A Multilevel Fast Multipole Method with Plane Waves Stable at All Scale

Francis Collino

Project EMA: www.cerfacs.fr/emc

REI (DGA) : No 2010 34 0001

July 8 2012

Outline

- The 3 Ingredients for getting nice Multilevel Fast Multipol algorithm with plane waves
- Limitation of the classical MLFMM
- MLFMM with both propagative an evanescent waves
  - Main formula
  - Treatment of the propagative waves (new computation of the translation function)
  - Treatment of the evanescent waves (new integration rules and interpolations)

MLFMM

The problem is to compute in a fast way the matrix vector product

$$V(x_t) = \sum_{x_t \in X \neq x_s} G(x_t, x_s)\rho(x_s), \quad x_t \in X$$

- X is a finite set of points of $\mathbb{R}^3$
- G is the 3-D Helmholtz Green Function

$$G(x_t, x_s) = \frac{e^{ik|x_s - x_t|}}{|x_s - x_t|}$$

- V potential, ρ ponctual distribution of charges

`MLFMM` uses finite rank approximation for block matrices

$$\mathbb{G}^{B_t, B_s}, \quad x_t \in B_t, x_s \in B_s \text{ packets of points}$$

Points in boxes

The `MLFMM` is based on an `oct-tree`

$B_p^{\pounds}$ cubic box of size $2^{-\pounds}\,\ell_0$

$$X \subset \bigoplus_p B_p^{\pounds}, \quad \pounds = 1, \ldots, Ł$$

$$\text{with } B_p^{\pounds} = \bigoplus_{b_q^{\pounds+1} \subset B_p^{\pounds}} b_q^{\pounds+1}$$

(at most 8 small boxes in a large box)

(x_t, x_s) is inside a series of pair of boxes

$$x_t \in {B_t}^1 \subset {B_t}^2 \subset .. \subset {B_t}^{Ł}$$

$$x_s \in {B_s}^1 \subset {B_s}^2 \subset .. \subset {B_s}^{Ł}$$

pair of boxes $(B_t, B_s) \Rightarrow$ matricial block $\mathbb{G}^{B_t, B_s}$

At which level do I associate the pair (x_t, x_s)?

