Quincke resonator



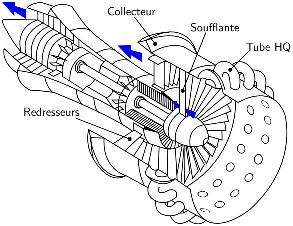
Fan noise



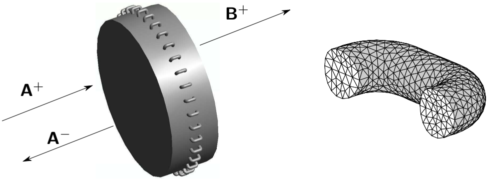
The fan noise is one of the dominant components at take-off and landing for aircraft with modern high bypass ratio turbofan engines : broadband noise + Blade Passing Frequency (BPF) tones

Fan noise

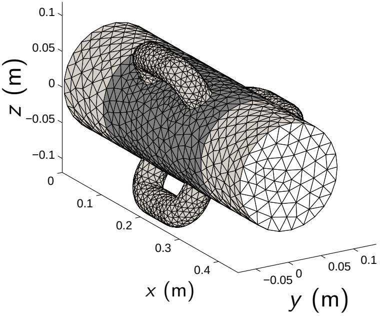
Actual configuration...

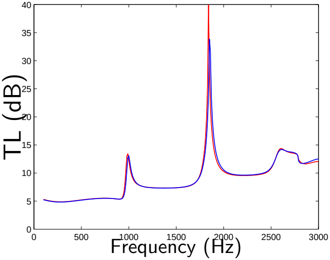


Model...



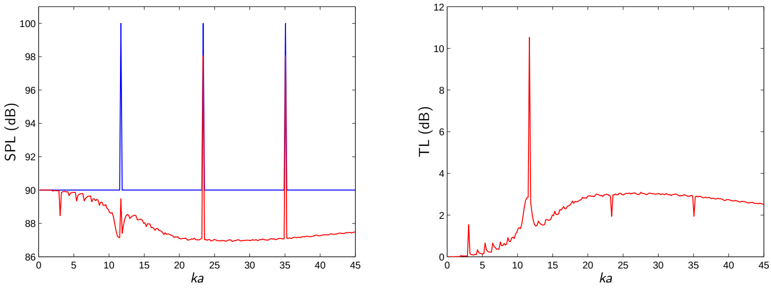
Validation on a small size configuration





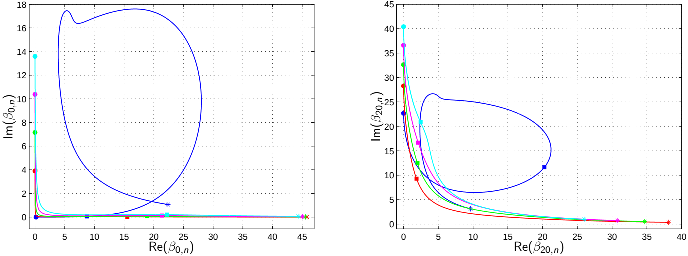
	Matrix size	CPU time (Matlab)
Our model	500	1 h 50 min
FEM	82 000	31 h 15 min

Optimal configuration (36 HQ tubes)

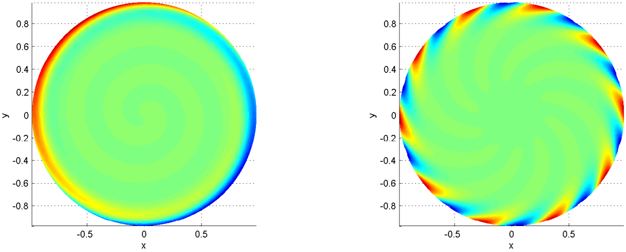


- Incident
- Liner-HQ system

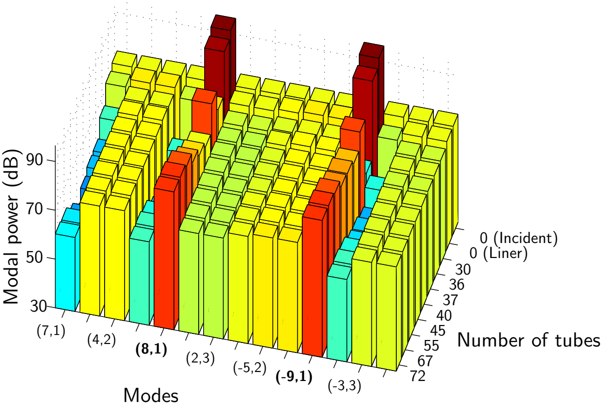
What does the liner do ?



Surface wave modes at 1530 Hz :



Influence of the number of tubes on the first BPF (iso-surface)



Conclusions and prospects

The proposed Green's function based method allows to reduce the computational effort as only the acoustic velocity at the interface needs to be calculated.
A very high number of propagative modes (few hundreds) can be handled easily on a single PC.
Gives access to physical interpretation in the low-frequency regime.

In prospect : - could be used for designing taylor-made resonators using optimization procedures. - viscosity effects should be included