ABOUT TRAPPED MODES IN OPEN WAVEGUIDES

Christophe Hazard

POEMS (Propagation d'Ondes: Etude Mathématique et Simulation)

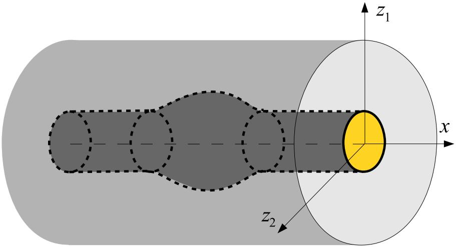
CNRS / ENSTA / INRIA, Paris

MATHmONDES

Reading, July 2012

Introduction

CONTEXT:
time-harmonic waves in locally perturbed uniform open waveguides (for instance, a defect in an optical fiber, or in an immersed pipe . . .).



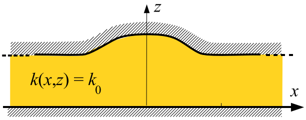
ISSUE :

Are there trapped modes, i.e., localized oscillations of the system which do not radiate towards infinity?

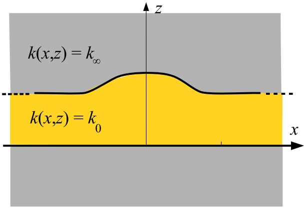
Surprisingly...

Trapped modes...

... may occur in closed waveguides (*),



... but not in open waveguides!



(*) See, e.g., Linton and McIver (2007).

Our 3-dimensional acoustic waveguide

Defined by a wavenumber function

$$k = k(x,z) \quad \text{where} \quad \begin{cases} x = \text{longitudinal direction}, \\ z := (z_1, z_2) = \text{transverse directions}, \end{cases}$$

such that

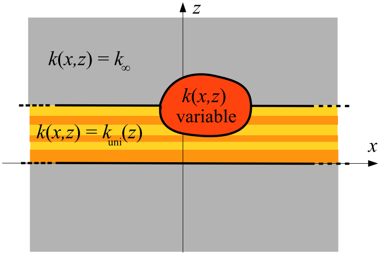
$$0 < \inf_{(x,z)\in\mathbb{R}^3} k(x,z) \quad \text{and} \quad \sup_{(x,z)\in\mathbb{R}^3} k(x,z) < \infty,$$

and k is a localized perturbation of a uniform waveguide:

$k - k_{\text{uni}}$ is compactly supported,

where $k_{\text{uni}} = k_{\text{uni}}(z)$ and

$k_{\text{uni}}(z) = k_\infty > 0 \quad \text{if } |z| > d > 0.$



Main result

Theorem (absence of trapped modes)

With the above assumptions on $k = k(x, z)$, the only solution $u \in H^2(\mathbb{R}^3)$ to the Helmholtz equation

$$-\Delta u - k^2 u = 0 \quad \text{in } \mathbb{R}^3,$$

is $u \equiv 0$.

Basic ideas for the proof:

- Modal decomposition of u resulting from a generalized Fourier transform in the transverse direction (instead of a usual Fourier transform in the longitudinal direction).
- Argument of analyticity with respect to the generalized Fourier variable.

Related works

Rough media

- Chandler-Wilde and Zhang (1998)
- Chandler-Wilde and Monk (2005)
- Lechleiter and Ritterbusch (2010)
- ...

} No guided wave

Perturbed stratified media

- Weder (1991) } Analyticity argument
- Bonnet-Ben Dhia, Chorfi, Dakia, H. (2009)
- Bonnet-Ben Dhia, Goursaud, H. (2011)

} 2D step-index

Outline

1. Modal analysis

2. Proof of the absence of trapped modes

1. Modal analysis

2. Proof of the absence of trapped modes

Modes of a uniform waveguide

Separation of variables: $u(x,z) = \Phi(z)\, e^{px}$ for $p \in \mathbb{C}$ solution to

$$-\Delta_{x,z} u - k_{\text{uni}}^2\, u = 0 \quad \text{in } \mathbb{R}^3,$$

$$\Longrightarrow \text{Eigenvalue problem} \quad \begin{cases} \text{Find } \lambda = p^2 \in \mathbb{C} \text{ and } \Phi \text{ bounded such that} \\ -\Delta_z \Phi - k_{\text{uni}}^2\, \Phi = \lambda\, \Phi \text{ in } \mathbb{R}^2. \end{cases}$$

Assuming $k_\infty < k_{\text{sup}} := \sup\limits_{z \in \mathbb{R}^2} k_{\text{uni}}(z)$, there are two kinds of solutions:

- Finite set of isolated $\lambda \in (-k_{\text{sup}}^2, -k_\infty^2)$ associated with evanescent Φ (as $|z| \to +\infty$).