Illustration

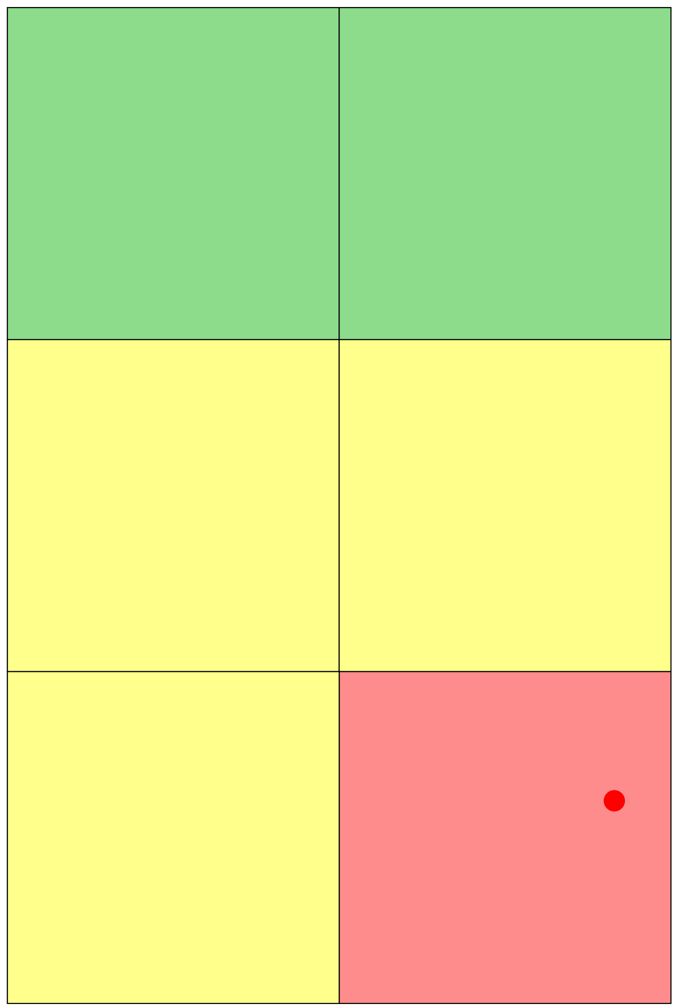


Figure 1: The gathering of source points in cubes at level 3

Illustration

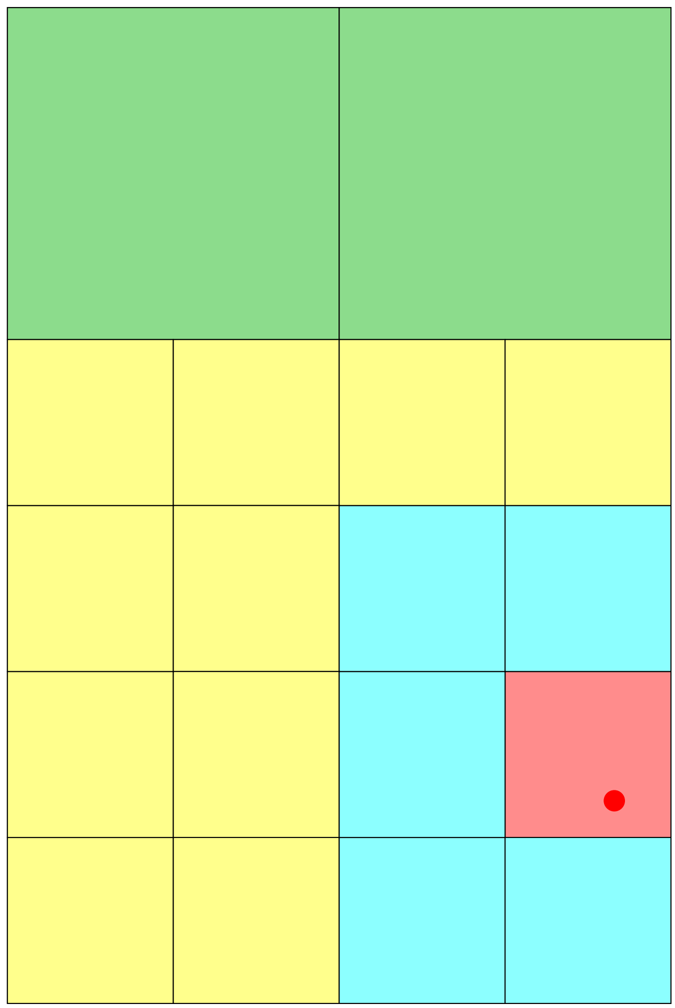


Figure 2: The gathering of source points in cubes at level 4

Illustration

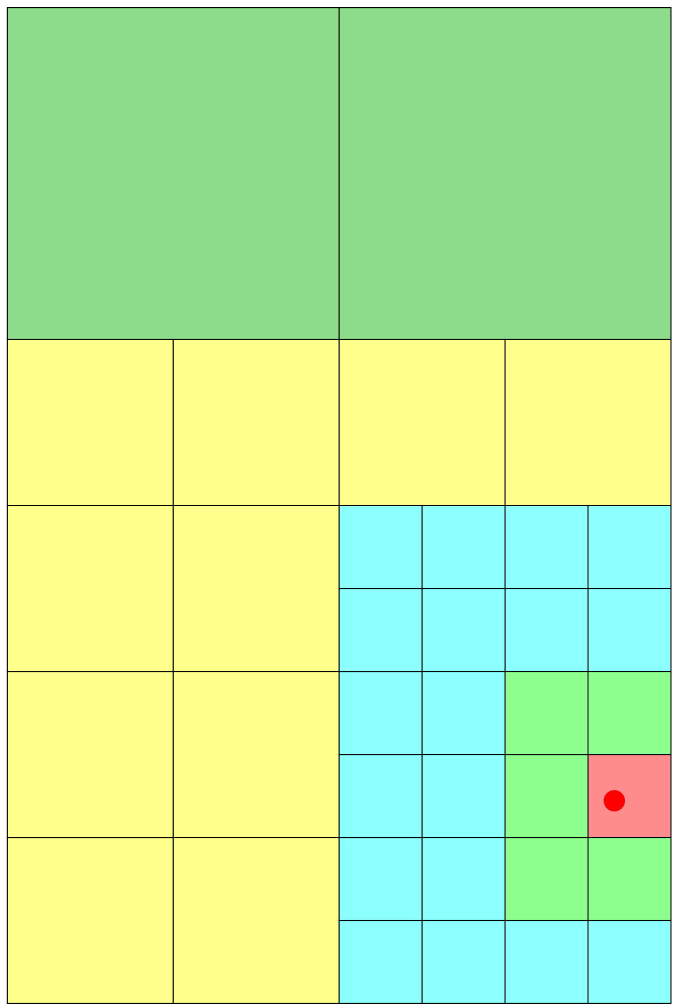
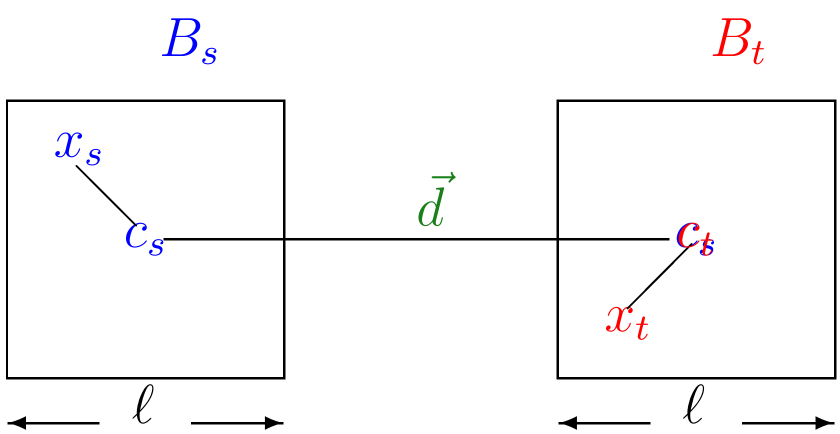


Figure 3: The gathering of source points in cubes at level 5

1: Computations for a couple of cubes

$$V^{B_s}(x_t) = \sum_{x_s \in B_s} G(x_t, x_s)\,\rho(x_s), \quad x_t \in B_t$$



$$\vec{d} = \ell(i\hat{x} + j\hat{y} + k\hat{z}), \quad \ell = 2^{-\pounds}\ell_0$$

i, j, k integers of modulus ≤ 3, $\quad |i| + |j| + |k| > 3$

$\Rightarrow$ 316 directions per level

1: Approximations by Herglotz waves

The approximation is

$$G(x_t, x_s) \simeq \int_{\hat{K}} e^{i\vec{k}\cdot(x_t - c_t)}\, T^L(\vec{d}, \vec{k})\, e^{i\vec{k}\cdot(c_s - x_s)}$$

$$\text{where } \hat{K} \subset \left\{ \vec{k} \in \mathbb{C}^3,\ k_x^2 + k_y^2 + k_z^2 = k^2 \right\}$$

we will consider 2 cases

1. Propagative waves: $\hat{K} = kS^2$
2. Propagative waves + evanescent waves in the $\hat{a}$ direction

$$(\hat{a} = \pm\hat{x},\ \pm\hat{y},\ \pm\hat{z})$$

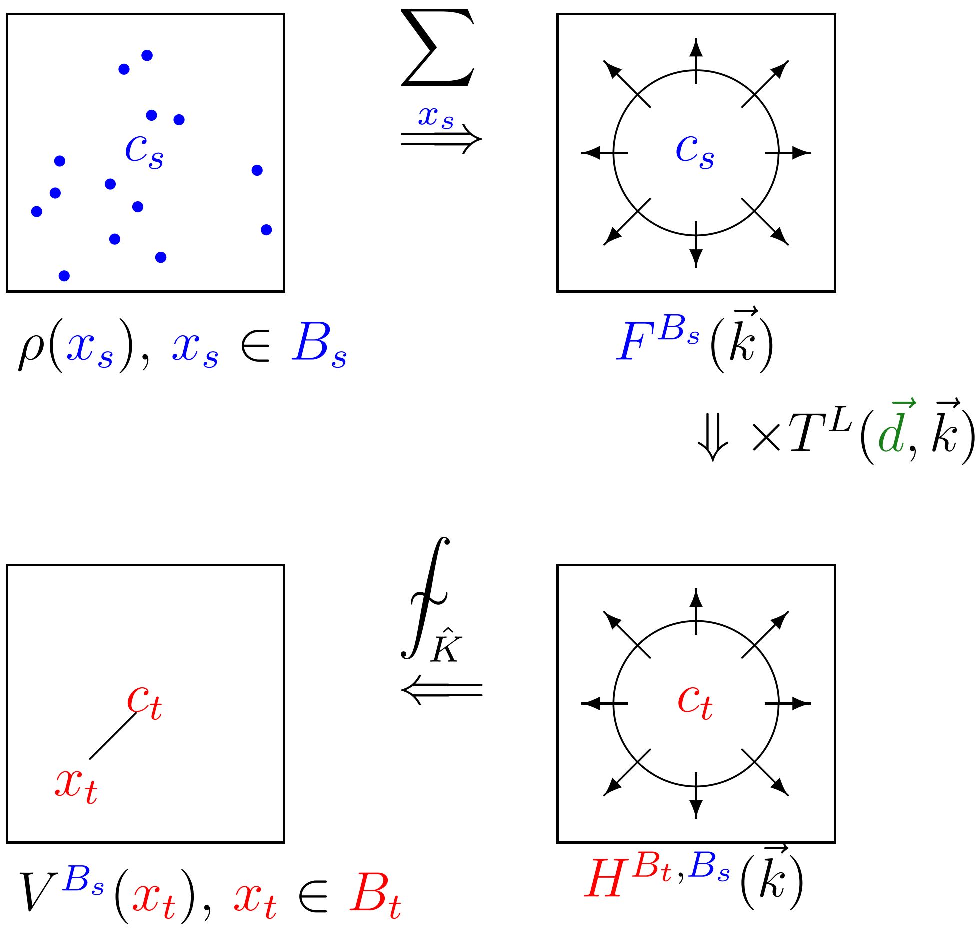
$\int_{\hat{K}}$ is a quadrature rule on $\hat{K}$
(one for each level $\mathcal{L}$)

$$\int_{\hat{K}} F(\vec{k}) = \sum_{p=1}^{P^\ell} \varpi_p^\ell F(\vec{k}_p^\ell).$$

1: One-level FMM

$$V^{B_s}(x_t) = \sum_{x_s \in B_s} G(x_t, x_s)\,\rho(x_s)$$

$$= \int_{\hat{K}} e^{i\vec{k}\cdot(x_t - c_t)} \underbrace{T^L(\vec{d}, \vec{k}) \underbrace{\sum_{x_s \in B_s} e^{i\vec{k}\cdot(c_s - x_s)}\,\rho(x_s)}_{F^{B_s}(\vec{k})}}_{H^{B_t,B_s}(\vec{k})}$$



1: Multilevel needs Interpolation

Idea: Reuse the far fields computed for smaller boxes.

