Hybrid numerical-asymptotic boundary element methods for high frequency wave scattering

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There has been considerable interest in recent years in the development of numerical methods for time-harmonic acoustic and electromagnetic wave scattering problems that can efficiently resolve the scattered field at high frequencies. Standard finite or boundary element methods (FEMs and BEMs), with piecewise polynomial approximation spaces, suffer from the restriction that a fixed number of degrees of freedom is required per wavelength in order to represent the oscillatory solution, leading to excessive computational cost when the scatterer is large compared to the wavelength.

The *hybrid numerical-asymptotic* (HNA) approach aims to reduce the number of degrees of freedom required, by enriching the numerical approximation space with oscillatory functions, chosen using partial knowledge of the high frequency (short wavelength) asymptotic behaviour of the solution. The BEM setting is particularly attractive for such an approach, since knowledge of the high frequency asymptotics is required only on the boundary of the scatterer; for a recent review of the HNA methodology in the BEM context see [3]. In this setting one takes the relevant boundary value problem, which in the acoustic case involves the Helmholtz equation

$$(\Delta + k^2)u = 0, \tag{1}$$

where the wavenumber k is proportional to the frequency of the incident wave, and reformulates it as a boundary integral equation on the boundary Γ of the scatterer, with frequency dependent solution V. Then, informed by a high frequency asymptotic theory such as the Geometrical Theory of Diffraction (GTD) (see e.g. [11, 2]), one seeks to approximate V using an HNA ansatz of the form

$$V(x,k) \approx V_0(x,k) + \sum_{m=1}^{M} V_m(x,k)\,\exp(\mathrm{i}k\psi_m(x)), \quad x \in \Gamma, \tag{2}$$

where V_0 is a known oscillatory function (e.g., the leading order term in GTD approximation), the phases ψ_m are chosen *a-priori* (e.g., from partial knowledge of the higher order GTD components) and the amplitudes V_m, $m = 1, \ldots, M$, are approximated numerically. The key idea is that if V_0 and ψ_m, $m = 1, \ldots, M$, in (2) are chosen wisely, then V_m, $m = 1, \ldots, M$, will be much less oscillatory than V and so can be better approximated by piecewise polynomials than V itself.

The nature and complexity of the HNA ansatz (2) is inherently problem-dependent, being governed by the underlying high frequency asymptotics of the solution, which themselves depend strongly on the geometry of the scatterer and the form of the incident wave. As a result, the HNA approach has been applied so far mainly to problems for which these asymptotics are relatively simple (mostly 2D problems, with the exception of [6] and [3, §7.6], and mostly convex scatterers, with the exception of [4]). But for many such problems (e.g., scattering by sound-soft smooth convex obstacles in 2D [5, 1], convex [10] and nonconvex [4] polygons and 2D planar screens [9] - see [3] for further examples) the HNA approach has proved to be very effective, providing a dramatic reduction in the number of degrees of freedom at high frequencies, and in some cases even frequency-independent computational cost (when the numerical integration required for practical implementation is carried out using appropriate oscillatory quadrature routines), see, e.g., [9].

In my talk I will outline the basic HNA methodology in the BEM context, and will highlight some of the interesting analytical and numerical challenges it presents. One such challenge is that to design HNA approximation spaces optimally, and to prove their effectiveness by rigorous numerical analysis, one needs to derive regularity estimates for the amplitudes V_m, $m = 1, \ldots, M$, which are explicit in their wavenumber dependence. This requires rigorous high frequency asymptotics of a type not typically available in the asymptotics literature. For instance, the HNA BEMs presented in [10, 4, 8] for scattering of acoustic plane waves by sound-soft polygons adopt an *hp* approximation strategy for the amplitudes V_m, $m = 1, \ldots, M$, with mesh refinement towards corner singularities (and towards geometrical shadow boundaries in the case of [8]). In order to apply standard *hp* techniques to obtain best approximation error estimates, one first has to derive non-standard wavenumber-explicit bounds on the analytic continuation of V_m, $m = 1, \ldots, M$, into the complex plane.

I will also give an overview of current research into the development and analysis of HNA methods for more general scattering problems involving nonconvex scatterers [4, 8], 3D scatterers, and transmission problems [7], where complicated multiple scattering and shadowing effects present interesting challenges.

References

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