Function spaces for integral equations on fractal domains

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Abstract

We report some results arising from our investigations into boundary integral equation formulations of acoustic scattering problems involving planar screens with fractal boundaries. Our focus is on determining the correct Sobolev space setting in which to pose the integral equations. This is well understood when the screen is smooth (Lipschitz). But for non-Lipschitz (e.g. fractal) screens the situation is less clear, because many of the equivalences and relations between the standard definitions of Sobolev spaces on subsets of Euclidean space (e.g. restriction, completion of spaces of smooth functions, interpolation...) that hold in the Lipschitz case, fail to hold in general. We point to concrete counterexamples for which the standard equivalences fail, as well as discussing the implications of this failure for the well-posedness (or otherwise) of the classical screen scattering problem.

Keywords: Integral equations, Sobolev spaces, Screen problems, Non-Lipschitz domains

1 Motivation

This paper concerns properties of Sobolev spaces relevant to the study of integral equations on non-Lipschitz domains. Our motivating example is time-harmonic acoustic scattering in $\mathbb{R}^{d+1}$ by a planar screen $\Gamma \times \{0\}$, where Γ is a non-empty bounded open subset of $\mathbb{R}^d$, $d = 1$ or 2. Such problems could represent simplified models for the performance of fractal antennas in electrical engineering applications [1].

The Sobolev spaces we study are derived from the Bessel potential spaces $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$. Following the notation of [2], let $H^s(\Gamma) := \{U|_\Gamma : U \in H^s(\mathbb{R}^d)\}$, where $|_\Omega$ denotes the (distributional) restriction to Γ. Let $\tilde{H}^s(\Gamma)$ denote the closure of $C_0^\infty(\Gamma)$ in $H^s(\mathbb{R}^d)$; we note that $H^s(\Gamma)$ is the dual space of $\tilde{H}^{-s}(\Gamma)$. For compact $K \subset \mathbb{R}^d$ let $H^s_K := \{u \in H^s(\mathbb{R}^d) : \operatorname{supp} u \subset K\}$.

For the screen scattering problem with Neumann boundary conditions, the scattered wave satisfies the following boundary value problem: Given $g_{\mathsf{N}} \in H^{-1/2}(\Gamma)$ (arising from the incident wave), find $u \in W^1_{\mathrm{loc}}(\mathbb{R}^{d+1} \setminus (\overline{\Gamma} \times \{0\}))$ such that $\Delta u + k^2 u = 0$ in $\mathbb{R}^{d+1} \setminus (\overline{\Gamma} \times \{0\})$, u is outgoing at infinity, and $\partial u/\partial \mathbf{n} = g_{\mathsf{N}}$ on Γ. (The latter condition can be written more precisely as $(\partial^\pm_{\mathbf{n}}(\chi u))|_\Gamma = g_{\mathsf{N}}$, where $\partial^+_{\mathbf{n}}$ and $\partial^-_{\mathbf{n}}$ are Neumann trace operators onto $\mathbb{R}^d \times \{0\}$ from the half spaces $x_{d+1} > 0$ and $x_{d+1} < 0$ respectively, and $\chi \in \mathbb{C}_0^\infty(\mathbb{R}^d)$ is any cut-off function which equals one in a neighbourhood of $\Gamma \times \{0\}$.)

This problem is well-posed whenever Γ is a Lipschitz subset of $\mathbb{R}^d$. In [3] it is shown that to ensure well-posedness (specifically, uniqueness) for arbitrary Γ one has to impose the following two additional conditions (with χ as above, and $\gamma^\pm$ denoting Dirichlet traces):

$$[\partial u/\partial \mathbf{n}] := \partial^+_{\mathbf{n}}(\chi u) - \partial^-_{\mathbf{n}}(\chi u) = 0, \quad (1)$$

$$[u] := \gamma^+(\chi u) - \gamma^-(\chi u) \in \tilde{H}^{1/2}(\Gamma). \quad (2)$$

Condition (1) ensures that u can be represented as a double layer potential $u = \mathcal{D}[u]$ (with no single layer potential component). A priori (from the Helmholtz equation and boundary condition) $[\partial u/\partial \mathbf{n}] \in H^{-1/2}_{\partial\Gamma}$, so (1) is required whenever Γ is rough enough that $H^{-1/2}_{\partial\Gamma} \neq \{0\}$.

Condition (2) ensures that the resulting first-kind boundary integral equation on Γ, involving the hypersingular operator, has a unique solution: as shown in [3], this operator is invertible between $\tilde{H}^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. A priori, $[u] \in H^{1/2}_{\overline{\Gamma}} \supset \tilde{H}^{1/2}(\Gamma)$, so (2) is required whenever Γ is rough enough that $\tilde{H}^{1/2}(\Gamma) \subsetneq H^{1/2}_{\overline{\Gamma}}$.

