Electromagnetic shielding by thin periodic structures and the Faraday cage effect

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Abstract

The ability of wire meshes to block electromagnetic waves (the "Faraday cage" effect) is well known to physicists and engineers. We consider the scattering of electromagnetic waves (governed by the time-harmonic Maxwell equations) by a thin periodic layer of perfectly conducting obstacles. The size of the obstacles and the distance between neighbouring obstacles are both small compared to the wavelength. We derive homogenized interface conditions for three model configurations, namely (i) discrete obstacles, (ii) parallel wires, (iii) a wire mesh, and observe that the leading order behaviour depends strongly on the topology of the periodic layer, with shielding of incident waves of all polarizations occurring only in the case of a wire mesh.

Keywords: Homogenized interface conditions, Maxwell equations, Faraday cage

Our aim is to derive homogenized interface conditions for electromagnetic scattering by a thin periodic layer of perfectly-conducting obstacles

$$\mathscr{L}^\delta = \mathrm{int}\left(\bigcup_{(i,j)\in\mathbb{Z}^2} \delta\left\{\overline{\hat\Omega} + i\mathbf{e}_1 + j\mathbf{e}_2\right\}\right),$$

where $\delta > 0$ is small and $\hat\Omega \subset [0,1]^2 \times \mathbb{R} \subset \mathbb{R}^3$ is the canonical obstacle in the period cell. In particular we consider three model cases:

(i) $\hat\Omega = (\frac{3}{8},\frac{5}{8})^2 \times (-\frac{1}{8},\frac{1}{8})$, i.e. a cube;

(ii) $\hat\Omega = [0,1] \times (\frac{3}{8},\frac{5}{8}) \times (-\frac{1}{8},\frac{1}{8})$, i.e. a wire (of square section) parallel to the direction $\mathbf{e}_1$.

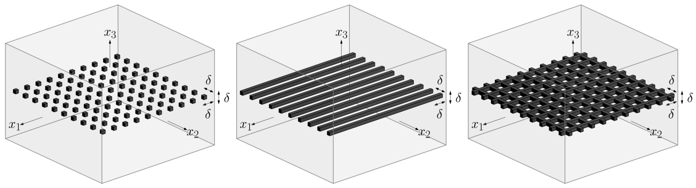
(iii) $\hat\Omega = \{[0,1] \times (\frac{3}{8},\frac{5}{8}) \times (-\frac{1}{8},\frac{1}{8})\} \cup \{(\frac{3}{8},\frac{5}{8}) \times [0,1]\times(-\frac{1}{8},\frac{1}{8})\}$, i.e. a cross-shaped domain with branches parallel to $\mathbf{e}_1$ and $\mathbf{e}_2$.

We seek a solution $\mathbf{u}^\delta$ of the Maxwell equations

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}^\delta - \omega^2\varepsilon\mathbf{u}^\delta = \mathbf{f} \quad \text{in } \Omega^\delta := \mathbb{R}^3 \setminus \overline{\mathscr{L}^\delta},$$

subject to the PEC boundary condition

$$\mathbf{u}^\delta \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma^\delta := \partial\Omega^\delta.$$



(i) (ii) (iii)

Figure 1: The domain Ω^δ in cases (i)-(iii).

It is well-known that, under appropriate assumptions on $\mathbf{f}$, ω and ε and the far-field behaviour, this problem is well-posed. Our aim is to identify the limit $\mathbf{u}_0$ of $\mathbf{u}^\delta$ as δ tends to 0. This limit solution is defined in the union of two distinct domains $\Omega^\pm = \{\mathbf{x} \in \mathbb{R}^3 : \pm x_3 > 0\}$, whose common interface is $\Gamma = \{\mathbf{x} \in \mathbb{R}^3 : x_3 = 0\}$.

Theorem 1 *The limit solution* $\mathbf{u}^0$ *satisfies*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}^0 - \omega^2\varepsilon\mathbf{u}^0 = \mathbf{f} \quad \text{in } \Omega^+ \cup \Omega^-,$$

and the following interface conditions on Γ *(where* $[\cdot]_\Gamma$ *denotes the jump across* Γ*):*
Case (i): $[\mathbf{u}_0 \times \mathbf{e}_3]_\Gamma = \mathbf{0}$ *and* $[\mathbf{curl}\,\mathbf{u}_0 \times \mathbf{e}_3]_\Gamma = 0$, *so the interface is transparent to leading order.*
Case (ii): $\mathbf{u}_0 \cdot \mathbf{e}_1 = 0$ *on* Γ, $[\mathbf{u}_0 \cdot \mathbf{e}_2]_\Gamma = 0$, *and* $[(\mathbf{curl}\,\mathbf{u}_0 \times \mathbf{e}_3)\cdot\mathbf{e}_2]_\Gamma = 0$, *so the interface reflects waves polarized parallel to the wires.*
Case (iii): $\mathbf{u}_0 \times \mathbf{e}_3 = \mathbf{0}$ *on* Γ, *so the interface reflects waves of all polarizations.*

To prove Theorem 1 we approximate $\mathbf{u}^\delta$ using matched asymptotic expansions, as in [1–7] where closely related problems are studied. It is convenient to work with the first-order formulation

$$\begin{aligned} -\mathrm{i}\omega\mathbf{h}^\delta + \mathbf{curl}\,\mathbf{u}^\delta &= 0 && \text{in } \Omega^\delta,\\ -\mathrm{i}\omega\mathbf{u}^\delta - \mathbf{curl}\,\mathbf{h}^\delta &= -\frac{1}{\mathrm{i}\omega}\mathbf{f} && \text{in } \Omega^\delta,\\ \mathbf{u}^\delta \times \mathbf{n} = 0 \text{ and } \mathbf{h}^\delta \cdot \mathbf{n} &= 0 && \text{on } \Gamma^\delta. \end{aligned}$$