  $\Longrightarrow$ Guided modes $\Phi(z)\, e^{\pm\sqrt{\lambda}x}$.

- Continuous set $\lambda \in [-k_\infty^2, +\infty)$ associated with oscillating Φ (as $|z| \to +\infty$)

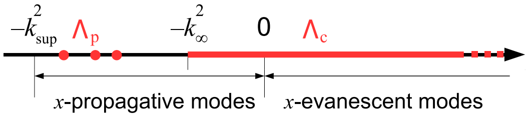
  $$\Longrightarrow \text{Radiation modes} \begin{cases} \text{propagative as } x \to \pm\infty \text{ if } \lambda < 0, \\ \text{exponentially } \nearrow \text{ or } \searrow \text{ if } \lambda > 0. \end{cases}$$

Spectral interpretation

The unbounded operator A defined in $L^2(\mathbb{R}^2)$ by

$$A\varphi := -\Delta_z \varphi - k_{\text{uni}}^2\,\varphi \quad \forall \varphi \in \mathrm{D}(A) := H^2(\mathbb{R}^2)$$

is selfadjoint. Its spectrum Λ is composed of two parts:



- A finite point spectrum $\Lambda_{\text{p}} = \{\text{eigenvalues}\} \subset (-k_{\text{sup}}^2, -k_{\infty}^2)$.
  $\Longrightarrow$ Associated $\Phi \in L^2(\mathbb{R}^2)$: eigenfunctions.
- A continuous spectrum $\Lambda_{\text{c}} = [-k_{\infty}^2, +\infty)$.
  $\Longrightarrow$ Associated $\Phi \notin L^2(\mathbb{R}^2)$: generalized eigenfunctions.

A natural question

Can we find a family of eigenfunctions and generalized eigenfunctions such that

- any $\varphi \in L^2(\mathbb{R}^2)$ can be represented by a discrete + continuous superposition, and
- A becomes diagonal in this "basis"?

A natural question

Can we find a family of eigenfunctions and generalized eigenfunctions such that

- any $\varphi \in L^2(\mathbb{R}^2)$ can be represented by a discrete + continuous superposition, and
- A becomes diagonal in this "basis"?

YES !

- easy for the eigenfunctions, but ...
- more involved for the generalized eigenfunctions ($\Longrightarrow$ scattering theory).

A generalized spectral basis

- A family of eigenfunctions (guided modes):
  For $\lambda \in \Lambda_{\mathrm{p}}$, choose an orthonormal basis $\{\Phi_{\lambda,\kappa};\ \kappa = 1, \ldots, m_\lambda\}$ of the associated eigenspace (m_λ = multiplicity of the eigenvalue λ).

- A family of generalized eigenfunctions (radiation modes):
  For $\lambda \in \Lambda_{\mathrm{c}} = [-k_\infty^2, +\infty)$ and $\kappa \in S^1$ (= unit circle),

$$\Phi_{\lambda,\kappa} := \underbrace{\Phi_{\lambda,\kappa}^{\infty}}_{\text{incident plane wave of direction } \kappa} + \underbrace{\Phi_{\lambda,\kappa}^{\mathrm{sc}}}_{\text{outgoing scattered wave}}$$

A key property of generalized eigenfunctions: analyticity

For all fixed $\kappa \in S^1$ and $z \in \mathbb{R}^2$, the function $\lambda \mapsto \Phi_{\lambda,\kappa}(z)$ extends to a meromorphic function of λ in the complex half plane $\mathrm{Re}\,\lambda > -k_\infty^2$.