If $\vec{\kappa}_p$ is a quadrature point, then

$$F^{B_s}(\vec{\kappa}_p) = \sum_{b \subset B_s} e^{i\vec{\kappa}_p \cdot (c_{B_s} - c_b)} F^b(\vec{\kappa}_p)$$

The problem is that the quadrature rules depends on the size of the box (i.e. on the level)

Require an interpolation procedure :

$$F^{B_s}(\vec{\kappa}_p) = \sum_{b \subset B_s} e^{i\vec{\kappa}_p \cdot (c_{B_s} - c_b)} \sum_q \mathbb{I}_{p,q} F^b(\vec{\kappa}_q)$$

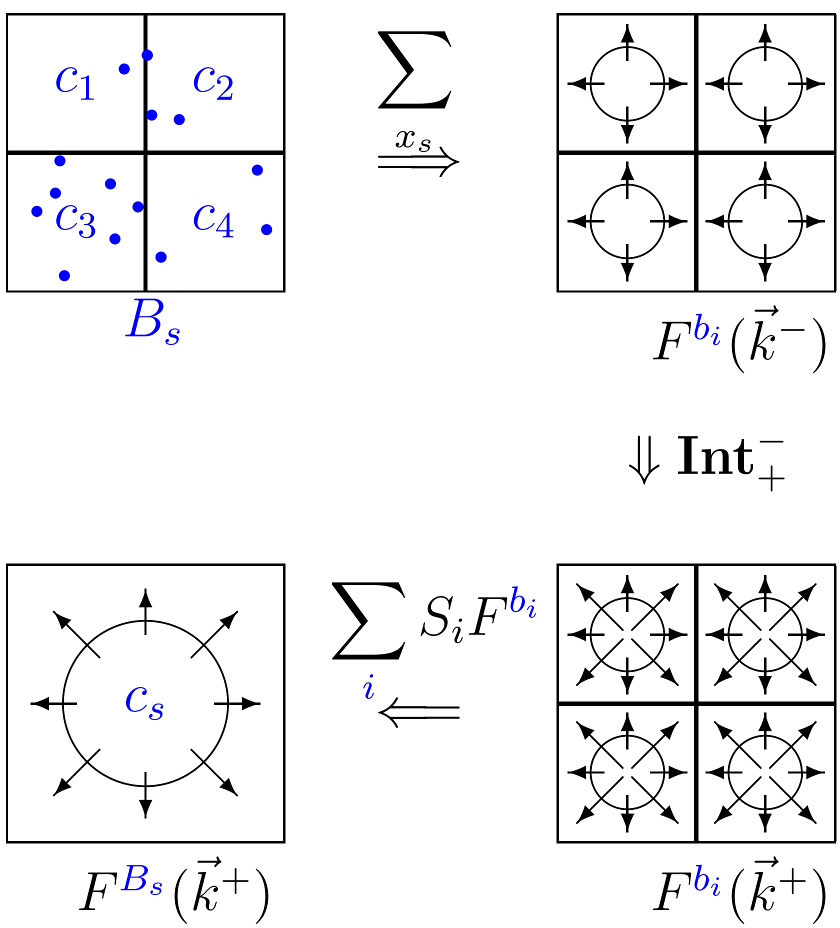
$\mathbb{I}$ must interpolates accuratly all the $e^{i\vec{\kappa} \cdot (x_s - c_b)}$'s

1: Interpolation

Idea: Reuse the fields computed for smaller boxes.

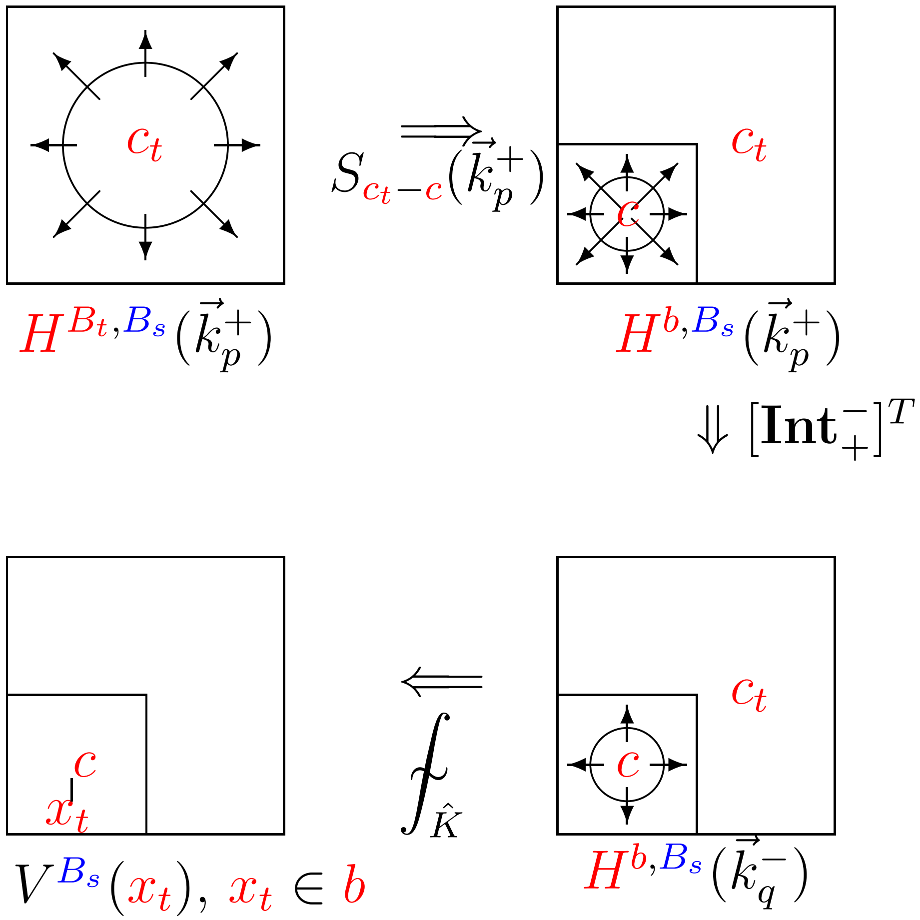
$$e^{i\vec{k}_p^+\cdot(x-c)} = e^{i\vec{k}_p^+\cdot(c_i-c)}\, e^{i\vec{k}_p^+\cdot(x-c_i)}$$

$$\simeq \underbrace{e^{i\vec{k}_p^+\cdot(c_i-c)}}_{S^i(\vec{k}_p^+)}\ \mathbf{Int}_+^-\left(e^{i\vec{k}_q^-\cdot(x-c_i)}\right)$$



1: Anterpolation

$$H^{b,B_s}(\vec{k}_p^+) = e^{-i\vec{k}_p^+\cdot(c_t-b)} H^{B_t,B_s}(\vec{k}_p^+), \quad x_t \in b \subset B_t$$



$$V^{B_s}(x_t) = \sum_p \varpi_p^+ H^{b,B_s}(\vec{k}_p^+)\ \mathbf{Int}_+^-\left[e^{i\vec{k}_q^-\cdot(x_t-c_t)}\right](\vec{k}_p^+)$$

$$= \sum_q \varpi_q^- [\mathbf{Int}_+^-]^T\left[H^{b,B_s}(\vec{k}_p^+)\right](\vec{k}_q^-)\ e^{i\vec{k}_q^-\cdot(x_t-c_t)}$$

1: What do we need?

