A high frequency boundary element method for scattering by three-dimensional screens

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Abstract

We propose a numerical-asymptotic boundary element method for time-harmonic acoustic scattering of an incident plane wave by sound-soft three-dimensional (3D) screens. Standard numerical schemes require the number of degrees of freedom to grow rapidly in order to maintain accuracy as frequency increases. Here, we enrich our approximation space away from the edges of the screen with oscillatory basis functions carefully designed to capture the high frequency behaviour of the solution. We show that reasonable accuracy can be achieved for a range of frequencies using relatively few degrees of freedom.

Keywords: Helmholtz, high frequency, hybrid numerical-asympotic boundary element method

1 Problem statement

We consider the 3D problem of scattering of the time harmonic incident plane wave $u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$, where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $k > 0$ is the wavenumber and $\mathbf{d}$ is a unit direction vector, by a sound soft screen $\Gamma := \{(x_1, x_2, 0) \in \mathbb{R}^3 : (x_1, x_2) \in (0, 2\pi) \times (0, 2\pi)\}$. The boundary value problem (BVP) we wish to solve is: given u^i, determine $u \in C^2(D) \cap W^1_{loc}(D)$ such that

$$\Delta u + k^2 u = 0 \text{ in } D := \mathbb{R}^3 \backslash \bar{\Gamma},\ u = 0 \text{ on } \Gamma,$$

and the scattered field $u^s := u - u^i$ satisfies the Sommerfeld radiation condition. For the solution of the above BVP, a form of Green's representation theorem holds:

$$u(\mathbf{x}) = u^i(\mathbf{x}) + \int_\Gamma \Phi_k(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial \mathbf{n}}\right](\mathbf{y})\,\mathrm{d}\mathbf{y},\ \mathbf{x} \in D,$$

where $\Phi_k(\mathbf{x}, \mathbf{y}) = \exp(\mathrm{i}k|\mathbf{x} - \mathbf{y}|)/4\pi|\mathbf{x} - \mathbf{y}|$ and $[\partial u/\partial \mathbf{n}] =: \phi$ is the jump in the normal derivative $\partial u/\partial \mathbf{n}$ across Γ. Then ϕ satisfies the boundary integral equation (see, e.g., [1, §7.6])

$$S_k\phi(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\,\mathrm{d}\mathbf{y} = u^i(\mathbf{x}),\ \mathbf{x} \in \Gamma. \quad (1)$$

2 Approximation space

The key idea of our approach is to adapt our approximation space for the solution of (1) to the high frequency asymptotic behaviour of the solution. Specifically, for $\mathbf{x} \in \Gamma$ we write

$$\phi(\mathbf{x}) = \Psi(\mathbf{x}) + \sum_{m=1}^{M} V_m(\mathbf{x}, k) \exp(\mathrm{i}k\psi_m(\mathbf{x})), \quad (2)$$

where $\Psi := 2\partial u^i/\partial \mathbf{n}$ represents the Geometrical Optics approximation, and our aim is to choose the phase functions ψ_m, $m = 1, \ldots, M$, in such a way that the corresponding amplitudes $V_m(\cdot, k)$ are (relatively) non-oscillatory. For the equivalent 2D problem (see [2]) it is sufficient to take $M = 2$, in which case V_1 and V_2 are provably non-oscillatory (i.e. all of the oscillations are captured completely by a small number of phase functions). For the 3D problem this is not the case, since the waves diffracted by the edges and corners of the screen are rediffracted infinitely often by the other edges and corners of the screen, taking a different direction of travel after each rediffraction. However, it turns out that with a judicious choice of ψ_m in (2) we can represent ϕ to a reasonable degree of accuracy away from the edges of the screen (where the solution is singular and a standard approximation space is used) using only a small value of M. Specifically, we choose $M = 8$, and ψ_m, $m = 1, \ldots, 8$, so that $\exp(\mathrm{i}k\psi_m(\mathbf{x}))$, $m = 1, \ldots, 4$, represent plane waves propagating in the direction of the singly-diffracted rays predicted by the Geometrical Theory of Diffraction (see [1, §7.6]), with one such wave associated to each of the four edges of the screen, and $\exp(\mathrm{i}k\psi_m(\mathbf{x}))$, $m = 5, \ldots, 8$, represent their reflections by other edges (this being sufficient to capture all rereflections from edges, due to the regular nature of the screen). Using (2) we can design an appropriate approximation space to represent $\varphi := \phi - \Psi$, the difference between $[\partial u/\partial \mathbf{n}]$ and its Geometrical Optics approximation. Precisely: within a tenth of a wavelength of each edge of the screen we use a standard (piecewise polynomial) approximation space on an appropriate graded mesh (in order to capture the singular behaviour near the edges); away from the edge of the screen we divide the screen into nine elements, and on each element our approximation space consists of piecewise polynomials (of maximum order p) multiplied by each of the four plane waves described above, as shown in Figure 1.

