Acoustic scattering by impedance screens with fractal boundary

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Abstract

We present a well-posedness analysis of boundary value problem and boundary integral equation formulations for acoustic scattering by planar screens/cracks on which impedance boundary conditions are imposed. In contrast to previous works, our analysis is valid for screens with arbitrarily rough (possibly fractal) boundary.

Keywords: Boundary integral equations, Sobolev spaces, screen problems, Mosco convergence

The scattering of time-harmonic acoustic waves by thin "screens" or "cracks" is important in many applications, including noise barrier design and defect detection in non-destructive testing. To model the case where the screen acts as an absorber of wave energy one formulates the scattering problem as an impedance boundary value problem (BVP), which itself can be reformulated as a boundary integral equation (BIE) on the screen. Previous studies [1,2] establish well-posedness of such formulations when the boundary of the screen is smooth (in an unspecified sense). In this study we clarify the smoothness assumptions required for the analysis of [1, 2] to be valid, and show how the addition of extra conditions in the BVP, along with a simple modification of the functional setting for the BIE, produce formulations which are well-posed for completely arbitrary bounded open planar screens, as was done previously for Dirichlet and Neumann screen problems in [3].

In order to focus attention on the regularity of the screen boundary, we simplify the problem as much as possible by restricting attention to a planar screen Γ, assumed to be a bounded open subset of the hyperplane $\mathbb{R}^{n-1}\times\{0\}\subset\mathbb{R}^n$, $n=2,3$. For $s\in\mathbb{R}$ let $H^s(\mathbb{R}^{n-1})$ be the standard Bessel potential Sobolev space on $\mathbb{R}^{n-1}$, normed by $\|u\|^2_{H^s(\mathbb{R}^{n-1})}=\int_{\mathbb{R}^{n-1}}(1+|\xi|^2)^s|\hat{u}(\xi)|^2\,\mathrm{d}\xi$, and for open $\Gamma\subset\mathbb{R}^{n-1}$ and closed $F\subset\mathbb{R}^{n-1}$ let

$$\widetilde{H}^s(\Gamma):=\overline{C_0^\infty(\Gamma)}^{H^s(\mathbb{R}^{n-1})},$$

$$H^s_F:=\{u\in H^s(\mathbb{R}^{n-1}):\operatorname{supp}u\subset F\},$$

and

$$H^s(\Gamma)=\{u|_\Gamma : u\in H^s(\mathbb{R}^{n-1})\},$$

$|_\Gamma$ denoting (distributional) restriction to Γ.

Given $k>0$ and an incident field u^i (e.g. a plane wave $u^i=\mathrm{e}^{\mathrm{i}kd\cdot x}$), the standard BVP for the scattered field is [2]: find $u\in C^2(\mathbb{R}^n\setminus\overline{\Gamma}))\cap W^{1,\mathrm{loc}}(\mathbb{R}^n\setminus\overline{\Gamma})$, outgoing at infinity, s.t.

$$(\Delta+k^2)u=0,\quad \text{in } \mathbb{R}^n\setminus\overline{\Gamma}, \tag{1}$$

$$\partial_n^\pm u|_\Gamma \pm \lambda^\pm\gamma^\pm u|_\Gamma = g^\pm, \tag{2}$$

where $\gamma^\pm$ and $\partial_n^\pm$ are the Dirichlet and Neumann traces from the half spaces $\{\pm x_n>0\}$ to the hyperplane $\mathbb{R}^{n-1}$, $\lambda^\pm$ are impedance parameters (constants with $\Im\lambda^\pm\geq 0$ and $\lambda^++\lambda^-\neq 0$) and $g^\pm=-(\partial_n^\pm u^i|_\Gamma \pm \lambda^\pm\gamma^\pm u^i|_\Gamma)\in H^{-1/2}(\Gamma)$.

Theorem 1 *Let $g^+-g^-\in\widetilde{H}^{-1/2}(\Gamma)|_\Gamma$. The BVP (1)-(2) is well-posed if $\widetilde{H}^{\pm1/2}(\Gamma)=H^{\pm1/2}_{\overline{\Gamma}}$ and $H^{-1/2}_{\partial\Omega}=\{0\}$. In particular, these conditions hold when Γ is "C^0 except at a countable set of points with finitely many limit points" [4].*