- Approximation of the Green function as an integral of weighted plane waves for well separated cubes.
- Precise quadrature rules (with few points) of this integral
- Fast interpolation process from and onto the quadrature points associated to adjacent levels
- (+Error estimates)

Those are the only 3 ingredients needed for a good `MLFMM`

Link with other methods

- 1-level FMM $\mathbb{G}^{b_s,b_t} \simeq {}^*\mathbb{F}^{b_t}\mathbb{T}^{b_s-b_t}\mathbb{F}^{b_s} + \text{error}$
- 2-levels FMM

  Instead of

  $$\mathbb{G} \simeq \mathbb{G}_{close} + {}^*\mathbb{F}^{(1)}\mathbb{T}^{(1)}\mathbb{F}^{(1)} + {}^*\mathbb{F}^{(2)}\mathbb{T}^{(2)}\mathbb{F}^{(2)}$$

  use

  $$\mathbb{G} \simeq \mathbb{G}_{close} + {}^*\mathbb{F}^{(1)}\left(\mathbb{T}^{(1)} + {}^*\mathbb{Y}^{(1\to 2)}\mathbb{T}^{(2)}\mathbb{Y}^{(1\to 2)}\right)\mathbb{F}^{(1)}$$

- Truncated SVD :

  $\mathbb{G}^{b_s,b_t} \simeq {}^*\mathbb{V}^{b_s,b_t}\mathbb{T}^{b_s,b_t}\mathbb{U}^{b_s,b_t} + \text{error}$

- ACA : same form as SVD but does not require all the $\mathbb{G}^{b_s,b_t}(x_s, x_t)$'s

III

The classical FMM and its Limitations

3: Addition Theorem

when $D = |\vec{D}| > v = |\vec{v}|$, then

$$\frac{e^{i|\vec{D}+\vec{v}|}}{i|\vec{D}+\vec{v}|} = \sum_{n=0}^{\infty}(-1)^n(2n+1)h_n^{(1)}(D)[j_n(v)P_n(\hat{D}\cdot\hat{v})]$$

series of Multipoles

Funk-Ecke Formula

$$i^n j_n(v)P_n(\hat{D}\cdot\hat{v}) = \frac{1}{4\pi}\int_{S^2} e^{i\hat{s}\cdot\hat{v}}\, P_n(\hat{D}\cdot\hat{s})d\sigma(\hat{s})$$

Truncation of the series to L terms

3: classical HF FMM

$$\hat{K} = kS^2 \quad \text{(Propagative Waves)}$$

when $D = |\vec{D}| > v = |\vec{v}|$, then

Formula (1)
$$\frac{e^{i|\vec{D}+\vec{v}|}}{|\vec{D}+\vec{v}|} \simeq \int_{S^2} e^{i\vec{v}\cdot\hat{s}}\, T^L(\vec{D},\hat{s})\, d\sigma(\hat{s})$$

$$T^L(\vec{D},\hat{s}) = \frac{i}{4\pi}\sum_{n=0}^{L} i^n(2n+1)h_n^{(1)}(D)P_n(\hat{D}\cdot\hat{s})$$

L is chosen such that

$$\sup_{\vec{v}\in B_{k\ell}} \underbrace{\left(|\vec{D}+\vec{v}|\sum_{n>L}(-1)^n h_n^{(1)}(D)j_n(v)P_n(\hat{D}\cdot\hat{v})]\right)}_{\varepsilon^L(\vec{v},\vec{D})} \leq \epsilon$$

use (1) with $\vec{D} = k\vec{d} = k(c_s - c_t)$

and $\vec{v} = k[(x_t - c_t) - (x_s - c_s)]$

3: Spectral quadrature rules

Let $\hat{K}$ some bounded set, and
$Y_L \subset L^2(\hat{K})$, $L = 0, 1, 2, \ldots$, such that
$Y_L Y_M \subset Y_{L+M}$;

Let

- Π^L the L^2 projector onto Y_L
- $⨍_{\hat{K}}$ some quadrature law exact in Y_{2L}

$$\int_{\hat{K}} t(x)e(x)dx \simeq ⨍_{\hat{K}} \Pi^L t(x)e(x)dx$$

then

$$\|\mathrm{err}\|_\infty \leq \|t\|_2 \left(\|\Pi^L e - e\|_2 + \|\Pi^L e - e\|_\infty\right)$$

$$(+\epsilon_{mach}\|t\|_\infty\|e\|_\infty)$$

3: The classical FMM

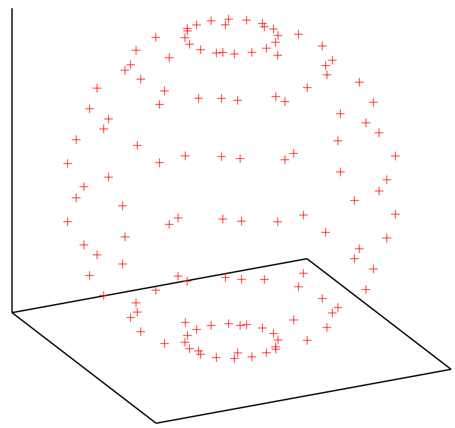
Formula : Formula (1)

Quadrature : exact for harmonic functions

of degree $< L + L_e \simeq 2L$

require $\sim L^2$ Gauss points

Interpolation : FFT + Alpert algo.



3: Link with the far field approximation

Use the formula to compute the potential at point x_t for all charges in B_s leads to form

$$F^{B_s}(\hat{s}) = \sum_{x_s \in B_s} e^{-ik(x_s - c_{B_s})\hat{s}} \rho(x_s)$$

i.e. the far field ! and compute

$$V^{B_t}(x_t) \simeq k \int_{S^2} e^{ik(x_t - c_{B_t})\cdot\hat{s}} T^L(kd, \hat{s}) F^{B_s}(\hat{s}) d\sigma(\hat{s})$$

$$d = (c_{B_s} - c_{B_t})$$

For L given and $|d| \to \infty$

$$T^L(\vec{d}, \hat{s}) \to \frac{e^{ik|d|}}{k|d|} \Pi^L \left(\delta(\hat{s} - \hat{d}) \right), \hat{d} = \frac{d}{|d|}$$

and we obtain the far field approximation

$$V^{B_t}(x_t) \simeq \frac{e^{ik|d|}}{|d|} F^{B_s}(\hat{d}) e^{ik(x_t - c_{B_s})\cdot\hat{d}}$$

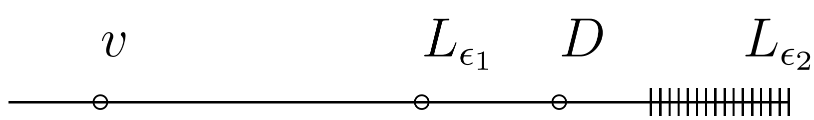
3: Breakdown Phenomenum

Problem : the terms in the series

$$T^L(\vec{D}, \hat{s}) = \frac{i}{4\pi} \sum_{n=0}^{L} i^n (2n+1) h_n^{(1)}(D) P_n(\hat{D} \cdot \hat{s})$$

increase exponentially when $n \to \infty$

$$|h_n^{(1)}(D)| \simeq \frac{1}{\epsilon} \text{ when } D < n + C_\epsilon n^{1/3},$$