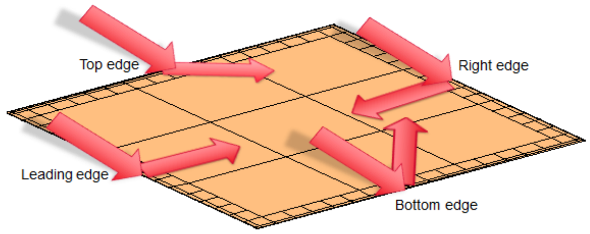

Figure 1: Coarse mesh away from edges, standard (graded) mesh near edges

3 Numerical results

We use a Galerkin method to select an element from our approximation space, denoted by $V_{p,k}$. That is, we seek $\varphi_p \in V_{p,k}$ such that

$$\langle S_k \varphi_p, v \rangle_\Gamma = \left\langle u^i - S_k \Psi, v \right\rangle_\Gamma, \ \forall v \in V_{p,k}, \quad (3)$$

where the duality pairings in (3) are L^2 inner products. With $\mathbf{d} = (3,1,1)/\sqrt{11}$ we solve (3) for $p = 0,1,2$, giving up to $1,4,9$ polynomials respectively on each coarse element. So, with eight wave directions and nine coarse elements, we have, for $p = 0,1,2$, respectively 72, 288, and 648 total degrees of freedom on our coarse mesh, and we keep this value fixed for each different value of k tested. Note that this central part of the screen covers $k - 0.2$ wavelengths in each direction, so a standard scheme requiring, say, 10 degrees of freedom per wavelength, might require of the order of $100(k-0.2)^2$ degrees of freedom on this region in order to represent the solution to "engineering accuracy". In Figure 2 we plot on a logarithmic scale the relative L^2 errors in ϕ on this central section (we restrict attention here to this central portion because we are primarily interested in understanding how well we can represent the oscillatory behaviour), against k, for $p = 0,1,2$, demonstrating that we can achieve a reasonable level of accuracy using this approach with a very small number of degrees of freedom compared to standard methods.

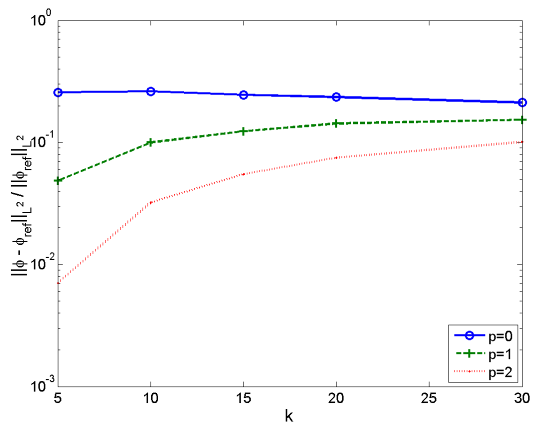

Figure 2: Convergence results

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