Least squares collocation for a high-frequency scattering problem

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Abstract

Hybrid numerical asymptotic methods for integral equations in scattering problems are based on incorporating asymptotic information about the solution at high frequencies into the discretization. However, in several cases this leads to an overcomplete set of functions, rather than to a basis, and this in turn leads to conditioning problems at small or moderately large frequencies. We show that such ill-conditioning is remedied effectively by considering a least squares collocation approach. The oversampled collocation scheme is found to be numerically stable regardless of the frequency and of the order of the discretization. Since it is based on single integrals rather than double integrals in the discretization, it is also computationally much more efficient than a Galerkin approach. We show examples for scattering by a screen.

Keywords: boundary element method, collocation, least squares, high-frequency

Introduction

High-frequency scattering problems necessitate a large number of degrees of freedom in their numerical discretization. The size of the discretization can be reduced by adding well chosen oscillatory basis functions to the approximation space, modelling for example reflected, refracted and diffracted waves. This approach leads to two distinct computational challenges:

1. The discretization of an integral operator with oscillatory kernel and oscillatory basis functions requires the evaluation of a large number of highly oscillatory integrals;

2. Function approximation by a collection of wavelike basis functions often seems ill-conditioned.

We tackle the first challenge by considering effective modern numerical methods for highly oscillatory integrals [1], noting that the collocation approach yields single integrals only. In particular, all integrals are evaluated at a cost that is independent of the frequency. In this talk we mainly focus on the second challenge, for which the proposed remedy is to oversample. Thus, we consider a least squares collocation approach. This follows from recent results on function approximation using redundant sets [2, 3].

Scattering by a screen

We consider 2D time-harmonic acoustic scattering by a sound-soft (Dirichlet) screen $\Gamma \subset \mathbb{R}^2$. This is modelled by the Helmholtz equation,

$$\Delta u + k^2 u = 0,$$

with appropriate boundary conditions (see [5]).

We use a first kind integral equation formulation

$$S\phi = f, \tag{1}$$

in which S is the single layer potential

$$S\phi(\mathbf{x}) = \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad \mathbf{x} \in \Gamma \tag{2}$$

with in 2D $\Phi(x, y) = \frac{i}{4} H_0^{(1)}(k\|x - y\|)$.

The HNA ansatz

An approximation space is proposed with the generic form

$$\phi(\mathbf{x}) \approx \sum_{m=1}^{d} \sum_{n=1}^{N_m} c_{m,n}\psi_{m,n}(\mathbf{x}) e^{ikg_m(\mathbf{x})}. \tag{3}$$

Here, the phases g_m are chosen based on the known high frequency asymptotics. In the case of a single flat screen (see Fig. 1), $d = 2$ and the two phases represent diffracted rays emanating from the corners. The corresponding amplitudes are known to be highly peaked: this is accommodated by chosing the basis functions $\psi_{m,n}(\mathbf{x})$ to be piecewise polynomials on a graded mesh, with geometric refinement towards the respective corners.

Suggested members of the Scientific Committee:
Oscar Bruno, Simon Chandler-Wilde, Xavier Antoine, Ralf Hiptmair

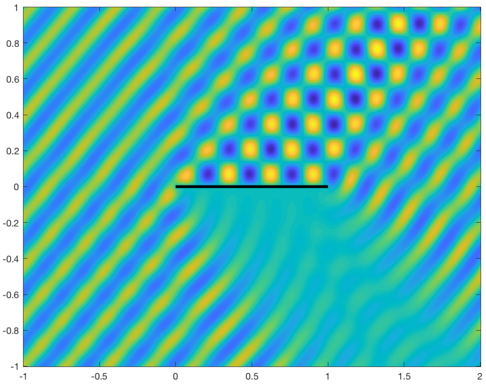

Figure 1: A typical screen scattering problem with $k = 32$.

Least squares collocation

The collocation method is described in [4] and is ongoing work by the authors of this talk. The redundancy of (3) is most evident in the limit $k \to 0$. Indeed, in that limit we have:

$$\phi(\mathbf{x}) \approx \sum_{m=1}^{d} \sum_{n=1}^{N_m} c_{m,n} \psi_{m,n}(\mathbf{x}). \qquad (4)$$

The unknown is described by a union of piecewise polynomials defined on the same domain. A function on that domain may have several representations with wildly different coefficients: this lack of uniqueness implies that any linear system to solve for $c_{m,n}$ is necessarily extremely ill-conditioned or even singular, even without an integral operator involved.

The ansatz (3) is plugged into (1). The collocation points are chosen to be Chebyshev nodes on each of the pieces of the graded meshes involved. We employ oversampling by a factor of $C_{OS} \geq 1$. The resulting rectangular linear system is solved by a regularization technique, in our implementation using the singular value decomposition with truncation of singular values smaller than a prescribed threshold ϵ.

Numerical results

For the single screen scattering problem illustrated in Fig. 1, it is shown in Fig. 2 that the number of degrees of freedom required to achieve a given accuracy in practice improves with increasing wavenumber. Fig. 3 shows that maximal accuracy for larger values of the polynomial degree is only achieved with oversampling.

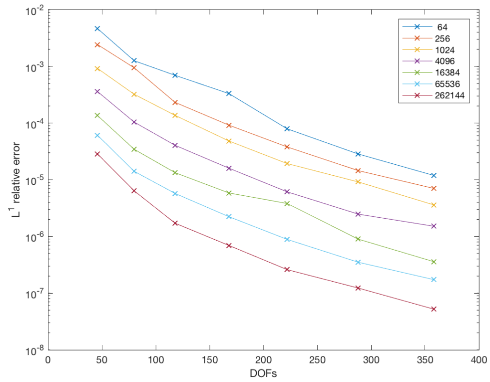

Figure 2: The number of degrees of freedom required to reach a certain accuracy for increasing values of the wavenumber.

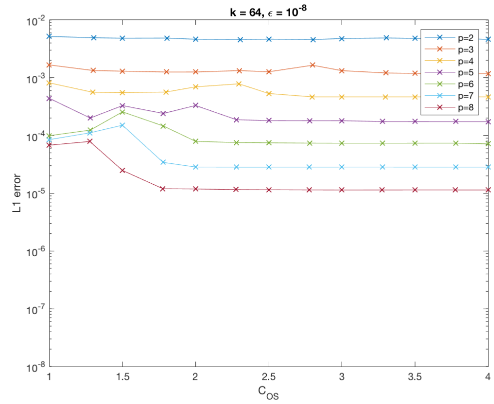

Figure 3: Accuracy as a function of the oversampling ratio C_{OS} for fixed $k = 64$ and increasing degree of the piecewise polynomials.

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