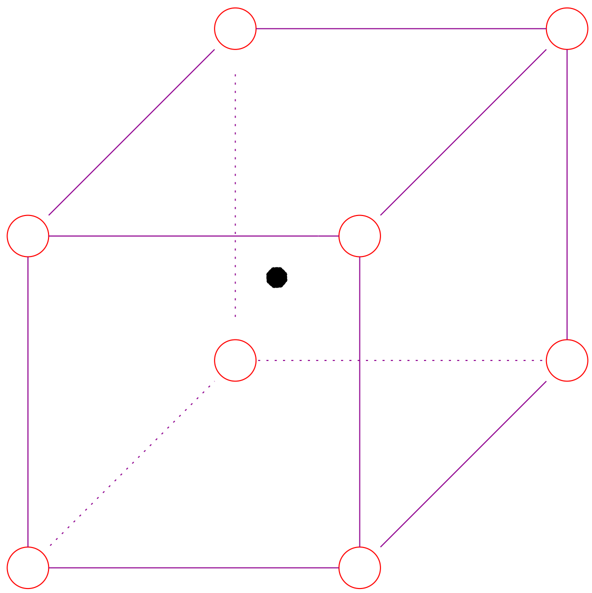
$$L \simeq v + 1.8 \left(v \log_{10}^2 \left(\frac{1}{\epsilon_{target}} \right) \right)^{1/3}$$

to have the bound

$$\varepsilon^L(\vec{v}, \vec{D}) \leq \epsilon_{target}.$$

Pb when D is too small with respect to L

3: Error estimate



We have

$$\sup_{\vec{v}\in B_{k\ell}} \varepsilon^L(\vec{v},\vec{D})) \leq \epsilon_{target} \quad \text{when}$$

$$\max_{\vec{v}_c} \max[\varepsilon^L(\vec{v}_c,\vec{D})), \varepsilon^{L+1}(\vec{v}_c,\vec{D}))] < \epsilon_{target}$$

where $\vec{v}_c$ is one of the 8 corner of the cube $B_{k\ell}$:

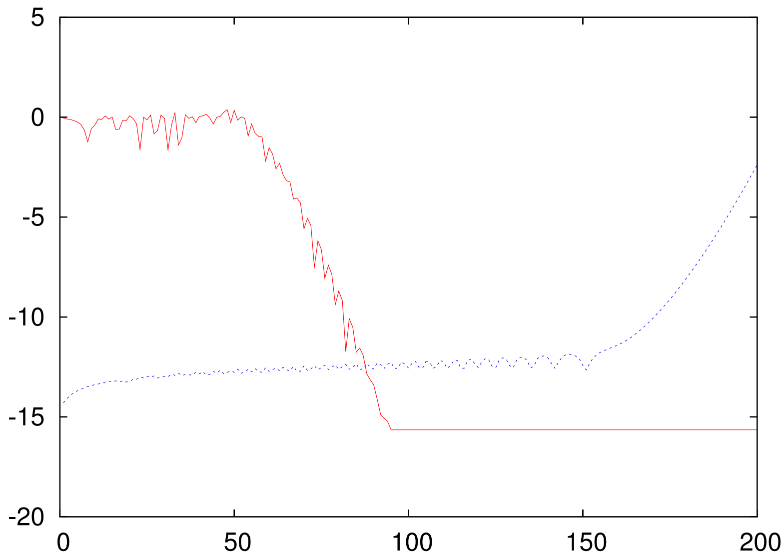
$$\vec{v}_c = \sqrt{3}k\ell(\pm 1, \pm 1, \pm 1)$$

- true when $k\ell$ large enough.
- conjecture for finite ℓ.

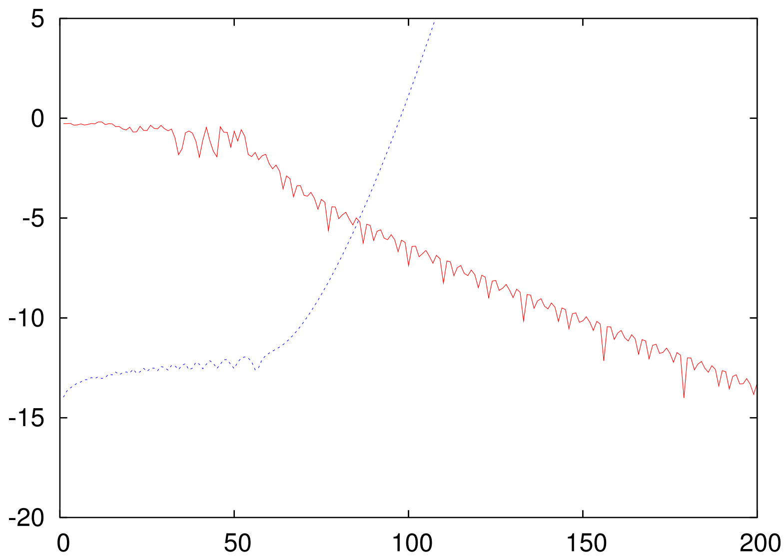
3: Breakdown Phenomenum

$$\vec{v} = 30(\sqrt{3}, \sqrt{3}, \sqrt{3}) \quad \text{(large box)}$$

— : $\log_{10}(\varepsilon^L(\vec{v}, \vec{D}))$, — : $\log_{10}(10^4 \epsilon_{mach} \max T^L(\vec{D}))$



$(\vec{D} = 30(3, 3, 3))$

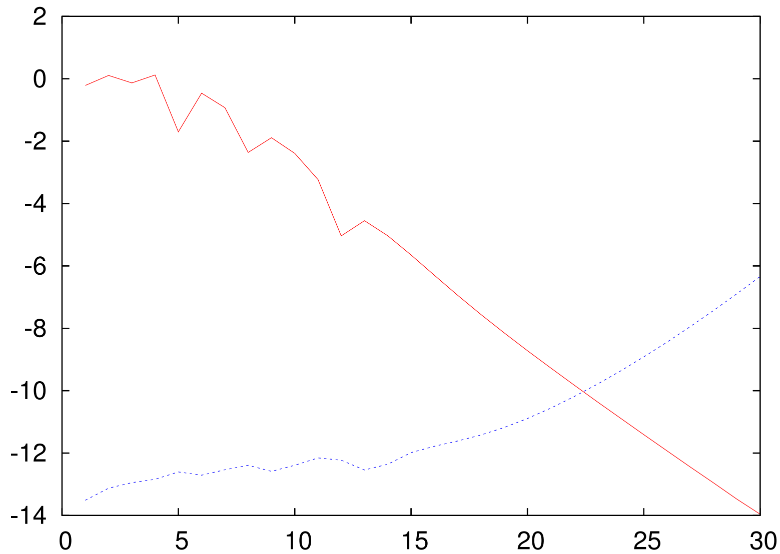


$(\vec{D} = 30(2, 0, 0))$

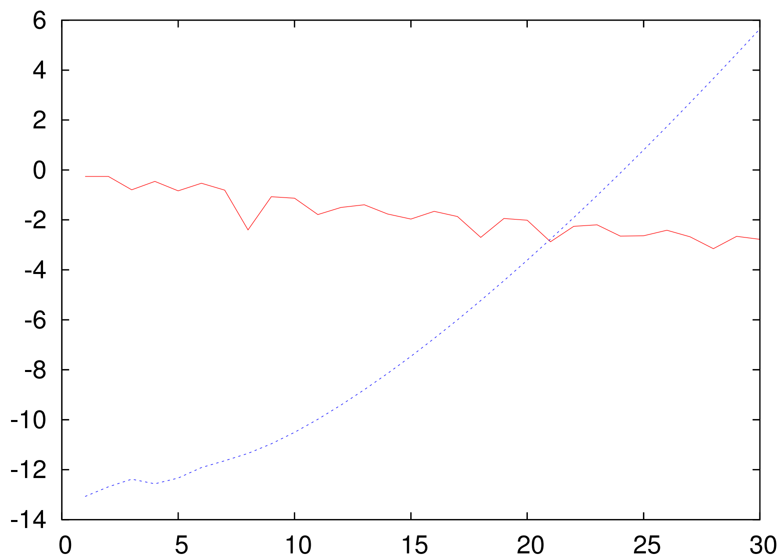
3: Breakdown Phenomenum

$\vec{v} = 3(\sqrt{3}, \sqrt{3}, \sqrt{3})$ (petite boîte)