The proof of Theorem 1 follows the approach in [2]. We first prove BVP uniqueness by a standard Rellich/unique continuation argument, then that the BVP solution (if it exists) is given by

$$u=\mathcal{D}\phi-\mathcal{S}\psi,$$

where $\mathcal{D}$ and $\mathcal{S}$ are the usual double- and single-layer potentials and $\phi:=[u]\in H^{1/2}_{\overline{\Gamma}}=\widetilde{H}^{1/2}(\Gamma)$ and $\psi:=[\partial_n u]\in H^{-1/2}_{\overline{\Gamma}}=\widetilde{H}^{-1/2}(\Gamma)$ are the Dirichlet and Neumann jumps, which must satisfy the BIE

$$A\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\lambda^-g^++\lambda^+g^-\\ g^+-g^-\end{pmatrix},$$

where

$$A=\begin{pmatrix}\lambda^+\lambda^-|_\Gamma+(\lambda^++\lambda^-)T & -\frac{1}{2}(\lambda^+-\lambda^-)|_\Gamma\\ \frac{1}{2}(\lambda^+-\lambda^-)|_\Gamma & |_\Gamma-(\lambda^++\lambda^-)S\end{pmatrix},$$

with $S:\widetilde{H}^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$ and $T:\widetilde{H}^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)$ denoting the usual single layer and hypersingular operators. (Note: the double layer contributions in [2] vanish when Γ is planar.)

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Viewing A as an operator

$$A : \widetilde{H}^{1/2}(\Gamma)\times\widetilde{H}^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)\times\widetilde{H}^{-1/2}(\Gamma)|_\Gamma,$$

the assumptions of Theorem 1 guarantee that $|_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is injective, implying that A is Fredholm of index zero, so that well-posedness follows from BVP uniqueness.

But for general open Γ the above analysis can fail. In particular, the spaces $\widetilde{H}^{\pm 1/2}(\Gamma)$ and $H^{\pm 1/2}_{\overline{\Gamma}}$ may differ, the space $H^{-1/2}_{\partial\Gamma}$ may be nontrivial (which holds if the Hausdorff dimension of $\partial\Gamma$ exceeds $n-2$), and $|_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ may have a nontrivial kernel [4].

Our proposed remedy is to demand a little extra smoothness, namely that $g^+ - g^- \in H^0(\Gamma) \cong L^2(\Gamma)$, and supplement (1)-(2) with two additional conditions, namely

$$[u] \in \widetilde{H}^{1/2}(\Gamma), \tag{3}$$

$$[\partial_n u] \in \widetilde{H}^0(\Gamma) \cong L^2(\Gamma). \tag{4}$$

It is easily checked that, when the assumptions of Theorem 1 hold, the extra assumptions (3)-(4) are superfluous, i.e. they hold automatically.

Theorem 2 *Let* $g^+ - g^- \in H^0(\Gamma) \cong L^2(\Gamma)$. *The BVP* (1)-(4) *is well-posed for any bounded open screen* Γ.

Our proof follows that of Theorem 1, except now we view A as an operator

$$A : \widetilde{H}^{1/2}(\Gamma) \times \widetilde{H}^0(\Gamma) \to H^{-1/2}(\Gamma) \times H^0(\Gamma).$$

(Note that both $\widetilde{H}^0(\Gamma)$ and $H^0(\Gamma)$ are isometrically isomorphic to $L^2(\Gamma)$ but we maintain the $\widetilde{H}^0(\Gamma)$ and $H^0(\Gamma)$ notation to distinguish distributions on $\mathbb{R}^{n-1}$ from those on Γ.) This new functional setting for A is attractive because now the codomain is the dual of the domain, so we have a symmetric variational framework. Furthermore, A is then not only Fredholm of index zero, but in fact one can decompose A as

$$A = A_{\text{coercive}} + A_{\text{compact}},$$

where, for a suitable choice of dual pairing on the product space,

$$A_{\text{coercive}} = \begin{pmatrix} (\lambda^+ + \lambda^-)T & 0 \\ 0 & |_\Gamma \end{pmatrix}$$

is coercive (strongly elliptic) [5] and

$$A_{\text{compact}} = \begin{pmatrix} \lambda^+\lambda^-|_\Gamma & -\frac{1}{2}(\lambda^+ - \lambda^-)|_\Gamma \\ \frac{1}{2}(\lambda^+ - \lambda^-)|_\Gamma & -(\lambda^+ + \lambda^-)S \end{pmatrix}$$

is compact.

This improved regularity result for A is a useful tool for the numerical analysis of boundary element approximations to impedance screen scattering problems. In particular, for a class of screens Γ with fractal boundary (including the von Koch snowflake) one can apply the theory of Mosco convergence for variational problems (detailed in [6]) and certain density results for function spaces on rough domains (derived in [7]) to prove that boundary element discretizations of scattering problems on smoother "prefractal" screens approximating Γ converge to the solution of (1)-(4) on the limiting fractal screen Γ.

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