The generalized Fourier transform

The operator of decomposition on the family $\{\Phi_{\lambda,\kappa}\}$:

$$(\mathcal{F}\varphi)(\lambda,\kappa) := \int_{\mathbb{R}^2} \varphi(z)\,\overline{\Phi_{\lambda,\kappa}(z)}\,\mathrm{d}z \quad \forall\lambda \in \Lambda,\ \forall\kappa \in \begin{cases} 1,\dots,m_\lambda & \text{if } \lambda \in \Lambda_{\mathrm{p}} \\ S^1 & \text{if } \lambda \in \Lambda_{\mathrm{c}} \end{cases}$$

defines (by density) a unitary transformation from $L^2(\mathbb{R}^2)$ to the spectral space

$$\widehat{\mathcal{H}} := \widehat{\mathcal{H}}_{\mathrm{p}} \oplus \widehat{\mathcal{H}}_{\mathrm{c}} \quad \text{where} \quad \widehat{\mathcal{H}}_{\mathrm{p}} := \oplus_{\lambda\in\Lambda_{\mathrm{p}}} \mathbb{C}^{m_\lambda} \text{ and } \widehat{\mathcal{H}}_{\mathrm{c}} := L^2(\Lambda_{\mathrm{c}} \times S^1).$$

It diagonalizes A in the sense that $A = \mathcal{F}^{-1}\lambda\,\mathcal{F}$.

1. Modal analysis

2. Proof of the absence of trapped modes

Getting rid of the defect!

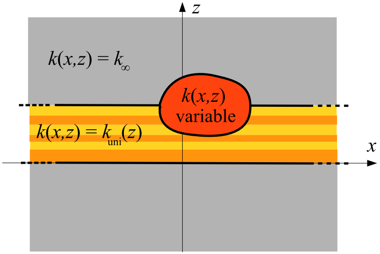
If $u \in H^2(\mathbb{R}^3)$ satisfies

$$\text{(H)} \quad -\Delta u - k^2\, u = 0 \quad \text{in } \mathbb{R}^3,$$

then

$$\text{(LS)} \quad -\Delta u - k_{\text{uni}}^2\, u = f(u) \quad \text{in } \mathbb{R}^3,$$

where $f(u) := (k^2 - k_{\text{uni}}^2)u$ is compactly supported.



Proof of the absence of trapped modes

1) Prove: (LS) $\Longrightarrow u = 0$ outside the support of $f(u)$.
2) Conclude by the unique continuation principle for (H).

Main theorem

Let $f \in L^2(\mathbb{R}^3)$ compactly supported. If $u \in H^2(\mathbb{R}^3)$ satifies

$$-\Delta u - k_{\text{uni}}^2\, u = f \quad \text{in } \mathbb{R}^3,$$

then $u = 0$ outside the support of f.

Proof: 3 steps . . .

Step 1: Using $\mathcal{F}$

Let $f \in L^2(\mathbb{R}^3)$ compactly supported and $u \in H^2(\mathbb{R}^3)$ solution to

$$-\Delta u - k_{\text{uni}}^2 \, u = f \quad \text{in } \mathbb{R}^3.$$

In other words,

$$-\frac{\partial^2 u}{\partial x^2} + A u = f \quad \text{in } \mathbb{R}.$$

Setting $\widehat{u}_{\lambda,\kappa}(x) := (\mathcal{F} u(x,\cdot))(\lambda,\kappa)$ and $\widehat{f}_{\lambda,\kappa}(x) := (\mathcal{F} f(x,\cdot))(\lambda,\kappa)$ (which makes sense since $u, f \in L^2(\mathbb{R}^3)$), we have

$$-\frac{\partial^2 \widehat{u}_{\lambda,\kappa}}{\partial x^2} + \lambda \, \widehat{u}_{\lambda,\kappa} = \widehat{f}_{\lambda,\kappa} \quad \text{in } \mathbb{R}, \quad \text{for a.e. } \lambda \text{ and } \kappa.$$

Step 1: Using $\mathcal{F}$ (contd)

Any solution to $-\dfrac{\partial^2 \widehat{u}_{\lambda,\kappa}}{\partial x^2} + \lambda\, \widehat{u}_{\lambda,\kappa} = \widehat{f}_{\lambda,\kappa}$ reads as

$$\widehat{u}_{\lambda,\kappa} = \widehat{u}^{\text{gen}}_{\lambda,\kappa} + \widehat{u}^{\text{part}}_{\lambda,\kappa}$$

where

$$\widehat{u}^{\text{gen}}_{\lambda,\kappa}(x) = \widehat{\alpha}^{+}_{\lambda,\kappa}\, \mathrm{e}^{-\sqrt{\lambda}\, x} + \widehat{\alpha}^{-}_{\lambda,\kappa}\, \mathrm{e}^{+\sqrt{\lambda}\, x},$$

