Boundary element methods for scattering by fractal screens

Simon N. Chandler-Wilde[1], **David P. Hewett**[2], **Andrea Moiola**[3,*]
[1]Department of Mathematics and Statistics, University of Reading, Reading, UK
[2]Department of Mathematics, University College London, London, UK
[3]Dipartimento di Matematica, Università degli Studi di Pavia, Pavia, Italy
*Email: andrea.moiola@unipv.it

Abstract

The scattering of time-harmonic acoustic waves by sound-soft (Dirichlet) screens, or cracks, its modeling via boundary integral equations (BIE) and discretisation with the boundary element method (BEM) are classical topics when the screens are sufficiently smooth. We extend the theory to include screens that are fractal or have fractal boundaries, such as the Sierpinski triangle, the Cantor dust and the Koch snowflake. Building on previous work on Sobolev spaces [1, 3] and BIE [2] we present well-posed BVPs for open and compact flat screens. We give conditions under which these problems can be approximated by similar ones on smoother "prefractal" sets, and results on the convergence of the associated BEM. Details, extensions, proofs and numerical experiments can be found in [4].

Keywords: Helmholtz equation, boundary element method, scattering, screen, sound-soft, fractal, non-Lipschitz set, Mosco convergence

1 Notation and Sobolev spaces

We consider a flat screen Γ that is a bounded subset of the plane $\Gamma_\infty := \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. (The case of 2-dimensional wave propagation $\Gamma \subset \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ can easily be treated in the same way.)

We use fractional (Bessel) Sobolev spaces. For $s \in \mathbb{R}$, $\Omega \subset \mathbb{R}^2$ open and $F \subset \mathbb{R}^2$ closed, let

$$\begin{aligned} H^s(\mathbb{R}^2) :=& \{u \in \mathcal{S}^*(\mathbb{R}^2) : \|u\|_{H^s(\mathbb{R}^2)} < \infty\} \\ \|u\|^2_{H^s(\mathbb{R}^2)} :=& \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi, \\ H^s(\Omega) :=& \{u|_\Omega : u \in H^s(\mathbb{R}^2)\}, \\ \widetilde{H}^s(\Omega) :=& \overline{C_0^\infty(\Omega)}^{H^s(\mathbb{R}^2)}, \\ H^s_F :=& \{u \in H^s(\mathbb{R}^2) : \mathrm{supp} u \subset F\}. \end{aligned}$$

In general $\widetilde{H}^s(\Omega) \subset H^s_{\overline{\Omega}}$; they coincide if Ω is sufficiently regular but examples with $\widetilde{H}^s(\Omega) \neq H^s_{\overline{\Omega}}$ can be constructed [3, Thm. 3.19]. We denote $\gamma^\pm$ the traces $\gamma^\pm : W^1(\mathbb{R}^3_\pm) \to H^{1/2}(\Gamma_\infty)$, where $\mathbb{R}^3_\pm$ are the upper and lower half-spaces.

2 Boundary value problems (BVP)

The classical sound-soft screen scattering BVP consists of looking for u satisfying the Helmholtz equation (1), the Sommerfeld condition (2) and the Dirichlet boundary condition (3):

$$\Delta u + k^2 u = 0 \qquad \text{in } D := \mathbb{R}^3 \setminus \overline{\Gamma}, \quad (1)$$

$$\partial_r u(\mathbf{x}) - \mathrm{i}ku(\mathbf{x}) = o(r^{-1}) \qquad r := |\mathbf{x}| \to \infty, \quad (2)$$

$$u = -u^i \qquad \text{on } \Gamma, \quad (3)$$

where $k > 0$ is the wavenumber and u^i is a given incident wave. To formulate a well-posed BVP, one needs to be more precise about the sense in which the boundary condition (3) holds.

We first describe the case when Γ is a relatively open subset of Γ_∞.