Far from the periodic layer $\mathscr{L}^\delta$ we expand

$$\begin{aligned} \mathbf{h}^\delta &= \mathbf{h}_0(\mathbf{x}) + \delta\mathbf{h}_1(\mathbf{x}) + \cdots,\\ \mathbf{u}^\delta &= \mathbf{u}_0(\mathbf{x}) + \delta\mathbf{u}_1(\mathbf{x}) + \cdots, \end{aligned}$$

and, in the vicinity of $\mathscr{L}^\delta$,

$$\mathbf{h}^\delta = \mathbf{H}_0(x_1, x_2, \frac{\mathbf{x}}{\delta}) + \delta \mathbf{H}_1(x_1, x_2, \frac{\mathbf{x}}{\delta}) + \cdots,$$
$$\mathbf{u}^\delta = \mathbf{U}_0(x_1, x_2, \frac{\mathbf{x}}{\delta}) + \delta \mathbf{U}_1(x_1, x_2, \frac{\mathbf{x}}{\delta}) + \cdots,$$

where, for $i \in \{0,1\}$, $\mathbf{H}_i(x_1,x_2,y_1,y_2,y_3)$ and $\mathbf{U}_i(x_1,x_2,y_1,y_2,y_3)$ are 1-periodic in both y_1 and y_2. The near and far field expansions satisfy matching conditions, the $O(1)$ one being

$$\lim_{x_3\to 0^\pm} \mathbf{h}_0 = \lim_{y_3\to\pm\infty} \mathbf{H}_0, \quad \lim_{x_3\to 0^\pm} \mathbf{u}_0 = \lim_{y_3\to\pm\infty} \mathbf{U}_0. \tag{1}$$

The near fields $\mathbf{U}_0$ and $\mathbf{H}_0$ satisfy

$$\begin{aligned} &\mathbf{curl}\,\mathbf{U}_0 = \mathbf{curl}\,\mathbf{H}_0 = \mathbf{0} && \text{in } \mathscr{B}_\infty, \\ &\operatorname{div}\mathbf{U}_0 = \operatorname{div}\mathbf{H}_0 = 0 && \text{in } \mathscr{B}_\infty, \\ &\mathbf{U}_0 \times \mathbf{n} = \mathbf{0} \text{ and } \mathbf{H}_0 \cdot \mathbf{n} = 0 && \text{on } \partial\mathscr{B}_\infty, \end{aligned}$$

where $\mathscr{B}_\infty = \mathbb{R}^3 \setminus \bigcup_{(i,j)\in\mathbb{Z}^2} \left\{ \overline{\hat{\Omega}} + i\mathbf{e}_1 + j\mathbf{e}_2 \right\}$. A key step in our analysis is to prove that $\mathbf{U}_0$ and $\mathbf{H}_0$ are uniquely defined respectively as elements of the normal and tangential cohomology spaces K_N and K_T [8, 9], which are subspaces of $H_{\text{loc}}(\mathbf{curl}; \mathscr{B}_\infty) \cap H_{\text{loc}}(\text{div}; \mathscr{B}_\infty)$ with a periodicity condition in y_1 and y_2 and appropriate decay conditions at $y_3 = \pm\infty$. In the following theorem let $\mathbf{U}^{i,\pm} := \lim_{y_3\to\pm\infty} \mathbf{U}\cdot\mathbf{e}_i$.

Theorem 2 *For each of the three cases (i)-(iii), K_N and K_T are spanned by gradients of certain scalar potentials satisfying the Laplace equation with appropriate boundary/decay conditions.*
Case (i): K_N and K_T have dimension 4 and 3 respectively. If $\mathbf{U} \in K_N$ and $\mathbf{H} \in K_T$ then $\mathbf{U}^{i,+} = \mathbf{U}^{i,-}$ and $\mathbf{H}^{i,+} = \mathbf{H}^{i,-}$ for $i \in \{1,2\}$.
Case (ii): K_N and K_T have dimension 3 and 4 respectively. If $\mathbf{U} \in K_N$ and $\mathbf{H} \in K_T$ then $\mathbf{U}^{1,\pm} = 0$, $\mathbf{U}^{2,+} = \mathbf{U}^{2,-}$, and $\mathbf{H}^{1,+} = \mathbf{H}^{1,-}$.
Case (iii): K_N and K_T have dimension 2 and 5 respectively. If $\mathbf{U} \in K_N$ then $\mathbf{U}^{1,\pm} = \mathbf{U}^{2,\pm} = 0$.

Theorem 1 then follows by combining Theorem 2 with the matching conditions (1). For details see [9]. We note that a study of case (iii), using a different approach to that outlined above, appeared recently in [4], where the first order correction terms were also considered.

The convergence of $\mathbf{u}^\delta$ to $\mathbf{u}_0$ in the limit $\delta \to 0$ can be made rigorous by justifying the asymptotic expansions considered above. This can be done a posteriori by constructing an approximation of $\mathbf{u}^\delta$ on Ω^δ (based on truncated expansions) and using a δ-explicit stability estimate, cf. [10]. This is the subject of ongoing work by the authors of the talk.

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