and

$$\widehat{u}^{\text{part}}_{\lambda,\kappa}(x) = \int_{\mathbb{R}} \gamma_\lambda(x - x')\, \widehat{f}_{\lambda,\kappa}(x')\, \mathrm{d}x',$$

where $\gamma_\lambda(x) := \dfrac{\mathrm{e}^{-\sqrt{\lambda}\,|x|}}{2\sqrt{\lambda}}$ is a Green's function of $-\dfrac{\partial^2}{\partial x^2} + \lambda$ (choose $\sqrt{\lambda}$ such that $\sqrt{\lambda} \in \mathbb{R}^+$ if $\lambda \in \mathbb{R}^+$).

Step 1: Using $\mathcal{F}$ (contd)

Outside the x-support of f,

$$\widehat{u}^{\text{part}}_{\lambda,\kappa}(x) = \widehat{\beta}^{\pm}_{\lambda,\kappa}\, \mathrm{e}^{-\sqrt{\lambda}\,|x|} \quad \text{as } x \to \pm\infty,$$

where

$$\widehat{\beta}^{\pm}_{\lambda,\kappa} := \int_{x\text{-supp}\, f} \frac{\mathrm{e}^{\pm\sqrt{\lambda}\, x'}}{2\sqrt{\lambda}}\, \widehat{f}_{\lambda,\kappa}(x')\,\mathrm{d}x'.$$

So

$$\widehat{u}_{\lambda,\kappa}(x) = \begin{cases} \widehat{\alpha}^{+}_{\lambda,\kappa}\, \mathrm{e}^{-\sqrt{\lambda}\,x} + \left(\widehat{\alpha}^{-}_{\lambda,\kappa} + \widehat{\beta}^{-}_{\lambda,\kappa}\right) \mathrm{e}^{+\sqrt{\lambda}\,x} & \text{as } x \to -\infty, \\ \left(\widehat{\alpha}^{+}_{\lambda,\kappa} + \widehat{\beta}^{+}_{\lambda,\kappa}\right) \mathrm{e}^{-\sqrt{\lambda}\,x} + \widehat{\alpha}^{-}_{\lambda,\kappa}\, \mathrm{e}^{+\sqrt{\lambda}\,x} & \text{as } x \to +\infty. \end{cases}$$

Step 2: Solutions with finite energy

Recall that $\mathcal{F}$ is unitary, hence

$$u \in L^2(\mathbb{R}^3) \Longrightarrow \widehat{u}_{\lambda,\kappa} \in L^2(\mathbb{R}) \quad \text{for a.e. } \lambda \text{ and } \kappa.$$

Among the possible $\widehat{u}_{\lambda,\kappa} = \widehat{u}^{\text{gen}}_{\lambda,\kappa} + \widehat{u}^{\text{part}}_{\lambda,\kappa}$, which ones belong to $L^2(\mathbb{R})$?

- Propagative modes: $\lambda < 0$.
  As $x \to \pm\infty$, $\widehat{u}_{\lambda,\kappa}$ = linear combination of oscillating exp. functions
  $$\Longrightarrow \begin{cases} \widehat{\alpha}^{+}_{\lambda,\kappa} = \widehat{\alpha}^{-}_{\lambda,\kappa} + \widehat{\beta}^{-}_{\lambda,\kappa} = 0, \\ \widehat{\alpha}^{+}_{\lambda,\kappa} + \widehat{\beta}^{+}_{\lambda,\kappa} = \widehat{\alpha}^{-}_{\lambda,\kappa} = 0, \end{cases}$$
  $$\Longrightarrow \widehat{\alpha}^{\pm}_{\lambda,\kappa} = \widehat{\beta}^{\pm}_{\lambda,\kappa} = 0.$$

- Evanescent modes: $\lambda > 0$.
  As $x \to \pm\infty$, only decreasing exp. functions are allowed
  $$\Longrightarrow \widehat{\alpha}^{+}_{\lambda,\kappa} = \widehat{\alpha}^{-}_{\lambda,\kappa} = 0.$$

Step 2: Solutions with finite energy (contd)

To sum up:

The only solutions with finite energy write as

$$\widehat{u}_{\lambda,\kappa}(x) = \widehat{u}^{\text{part}}_{\lambda,\kappa}(x) = \int_{\mathbb{R}} \gamma_\lambda(x - x')\, \widehat{f}_{\lambda,\kappa}(x')\, \mathrm{d}x'$$

with the condition

$$\widehat{u}_{\lambda,\kappa}(x) = 0 \quad \text{for } \lambda < 0,\ \kappa \in S^1 \text{ and } x \text{ outside the } x\text{-support of } f.$$

(i.e., the modal components of u associated with propagative modes vanish).