Definition 1 (BVP $\mathsf{D}^{op}(\Gamma)$) *Let $\Gamma \subset \Gamma_\infty$ be bounded and open and $g \in H^{1/2}(\Gamma)$. Find $u \in C^2(D) \cap W^{1,\mathrm{loc}}(D)$ satisfying (1)–(2) and*

$$(\gamma^\pm u)|_\Gamma = g.$$

Theorem 2 (Thm. 6.2 [2]) *If $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$, then $\mathsf{D}^{op}(\Gamma)$ admits a unique solution u.*

Moreover, u satisfies the representation formula $u(\mathbf{x}) = -\mathcal{S}_\Gamma\phi(\mathbf{x})$, $\mathbf{x} \in D$, where $\mathcal{S}_\Gamma$ is the single-layer potential and $\phi = [\partial_n u] := \partial^+_n u - \partial^-_n u \in \widetilde{H}^{-1/2}(\Gamma)$ is the unique solution of the BIE $S_\Gamma\phi = -g$, with $S_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ the single-layer operator.

The main assumption for the well-posedness of the BVP is $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$, equivalent to the density of $C^\infty_0(\Gamma)$ in $H^{-1/2}_{\overline{\Gamma}}$. This is guaranteed if, e.g., *(i)* Γ is C^0 up to countably many points $P \subset \partial\Gamma$ such that P has only finitely many limit points [3, Thm. 3.24], or *(ii)* Γ is "thick" in the sense of Triebel [1]. All Lipschitz Γ, but also classical and exotic snowflakes with fractal boundaries, satisfy these conditions [1].

If the screen Γ is a compact set, we substitute the restriction operator $|_\Gamma$ in the boundary conditions with the orthogonal projection $P_\Gamma : H^{1/2}(\mathbb{R}^2) \to (\widetilde{H}^{1/2}(\Gamma^c))^\perp$, where $\Gamma^c = \mathbb{R}^2 \setminus \Gamma$.

Definition 3 (BVP $\mathsf{D}^{co}(\Gamma)$) *Let $\Gamma \subset \Gamma_\infty$ be compact and $g \in (\widetilde{H}^{1/2}(\Gamma^c))^\perp$. Find $u \in C^2(D) \cap W^{1,\mathrm{loc}}(D)$ satisfying (1)–(2) and*

$$P_\Gamma \gamma^\pm u = g.$$

This choice of P_Γ ensures that if $\Omega \subset \Gamma_\infty$ is bounded, open, and $\widetilde{H}^{-1/2}(\Omega) = H^{-1/2}_{\overline{\Omega}}$, then the problems $\mathsf{D}^{op}(\Omega)$ and $\mathsf{D}^{co}(\overline{\Omega})$ are equivalent.

Theorem 4 (Thm. 6.4 [2]) *Problem $\mathsf{D}^{co}(\Gamma)$ admits a unique solution u.*

Moreover, u satisfies the representation formula $u(\mathbf{x}) = -\mathcal{S}_\Gamma\phi(\mathbf{x})$, $\mathbf{x} \in D$, with $\phi = [\partial_n u]$ the solution of the BIE $S_\Gamma\phi = -g$ for the single-layer operator $S_\Gamma : H^{-1/2}_\Gamma \to (\widetilde{H}^{1/2}(\Gamma^c))^\perp$.

3 Prefractal to fractal convergence

To study the scattering by a fractal screen Γ, we approximate it with simpler prefractal shapes $(\Gamma_j)_{j\in\mathbb{N}}$. The BVP $\mathsf{D}^x(\Gamma)$ ($^x \in \{^{op}, ^{co}\}$) is correctly approximated by a sequence of BVPs $\mathsf{D}^x(\Gamma_j)$ if the corresponding sequence of subspaces (either $\widetilde{H}^{-1/2}(\Gamma_j)$ or $H^{-1/2}_{\Gamma_j}$) of $H^{-1/2}(\mathbb{R}^2)$ converges in the sense of Mosco [4]. In [4] we show:

Theorem 5 *The solution ϕ_j of the BIE on Γ_j converges in $H^{-1/2}(\mathbb{R}^2)$ to the solution ϕ of the BIE on Γ and $\mathcal{S}\phi_j \to \mathcal{S}\phi$ in $W^{1,loc}(\mathbb{R}^2)$ if, e.g.,*
- *Γ and Γ_j are bounded, open with $\Gamma_j \subset \Gamma_{j+1}$ and $\Gamma = \bigcup_{j\in\mathbb{N}} \Gamma_j$, or*
- *Γ_j are compact, $\Gamma_j \supset \Gamma_{j+1}$ and $\Gamma = \bigcap_{j\in\mathbb{N}} \Gamma_j$.*

We also show convergence for a class of non-nested prefractals such as those in Figure 1.