— : $\log_{10}(\epsilon_L(\vec{v}, \vec{D}))$, — : $\log_{10}(10^4 \epsilon_{mach} \max T^L(\vec{D}))$



$(\vec{D} = 3(3, 3, 3))$



$(\vec{D} = 3(2, 0, 0))$

3: Number of Multipoles

$$k\ell = 30, \qquad \epsilon_{target} = 5\,10^{-3}$$

$\vec{D}/k\ell$	ε^L	L
(3, 3, 3)	$4.99\,10^{-3}$	63
(3, 2, 3)	$3.63\,10^{-3}$	61
(2, 2, 3)	$4.68\,10^{-3}$	60
(2, 2, 2)	$4.32\,10^{-3}$	64
(3, 1, 3)	$4.79\,10^{-3}$	60
(2, 1, 3)	$3.74\,10^{-3}$	60
(2, 1, 2)	$4.29\,10^{-3}$	61
(1, 1, 3)	$4.73\,10^{-3}$	60

$\vec{D}/k\ell$	ε^L	L
(1, 1, 2)	$4.76\,10^{-3}$	61
(3, 0, 3)	$3.78\,10^{-3}$	60
(2, 0, 3)	$3.51\,10^{-3}$	60
(2, 0, 2)	$4.46\,10^{-3}$	60
(1, 0, 3)	$4.43\,10^{-3}$	60
(1, 0, 2)	$4.30\,10^{-3}$	61
(0, 0, 3)	$2.26\,10^{-3}$	60
(0, 0, 2)	$3.02\,10^{-3}$	62

$$L^{Chew} = L^{Lambert} = 64!$$

3: Number of Multipoles

$k\ell = 0.5, \qquad \epsilon_{target} = 5\,10^{-3}, \qquad \epsilon_{machine} = 10^{-16}$

$\vec{D}/k\ell$	ε^L	L
(3 ,3, 3)	$2.32 10^{-3}$	5
(3 ,2, 3)	$4.85 10^{-3}$	5
(2 ,2, 3)	$4.30 10^{-3}$	5
(2 ,2, 2)	$4.85 10^{-3}$	7
(3 ,1, 3)	$4.49 10^{-3}$	4
(2 ,1, 3)	$3.30 10^{-3}$	5
(2 ,1, 2)	$3.04 10^{-3}$	7
(1 ,1, 3)	$4.72 10^{-3}$	6

$\vec{D}/k\ell$	ε^L	L
(1 ,1, 2)	$6.23 10^{-3}$	12
(3 ,0, 3)	$4.39 10^{-3}$	5
(2 ,0, 3)	$4.62 10^{-3}$	6
(2 ,0, 2)	$4.99 10^{-3}$	7
(1 ,0, 3)	$3.26 10^{-3}$	6
(1 ,0, 2)	$9.84 10^{-3}$	12
(0 ,0, 3)	$3.04 10^{-3}$	7
(0 ,0, 2)	$9.72 10^{-3}$	11

$\Rightarrow$ Classical FMM fails for 3 directions

IV

FMM with both propagative & evanescent plane waves

4: Sommerfeld Formula

Formula with evanescent waves

Privilegied Direction $\hat{z}$: $\vec{D} \cdot \hat{z} > 0$

Solve $k^2 u + \Delta u = -\delta(x, y, z)$ in $z > 0$

with Fourier transform in (x, y)

$$\frac{e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2 + y^2 + z^2}} = \int_0^\infty J_0(\lambda\sqrt{x^2 + y^2}) \frac{\lambda e^{z\sqrt{k^2-\lambda^2}}}{\sqrt{k^2 - \lambda^2}} \, d\lambda$$

- $\lambda \in [0, k[$: propagative waves
- $\lambda \in]k, \infty[$: evanescent waves

$$J_0(\lambda r) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\lambda r \cos(\psi - \psi_0)} d\psi, \text{ any } \psi_0$$

4: Sommerfeld Formula

Formula with evanescent waves

$$\vec{x} = x_t - c_{B_t} + (c_{B_s} - x_s), \ \vec{d} = c_{B_s} - c_{B_t}$$

$$G(\vec{d} + \vec{x}) = G_{prop}(\vec{d} + \vec{x}) + G_{evan}(\vec{d} + \vec{x})$$

prop. $\hat{s} \in S^2_+ = S^2 \cap \{\hat{z} \cdot \hat{s} > 0\}$

$$G_{prop}(\vec{d} + \vec{x}) = \int_{S^2_+} \frac{ik}{2\pi} e^{ik\hat{s}\cdot\vec{d}} \, e^{ik\hat{s}\cdot\vec{x}} \, d\sigma(\hat{s})$$

eva. $\vec{k} = \begin{bmatrix} \sqrt{k^2+\lambda^2}\cos\varphi \\ \sqrt{k^2+\lambda^2}\sin\varphi \\ -i\lambda \end{bmatrix}$

$$\mathcal{C} = [0, \infty[\times [0, 2\pi]$$

$$G_{evan}(\vec{d} + \vec{x}) = \int_{\mathcal{C}} \frac{1}{2\pi} e^{i\vec{k}\vec{d}} \, e^{i\vec{k}\cdot\vec{x}} \, d\sigma(\lambda, \varphi),$$

4: Propagative waves

Darve's trick...

replace S^2_+ by S^2

$$G_p(\vec{d}+\vec{x}) = \int_{S^2} e^{ik\hat{s}\cdot\vec{x}}\tilde{T}(\vec{d},\hat{s})\,d\sigma(\hat{s}).$$

$$\tilde{T}(\vec{d},\hat{s}) = \frac{ik}{2\pi}e^{ik\hat{s}\cdot\vec{d}}1_{S^2_+}(\hat{s})$$

Plane waves $\simeq$ harmonic functions

$$e^{ik\hat{s}\cdot\vec{x}} = \Pi^L\left(e^{ik\hat{s}\cdot\vec{x}}\right) + \epsilon^L(\vec{x},\hat{s})$$

$$\Rightarrow\ G_p(\vec{d}+\vec{x}) \simeq \int_{S^2} e^{ik\hat{s}\cdot\vec{x}}T^L(\vec{d},\hat{s})\,d\sigma(\hat{s}).$$

$$T(\vec{d},\hat{s}) = \Pi^L\left(\frac{ik}{2\pi}e^{ik\hat{s}\cdot\vec{d}}1_{S^2_+}(\hat{s})\right)$$

(Π^L Projector onto the sphe. harm. of degree $< L$.)

