Faraday cages, homogenized boundary conditions and resonance effects

David P. Hewett[1,*], **Ian J. Hewitt**[2]
[1]Department of Mathematics, University College London, London, UK
[2]Mathematical Institute, University of Oxford, Oxford, UK
*Email: d.hewett@ucl.ac.uk

Abstract

We study electromagnetic shielding by a cage of perfectly conducting wires - the 'Faraday cage effect'. In the limit as the number of wires tends to infinity we derive continuum models for the shielding, involving homogenized boundary conditions on an effective cage boundary. For wires of sufficiently large radius there are resonance effects: at wavenumbers close to the natural resonances of the equivalent solid shell, the cage actually amplifies the incident field, rather than shielding it. By modifying the continuum model we can calculate the wavenumbers giving the largest response, along with the associated peak amplitudes.

Keywords: Electromagnetic shielding, homogenization, multiple scales, resonance.

Introduction

The Faraday cage effect is a well-known phenomenon whereby electromagnetic waves can be blocked by a wire mesh 'cage'. Somewhat surprisingly, until recently there was apparently no widely-known mathematical analysis quantifying the effectiveness of the shielding as a function of basic parameters, such as the geometry of the cage, and the thickness, shape and spacing of the wires in the mesh from which it is constructed. The recent publications [1, 3] provide such an analysis for the two-dimensional case where the cage comprises a large number of equally-spaced perfectly-conducting 'wires' of the same shape and radius. Our analysis in [1,3] uses homogenized boundary conditions derived by the method of multiple scales and matched asymptotic expansions, as employed in [2, 4] to study similar problems.

For brevity we consider here only TE polarization, requiring the study of a scalar field satisfying the Helmholtz equation and zero Dirichlet boundary conditions on the wires. The TM case can be treated similarly [3].

Let Ω_- be a bounded simply connected open subset of the plane with smooth boundary $\Gamma = \partial\Omega_-$ and let $\Omega_+ := \mathbb{R}^2 \setminus \overline{\Omega}_-$ denote the complementary exterior domain. We consider a 'cage' of M non-intersecting wires $\{K_j\}_{j=1}^M$ (compact subsets of the plane, of identical radius r, shape and orientation relative to Γ) distributed along Γ with constant arc length separation

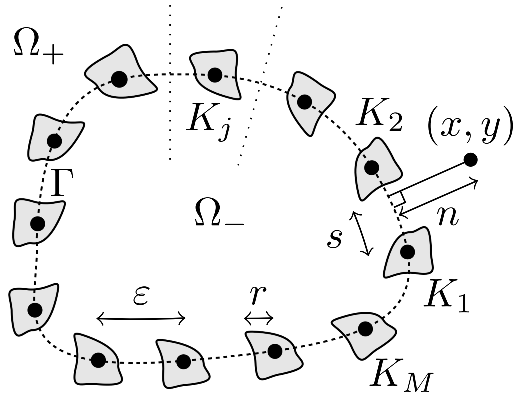

Figure 1: Faraday cage geometry

$$\varepsilon = |\Gamma|/M,$$

where $|\Gamma|$ is the total length of Γ, see Figure 1. We set $D := \mathbb{R}^2 \setminus \left(\bigcup_{j=1}^M K_j\right)$. Given an incident wave ϕ^i (e.g. a plane wave or point source) we seek a scattered field ϕ satisfying

$$\begin{aligned} (\nabla^2 + k^2)\phi &= 0, && \text{in } D, \\ \phi &= -\phi^i, && \text{on } \partial D, \end{aligned}$$

and an outgoing radiation condition at infinity.

Homogenization for $\varepsilon \to 0$

In the limit as $\varepsilon \to 0$ ($M \to \infty$) we look for outer approximations in $\Omega_\pm$ of the form

$$\phi = \phi_0^\pm + \varepsilon\phi_1^\pm + O(\varepsilon^2) \qquad \text{in } \Omega_\pm.$$

The rapid variation close to Γ is modelled by a boundary layer of width $O(\varepsilon)$. Here we look for a solution in multiple-scales form

$$\phi(n, s) = \Phi(N, S; s),$$

where (n, s) are normal and tangential coordinates (see Figure 1), (N, S) are rescaled versions

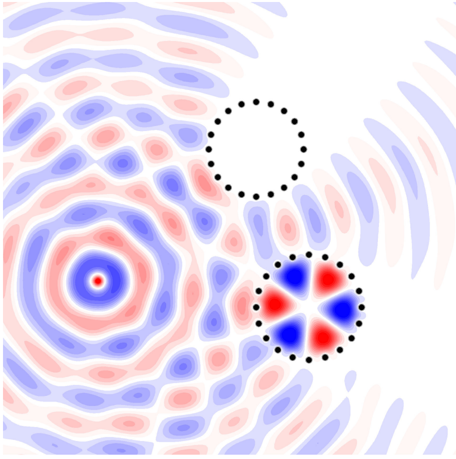

Figure 2: Resonance effects. Two slightly different sized cages of identical circular wires irradiated by a point source. The smaller cage is shielding while the larger cage is amplifying.

defined by $(n, s) = (\varepsilon N, \varepsilon S)$, and $\Phi(N, S; s)$ is assumed 1-periodic in the fast tangential variable S, satisfying an appropriate cell problem. Asymptotic matching gives homogenized boundary conditions for $\phi_j^\pm$ on Γ, the nature of which depend on the thickness of the wires.