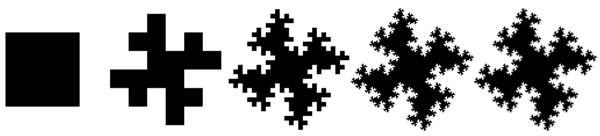

Figure 1: Non-nested prefractals Γ_j for the square snowflake Γ, for which $\phi_j \to \phi$ in $H^{-1/2}$.

4 BEM discretisation

We approximate the solution of scattering problems posed on a non-Lipschitz screen Γ (fractal or with fractal boundary) using a piecewise-constant boundary element methods (BEM) on prefractal screens Γ_j. In [4] we give general criteria on the mesh to guarantee convergence of the Galerkin BEM: the key is the Mosco convergence of the discrete spaces on Γ_j to the desired Sobolev space, either $\widetilde{H}^{-1/2}(\Gamma)$ or $H^{-1/2}_\Gamma$.

E.g., if Γ_j are the classical prefractal approximation of the Koch snowflake, or the square snowflake prefractals of Figure 1, then any sequence of meshes $\mathcal{T}_j$ on Γ_j with meshsize $h_j \searrow 0$ provides a provably convergent BEM scheme.

If Γ is a Cantor dust (the Cartesian product of two identical Cantor sets) then its scattered field is non-zero (for almost every incident plane waves) if and only if the Hausdorff dimension of Γ is larger than 1, [2]. We verify this numerically in Figure 2. For details and more extensive numerical tests, see [4].

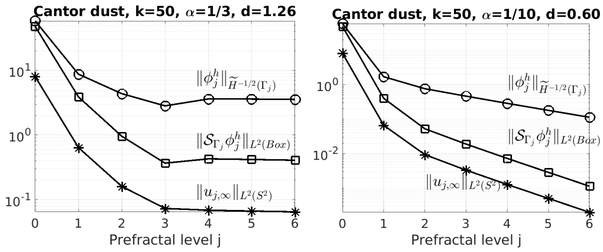

Figure 2: On-screen ($\circ$), near-field ($\Box$) and far-field ($*$) norms of the field scattered by Cantor dust prefractals $\Gamma_0, \ldots, \Gamma_6$ computed with BEM. When the prefractal level is refined, for Hausdorff dimension $d = \frac{\log 4}{\log 3} > 1$ (left) the norms provably converge to a positive value, while for $d = \frac{\log 4}{\log 10} < 1$ (right) they converge to 0.

References

[1] A. Caetano, D. P. Hewett and A. Moiola, Density results for Sobolev, Besov and Triebel-Lizorkin spaces on rough sets, *in preparation.*

[2] S. N. Chandler-Wilde and D. P. Hewett, Well-posed PDE and integral equation formulations for scattering by fractal screens, *SIAM J. Math. Anal.*, **50** (2018), pp. 677–717.

[3] S. N. Chandler-Wilde, D. P. Hewett and A. Moiola, Sobolev spaces on non-Lipschitz subsets of $\mathbb{R}^n$ with application to boundary integral equations on fractal screens, *Integr. Equat. Operat. Th.*, **87** (2017), pp. 179–224.

[4] S. N. Chandler-Wilde, D. P. Hewett and A. Moiola, Boundary element methods for acoustic scattering by fractal screens, *in preparation.*

Suggested members of the Scientific Committee:
Annalisa Buffa, Xavier Claeys, Patrick Joly