We want to understand the how the geometry of Γ affects whether or not these additional conditions are required. We might also ask the question: Given two screens Γ_1, Γ_2, under what conditions are the solutions u_1, u_2 for the respective scattering problems equal? It turns out that, under appropriate assumptions on the form of the incident wave, this holds for every incident wave if and only if $\tilde{H}^{1/2}(\Gamma_1) = \tilde{H}^{1/2}(\Gamma_2)$.

2 Function space results

Motivated by the above considerations, we pose the following general questions, with $s \in \mathbb{R}$.

Q1: When is $H^s_K = \{0\}$ for a compact set K?

Q2: When is $\tilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ for an open set Γ?

Q3: When is $\tilde{H}^s(\Gamma_1) = \tilde{H}^s(\Gamma_2)$ for $\Gamma_1 \neq \Gamma_2$?

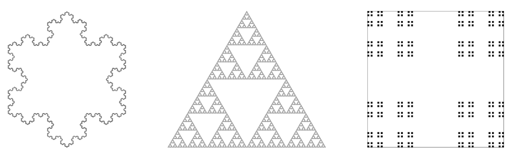
Q1 concerns the "negligibility" of the set K in terms of Sobolev regularity. It is straightforward to show that for every non-empty compact K there exists $s_K \in [-d/2, d/2]$ such that $H^s_K = \{0\}$ for $s > s_K$ and $H^s_K \neq \{0\}$ for $s < s_K$.

If K has zero Lesbegue measure then s_K can be expressed in terms of Hausdorff dimension. (Some partial results for sets with positive measure will be reported in the talk.)

Theorem 2.1 *Let K be non-empty, compact, and have zero Lesbegue measure. Then* $s_K = (\dim_{\mathrm{H}}(K) - d)/2$.

Theorem 2.2 *(i) If Γ is C^0 then $s_{\partial\Gamma} \in [-1/2, 0]$, and furthermore $H^0_{\partial\Gamma} = L^2(\partial\Gamma) = \{0\}$. (ii) If Γ is $C^{0,\alpha}$ for some $0 < \alpha < 1$ then $s_{\partial\Gamma} \in [-1/2, -\alpha/2]$. (iii) If Γ is Lipschitz then $s_{\partial\Gamma} = -1/2$, and furthermore $H^{-1/2}_{\partial\Gamma} = \{0\}$.*

These theorems provide open sets Γ for which $H^{-1/2}_{\partial\Gamma} \neq \{0\}$; for $d = 2$ this will hold whenever $\dim_{\mathrm{H}}(\partial\Gamma) > 1$. For instance, one can take Γ to be the interior of the Koch snowflake ($\dim_{\mathrm{H}}(\partial\Gamma) = \log 4/\log 3$), or the open set formed by removing from a unit equilateral triangle the points of the Sierpinski triangle ($\dim_{\mathrm{H}}(\partial\Gamma) = \log 3/\log 2$). A further example is $\Gamma = ((0,1) \setminus C_\lambda)^2$, for $1/4 < \lambda < 1/2$. Here $C_\lambda := \cap_{n=0}^{\infty} C_{\lambda,n}$ is the Cantor set where $C_{\lambda,0} = [0,1]$ and, for $n > 0$, $C_{\lambda,n}$ is formed by removing an open interval from the middle of each interval in $C_{\lambda,n-1}$ to leave two subintervals of length λ^n (see figure below).



Concerning **Q2**, it is well-known (see e.g. [2]) that if Γ is C^0 then $\tilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ for all $s \in \mathbb{R}$. However, this equality fails in general. The following result relates **Q2** to **Q1** and provides a way of constructing counterexamples.

Theorem 2.3 *If $\exists$ a compact set $K \subset \mathrm{int}(\overline{\Gamma})\backslash\Gamma$ for which $H^{-s}_K \neq \{0\}$, then $\tilde{H}^s(\Gamma) \subsetneq H^s_{\overline{\Gamma}}$.*

Q3 is also intimately related to **Q1**.

Theorem 2.4 *For $\Gamma_1 \neq \Gamma_2$, $\tilde{H}^s(\Gamma_1) = \tilde{H}^s(\Gamma_2)$ iff $H^{-s}_K = \{0\}$ for every compact $K \subset \Gamma_1 \ominus \Gamma_2$.*

We end with a remark on interpolation. For Lipschitz Γ, both $H^s(\Gamma)$ and $\tilde{H}^s(\Gamma)$ are interpolation scales over $s \in \mathbb{R}$. But for general open Γ this can fail - see the counterexamples in [4]. It would seem that interpolation is a somewhat unstable way of defining spaces on rough domains.

References

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