For 'thin' wires (radius $r \ll \varepsilon$) the two-term approximation $\phi_0 + \varepsilon\phi_1$ is continuous across Γ but undergoes a jump in normal derivative with

$$\left[\frac{\partial \phi_0}{\partial n} + \varepsilon \frac{\partial \phi_1}{\partial n}\right]_-^+ = \alpha\left(\phi_0 + \varepsilon\phi_1\right) \qquad \text{on } \Gamma,$$

where

$$\alpha = \frac{|\Gamma|}{\varepsilon \log\left(\varepsilon/(r|\Gamma|)\right) + a_0} = \frac{M}{\log\left(1/(rM)\right) + a_0},$$

and a_0 is a constant depending on the wire shape. (Specifically, a_0 can be expressed in terms of the logarithmic capacity of the scaled wire $(1/r)K_j$.) The distinguished scaling in which $\alpha = O(1)$ occurs when $r = O(\varepsilon \mathrm{e}^{-c/\varepsilon})$ for some $c > 0$.

For 'thick' wires (radius $r = O(\varepsilon)$) the leading order approximation satisfies

$$\phi_0^+ = \phi_0^- = 0 \qquad \text{on } \Gamma. \tag{1}$$

The $O(\varepsilon)$ corrections $\phi_1^\pm$ satisfy inhomogeneous Dirichlet boundary conditions involving $\partial\phi_0^\pm/\partial n$ and constants extracted from the boundary layer cell problem. For non-resonant k, the homogeneous boundary condition (1) implies that $\phi_0^- \equiv 0$ in Ω_-, so we predict a shielding effect, with $\phi = O(\varepsilon)$ in Ω_-.

However, if k is resonant, i.e. k^2 is a Dirichlet eigenvalue of the negative Laplacian in Ω_-, then the problem for ϕ_0^- is ill-posed and our approximation breaks down. In practice, close to resonance one observes a large excitation inside Ω_-; the cage *amplifies* the field rather than shielding from it, see Figure 2. For $k \approx k_*$ (a resonant wavenumber) we modify our ansatz to

$$\phi^-(x, y) = \frac{1}{\varepsilon}\phi_{-1}^- + \phi_0^- + \varepsilon\phi_1^- + O(\varepsilon^2) \qquad \text{in } \Omega_-.$$

Matching then reveals that

$$\phi_{-1}^- = C_{-1}\psi^*,$$

where ψ^* is the eigenmode corresponding to k^* (in general a superposition of eigenmodes), and solvability conditions for ϕ_0^- and ϕ_1^- give

$$|C_{-1}| = A_1 \left(1 + \left(\frac{k - \tilde{k}^*}{\varepsilon^2 A_2}\right)^2\right)^{-1/2},$$

where $A_1, A_2, \tilde{k}^*$ are constants depending on Γ (the cage geometry) and the wire shape/size (for details see [3]). The maximum amplitude occurs not at $k = k^*$ but rather at the shifted value $\tilde{k}^*$; in [3] we derive a three-term expansion $\tilde{k}^* = k^* + \varepsilon\tilde{k}_1^* + \varepsilon^2\tilde{k}_2^*$, with explicit formulas for $\tilde{k}_1^*$ and $\tilde{k}_2^*$. We also demonstrate the excellent agreement between these asymptotic approximations and full numerical simulations.

References

[1] S. J. Chapman, D. P. Hewett and L. N. Trefethen, Mathematics of the Faraday cage, *SIAM Review* **57(3)** (2015), 398–417.

[2] B. Delourme, H. Haddar and P. Joly, Approximate models for wave propagation across thin periodic interfaces, *J. Math. Pures Appl.* **98** (2012), 28–71.

[3] D. P. Hewett and I. J. Hewitt, Homogenized boundary conditions and resonance effects in Faraday cages, *Proc. Roy. Soc. A* **472** (2016), 2189.

[4] C.L. Holloway, E.F. Kuester and A. Dienstfrey, A homogenization technique for obtaining generalized sheet transition conditions for an arbitrarily shaped coated-wire grating, *Radio Sci.* **49(10)** (2014), 813–850.