4: Advantages

- Function to be integrated is almost an harmonic function (as for the usual FMM)

$$\Rightarrow \text{same quadrature rule}$$

- Usual Interpolation Procedure

- The set of $\vec{k}$ is the same for the whole 6 directions

$$\hat{K}_{propag} = kS^2$$

$$\Rightarrow F^{B_s}(\vec{k}) \text{ for } \pm\hat{x}, \pm\hat{y}, \pm\hat{z}$$

- We can use HF or BF for different $\vec{d}$

4: The translation function

New formula...

Sph. Harm. $Y_n^m(\theta,\varphi) = \sqrt{\dfrac{2n+1}{4\pi}}\tilde{P}_n^{|m|}(\cos\theta)e^{im\varphi}$

$\tilde{P}_n^{|m|}$: *associated Legendre functions.*

$$T^L(\vec{d},\hat{s}) = \sum_{n=0}^{L}\sum_{|m|\leq n} T_n^m(\vec{d})\, Y_n^m(\theta_s,\varphi_s)$$

$$T_n^m(\vec{d}) = \sum_{p=|m|}^{\infty} i^{p+1} j_p(k|\vec{d}|)\,\gamma_{p,n}^{|m|}\, Y_p^m(\theta_d,\varphi_d)$$

$$\gamma_{p,n}^{\mu} = \frac{\tilde{P}_p^{\mu\,\prime}(0)\tilde{P}_n^{\mu}(0) - \tilde{P}_p^{\mu}(0)\tilde{P}_n^{\mu\,\prime}(0)}{p(p+1)-n(n+1)}$$

$\Rightarrow$ fast computation of T^L

4: Generalized Jacobi Anger Series

$$i^m J_m(v \sin\theta \sin\psi) e^{iv\cos\theta\cos\psi}$$

$$= \sum_{p=|m|}^{\infty} i^p (2p+1) j_p(v) \tilde{P}_p^{|m|}(\cos\theta) \tilde{P}_p^{|m|}(\cos\psi).$$

$$\tilde{P}_\ell^m(t) = (-1)^m \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} (1-t^2)^{\frac{m}{2}} \frac{d^m P_\ell(t)}{dt^m},$$

Watson's result (in a hidden form!)

When $\psi = 0$ and $m = 0$,

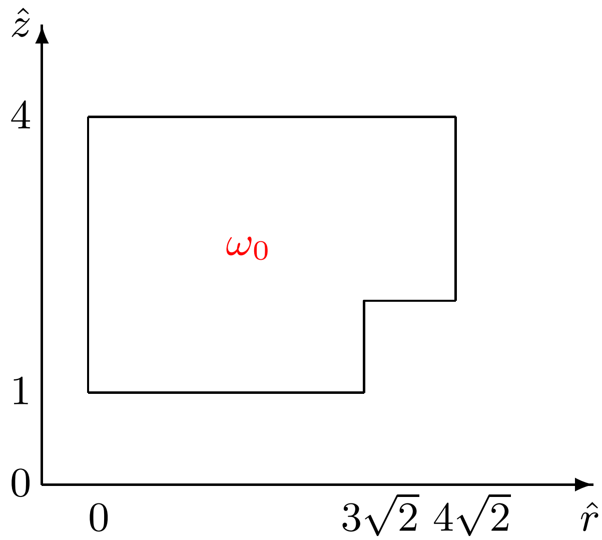
we get Jacobi Anger Series

$$e^{iv\cos\theta} = \sum_{p=0}^{\infty} i^p (2p+1) j_p(v) P_p(\cos\theta)$$

4: Evanescent Waves

$$\vec{w} = \vec{x} + \vec{d} \in \Omega_\ell :$$

$$\Omega_\ell = \{(r_w, z_w),\ (r_w, z_w) \in \ell\omega_0\} \times [0, 2\pi]$$



$$G_e(\vec{w}) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{i\sqrt{\lambda^2+k^2} r_w \cos(\varphi-\varphi_w)\psi)} e^{-\lambda z_w}\, d\varphi\, d\lambda$$

We must find the quadrature rule on

$[0, \infty[\times[0, 2\pi]$

and an interpolator ...

4: Quadrature ?

Havé : equidistributed in φ

$$\text{quad. for } \int_0^\infty e^{i\sqrt{\lambda^2+k^2}r_w \cos(\varphi_j-\varphi_w)} e^{-\lambda z_w}\, d\lambda ?$$

- The optimal law depends on $k\ell$ (Wallen) : Greengard & Yarvin's method

- Interpolation is possible but hard It implies (implicitely) full matrices :

$$[\mathbf{Int}]_{\lambda_p^+,\varphi_j^+}^{\lambda_q^-,\varphi_i^-}$$

Costly Algorithm

⇒Bottleneck of the method

4: A change of variables that changes a lot!

$$G_e(\vec{w}) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{i\sqrt{\lambda^2+k^2}r_w \cos(\varphi-\varphi_w-\theta_{\lambda,k})} e^{-\lambda z_w}$$

λ given, we set $\phi = \varphi + \theta_{\lambda,k}$

$$\cos\theta_{\lambda,k} = \frac{\lambda}{\sqrt{\lambda^2+k^2}},\ \sin\theta_{\lambda,k} = \frac{k}{\sqrt{\lambda^2+k^2}},$$

$$G_e(\vec{w}) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{i\vec{k}(\lambda,\phi)\cdot\vec{w}}$$

$$\text{with } \vec{k}(\lambda,\phi) = \lambda \begin{bmatrix} \cos\phi \\ \sin\phi \\ -i \end{bmatrix} + k \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$$

The phase is now linear in λ !

k does not occur in the integral in λ!

4: New Method

- Quadrature: same than the one for $k = 0$

$$(\lambda_q, \varpi_q) = \left(\frac{1}{\ell}\,\hat{\lambda}_q, \frac{1}{\ell}\hat{\varpi}_q\right), \, q = 1, \ldots Q$$

  where $(\hat{\lambda}_q, \varpi_q)$ is adapted for ω_0 (Greengard)

- $Q(\lambda_p)$ equidistributed points in ϕ
- Filtering the translation functions in ϕ
- Error estimates!
- Interpolation : `FFT` in ϕ + matrix vector product in λ

$$F(\frac{1}{2}\lambda_p, \vec{v}) \simeq \sum_{q=1}^{Q} \mathbb{M}_{p,q}\, F(\lambda_q, \vec{v}), \; \forall \vec{v} \in \text{some set}$$

