A high-frequency boundary element method for a transmission scattering problem

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Abstract

We consider time-harmonic scattering by penetrable polygonal obstacles in 2D. We present a boundary element method (BEM) based on a hybrid numerical-asymptotic (HNA) approximation space built from oscillatory basis functions. The basis functions are carefully chosen to capture the leading order terms in the high-frequency asymptotic behaviour of the boundary solution. Numerical experiments suggest that the method can achieve fixed accuracy using a small (and frequency-independent) number of degrees of freedom, even at high frequencies.

Keywords: Helmholtz, high-frequency scattering, transmission, boundary element method

1 Introduction

It is well-known that conventional numerical methods for time-harmonic scattering problems become computationally expensive when the size of the scattering obstacle is large relative to the wavelength of the incident wave. The HNA boundary element approach (see [1] and the references therein) aims to address this by building the high-frequency oscillatory behaviour of the solution directly into the approximation space.

This is done by making a high-frequency ansatz for the boundary solution of the form

$$V(\mathbf{x}) = V_{go}(\mathbf{x}, k) + \sum_{m=1}^{M} V_m(\mathbf{x}, k)\exp(ik\psi_m(\mathbf{x})), \tag{1}$$

where V_{go} is the geometrical optics (GO) approximation and the summation term represents the diffracted field. The phases ψ_m are chosen by referring to asymptotic methods, such as the Geometrical Theory of Diffraction (GTD). If the phases are chosen correctly, the amplitudes V_m will be slowly varying and hence may be efficiently approximated by low-order polynomials at all frequencies. To date, HNA methods have been applied successfully to problems for which the asymptotics are known and (1) requires only a small number of terms. In particular, to scattering by impenetrable convex, as well as a class of non-convex, polygons.

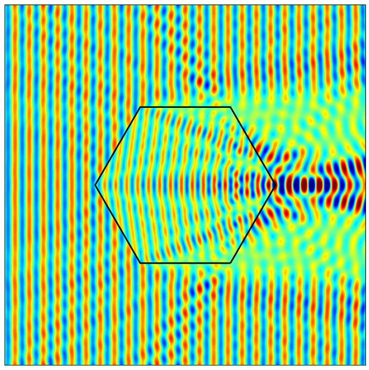

Figure 1: Hexagonal scatterer with $\mu = 1.31$.

The present work extends the HNA methodology to a transmission problem. In this problem, the GO and diffracted components are considerably more complicated owing to internal reflections. In fact, both components comprise infinitely many terms. Furthermore, the relevant canonical problem of scattering by an infinite penetrable wedge has, as yet, no known closed form or asymptotic solution.

In [2] the construction of a HNA approxmation space for this problem, using a heuristic adaptation of classical GTD, was discussed. A beam-tracing algorithm was employed to calculate V_{go} and, in the summation in (1), only the leading order diffracted waves were considered, i.e. both head waves and internal reflections of diffracted waves were ignored. Here we present the implementation of this approximation space within a Galerkin BEM.

2 The transmission problem

Consider the 2D problem of scattering of a time-harmonic wave u^i by a penetrable polygon Ω, as illustrated in Figure 1. We wish to determine the field u_1 in the exterior domain D and the field u_2 within Ω such that

$$\begin{aligned} &\Delta u_1 + k_1^2 u_1 = 0, \quad \text{in } D := \mathbb{R}^2 \backslash \Omega, \\ &\Delta u_2 + k_2^2 u_2 = 0, \quad \text{in } \Omega, \\ &u_1 = u_2 \text{ and } \frac{\partial u_1}{\partial \mathbf{n}} = \alpha \frac{\partial u_2}{\partial \mathbf{n}}, \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

in addition to an outgoing radiation condition for the scattered field $u^s := u_1 - u^i$ at infinity. Here $\alpha \in \mathbb{C}$, k_1, k_2 are the wavenumbers in the exterior and interior domains respectively. We shall write $k_2 = \mu k_1$, μ being the complex refractive index of the scatterer with $\mathrm{Im}(\mu) \geq 0$.

We may employ Green's representation theorem to reformulate (2) as a system of boundary integral equations, for example as

$$AV = f,$$

where $V = (u_1, \partial u_1/\partial \mathbf{n})^T$, $f = \left(u^i, \partial u^i/\partial \mathbf{n}\right)^T$,

$$A = \begin{pmatrix} \frac{1}{2}(I+\alpha) - (D_1 - \alpha D_2) & S_1 - S_2 \\ -\alpha(H_1 - H_2) & \frac{1}{2}(I+\alpha) + (\alpha D_1' - D_2') \end{pmatrix},$$

and S_i, D_i, D_i' and H_i, for $i = 1, 2$, are the single, double, adjoint-double and hypersingular integral operators.

3 HNA for the transmission problem

Our HNA ansatz for V on each side is (as in [2])

$$\begin{aligned} V(\mathbf{x}) \approx V_{go} &+ V_1^+ e^{ik_1 s} + V_1^- e^{-ik_1 s} \\ &+ V_2^+ e^{ik_2 s} + V_2^- e^{-ik_2 s} + \sum_{j=1}^{n-2} V_j^r e^{ik_2 r_j}, \end{aligned}$$

where s is the arc-length along the side, n is the number of corners, and $r_1, \ldots, r_{n-2}$ are the distances between $\mathbf{x}$ and the $n-2$ corners not adjacent to the side. Therefore, this ansatz aims to capture the "primary" diffraction on each side arising from all the corners of the polygon. By primary, we mean that we are excluding the effects of diffracted waves which have undergone at least one internal reflection, and head waves.

Numerically, we aim to approximate the amplitudes $V_i^{\pm}$ by low-order piecewise polynomials on overlapping meshes graded towards the corners. The amplitudes V_j^r are approximated by polynomials on a mesh with elements dictated by discontinuities in V_{go}.

4 Numerical results

The table below presents results for scattering by an equilateral triangle with $\alpha = 1$ at different size parameters ak_1, where a is the radius of the smallest circle which circumscribes the triangle. The maximum polynomial degree in the approximation space is 3. The scattered field u^s is calculated on a circle of radius $3a/2$ with its centre coinciding with that of the triangle. We also calculate the Kirchhoff approximation (KA) on this circle which is obtained by replacing u_1 and $\partial u_1/\partial \mathbf{n}$ in Green's representation formula by their GO approximations on $\partial\Omega$. The errors shown are relative errors calculated in the L^2-norm and the reference solutions are obtained using a standard BEM at high resolution (to ensure an accuracy of $\sim 1e-6$ relative error on Γ).

ak_1	$u_1\|_{\partial\Omega}$ error	u^s error	u_{KA} error	# DOF per λ
20	4.1e-3	2.9e-4	6.2e-2	5.5
40	2.5e-3	1.7e-3	4.6e-2	2.8
80	7.9e-4	4.0e-4	3.2e-2	1.4
160	3.0e-3	1.4e-3	2.3e-2	6.9e-1
320	1.2e-3	5.0e-4	1.64e-2	3.4e-1

We observe that an error of ~0.4% or less is maintained in both the field on the boundary $u_1|_{\partial\Omega}$ and the scattered field u^s as the size parameter is increased. The number of degrees of freedom (DOF) in the approximation space was fixed at 138 for all ak_1. We note that even with this relatively small number of DOF (corresponding to only 0.34 DOF per wavelength in the case $ak_1 = 320$) we gain a significant improvement over the KA, which is a widely employed asymptotic approach. Moreover, the error in the HNA method appears to be bounded independently of the size parameter.

References

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