Step 3: Analyticity of the modal components

$$\begin{aligned}\widehat{u}_{\lambda,\kappa}(x) &= \int_{\mathbb{R}} \frac{\mathrm{e}^{-\sqrt{\lambda}\,|x-x'|}}{2\sqrt{\lambda}}\, \widehat{f}_{\lambda,\kappa}(x')\,\mathrm{d}x' \\ &= \int_{\mathbb{R}} \frac{\mathrm{e}^{-\sqrt{\lambda}\,|x-x'|}}{2\sqrt{\lambda}} \int_{\mathbb{R}^2} f(x',z)\,\overline{\Phi_{\lambda,\kappa}(z)}\,\mathrm{d}z\,\mathrm{d}x'\end{aligned}$$

Noticing that

- For all fixed $\kappa \in S^1$ and $z \in \mathbb{R}^2$, the function $\lambda \mapsto \overline{\Phi_{\lambda,\kappa}(z)}$ extends to a meromorphic function of λ in the complex half plane $\mathrm{Re}\,\lambda > -k_\infty^2$,
- $\lambda \mapsto \sqrt{\lambda}$ is analytic outside the branch cut,
- f is compactly supported,

we deduce that

for all fixed $\kappa \in S^1$ and $x \in \mathbb{R}$, the function $\lambda \mapsto \widehat{u}_{\lambda,\kappa}(x)$ extends to a meromorphic function of λ in the complex half plane $\mathrm{Re}\,\lambda > -k_\infty^2$ outside the branch cut of $\sqrt{\lambda}$.

Step 3: Analyticity of the modal components (contd)

We already know that the modal components of u associated with propagative modes vanish:

$$\widehat{u}_{\lambda,\kappa}(x) = 0 \quad \text{for } \lambda < 0, \; \kappa \in S^1 \text{ and } x \text{ outside the } x\text{-support of } f.$$

The analyticity of $\lambda \mapsto \widehat{u}_{\lambda,\kappa}(x)$ then shows that this holds for $\lambda \in \Lambda_c$, i.e., the modal components of u associated with evanescent modes also vanish.

Finally:

$u(x,z) = 0$ for all x outside the x-support of f and all $z \in \mathbb{R}^2$.

Conclusion

Note that our method does not apply for closed waveguides because the transverse spectrum is discrete.

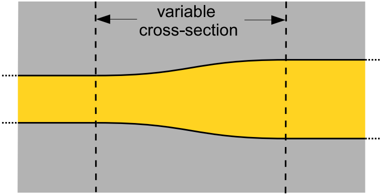
The idea to remember:

Energy deals with propagative modes,
whereas analyticity takes care of evanescent modes.

Here, analyticity means that propagative and evanescent components of a radiating wave are connected in a subtle but strong way in an open waveguide (whereas they are independent in a closed waveguide).

Conclusion (contd)

The same result holds for the junction of two semi-infinite uniform open waveguides:



Theorem (absence of trapped modes)

The only solution $u \in H^2(\mathbb{R}^3)$ to the Helmholtz equation

$$-\Delta u - k^2 u = 0 \quad \text{in } \mathbb{R}^3,$$

is $u \equiv 0$.

Proof: sames ideas as for the defect, but... far more intricate!

Conclusion (contd)

What about scattering in open waveguides?

Case of 2D step-index waveguides:

- Bonnet-Ben Dhia, Chorfi, Dakia, H. (2009) = defect
- Bonnet-Ben Dhia, Goursaud, H. (2011) = junction

Use of $\mathcal{F} \Longrightarrow$ Modal radiation condition + well-posedness.

More general waveguides?

Main difficulty: extension of the generalized Fourier transform to slowly decreasing functions (not in $L^2(\mathbb{R}^2)$).

Thank you for your (trapped?) attention !