An example

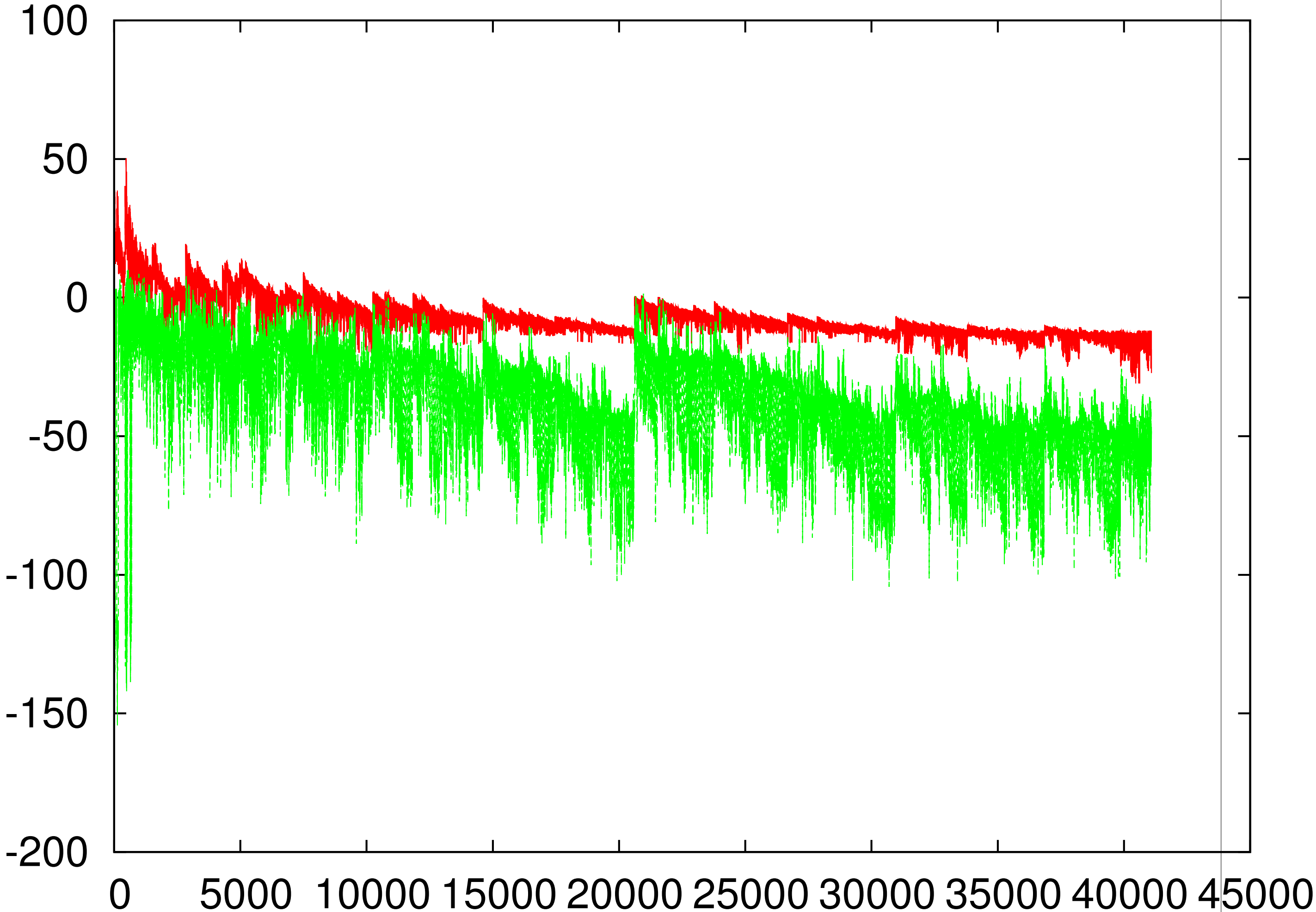
Matrix vector product for the EFIE

- Geometry: thin cylinder
- Radius: 1 wavelenght
- 30000 triangles, 45000 dofs
- 135000 Gauss points
- 5 levels of FMM

Visualization of the HF error

$$d \to 10\log_{10}|V_d|$$

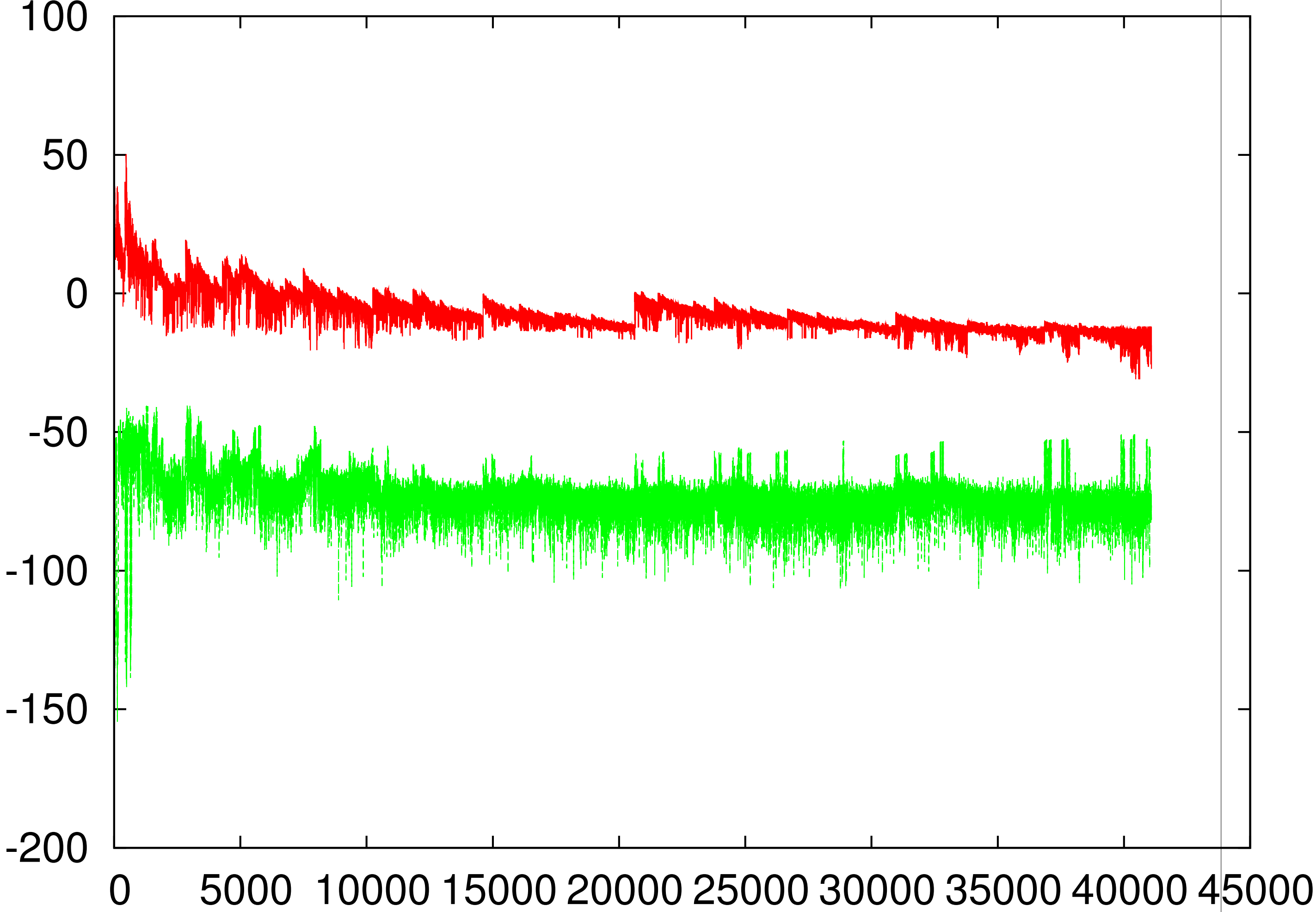
$$d \to 10\log_{10}|V_d - V_d^{fmm}|$$



Visualisation of the BF error

$$d \to 10\log_{10}|V_d|$$

$$d \to 10\log_{10}|V_d - V_d^{fmm}|$$



Comparison with FMM HF

Amazing!

L	Q	$\max T(\vec{\kappa}_q)$	t(s)	ϵ_{ℓ^2}
3	24	10^5	5	$2.3\ 10^{-3}$
4	60	10^7	6	$6.8\ 10^{-4}$
5	60	10^9	7	$3.7\ 10^{-4}$
6	96	10^{11}	12	$1.6\ 10^{-4}$
9	180	10^{19}	15	$1.6\ 310^3$!