A high frequency boundary element method for scattering by two-dimensional screens

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Abstract

We propose a numerical-asymptotic boundary element method for problems of time-harmonic acoustic scattering of an incident plane wave by a sound-soft two-dimensional (2D) screen. Standard numerical schemes have a computational cost that grows at least linearly with respect to the frequency of the incident wave. Here, we enrich our approximation space with oscillatory basis functions carefully designed to capture the high frequency behaviour of the solution. We show that in order to achieve any desired accuracy it is sufficient to increase the number of degrees of freedom only in proportion to the logarithm of the frequency, as the frequency increases, and for fixed frequency we demonstrate exponential convergence with respect to the number of degrees of freedom.

Introduction

There has been much recent interest (see, e.g., [1]) in the development of numerical-asymptotic boundary element methods for time-harmonic scattering problems. In these methods, knowledge of the high frequency asymptotic behavior of the solution is incorporated into the approximation space, leading to improved performance at high frequencies and, in many cases, rigorous error estimates demonstrating sublinear (often logarithmic) growth in the number of degrees of freedom required to maintain accuracy as frequency increases. Here, we apply this idea to the problem of scattering by a 2D screen. This represents the first application of this approach (supported by error estimates) to any problem of scattering by separated multiple scatterers (in this case the separate components of the screen).

1 Problem statement

We consider the 2D problem of scattering of the time harmonic incident plane wave $u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$, where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $k > 0$ is the wavenumber and $\mathbf{d}$ is a unit direction vector, by a sound soft screen $\Gamma := \{(x_1, 0) \in \mathbb{R}^2 : x_1 \in \tilde{\Gamma}\}$. Here $\tilde{\Gamma} \subset \mathbb{R}$ is a union of disjoint open intervals, i.e. $\tilde{\Gamma} = \cup_{j=1}^{n_i}(s_{2j-1}, s_{2j})$, where $0 = s_1 < \ldots < s_{2n_i} = L$, with n_i denoting the number of intervals making up $\tilde{\Gamma}$ and L being the length of the screen in the case $n_i = 1$. We denote the propagation domain by $D := \mathbb{R}^2\backslash\bar{\Gamma}$, where $\bar{\Gamma}$ is the closure of Γ.

The boundary value problem (BVP) we wish to solve is: given the incident field u^i, determine the total field $u \in C^2(D) \cap H^1_{loc}(D)$ such that

$$\Delta u + k^2 u = 0 \text{ in } D, \quad u = 0 \text{ on } \Gamma,$$

and the scattered field $u^s := u - u^i$ satisfies the Sommerfeld radiation condition. The precise sense in which $u = 0$ holds on Γ is explained in [2].

For the solution of the above BVP, a form of Green's representation theorem holds:

$$u(\mathbf{x}) = u^i(\mathbf{x}) + \int_\Gamma \Phi_k(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial \mathbf{n}}\right](\mathbf{y})\, \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D,$$

where $\Phi_k(\mathbf{x}, \mathbf{y}) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)$ is the fundamental solution of the Helmholtz equation and $\left[\frac{\partial u}{\partial \mathbf{n}}\right]$ is the jump in the normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ across Γ. It is shown in [2] that $\phi = \left[\frac{\partial u}{\partial \mathbf{n}}\right]$ satisfies the boundary integral equation

$$S_k\phi(\mathbf{x}) = u^i(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{1}$$

where $S_k\phi(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, \mathrm{d}s(\mathbf{y})$, $\mathbf{x} \in \Gamma$.

2 Analyticity and regularity of solutions

Our approximation space for the solution of (1) is adapted to the high frequency asymptotic behaviour of the solution, which we now consider. Representing $\mathbf{x} \in \Gamma$ parametrically by $\mathbf{x}(s) := (s, 0)$, where $s \in \tilde{\Gamma} \subset (0, L)$, the following theorem is proved in [2]:

Theorem 2.1 *Let $k \geq k_0 > 0$. Then for any $j = 1, \ldots, n_i$ there exists a constant $C > 0$, dependent only on k_0 and $\min_{m=1,\ldots,2n_i-1}\{s_{m+1} - s_m\}$, such that*

$$\phi(\mathbf{x}(s)) = \Psi(\mathbf{x}(s)) + v_j^+(s - s_{2j-1})\mathrm{e}^{\mathrm{i}ks} + v_j^-(s_{2j} - s)\,\mathrm{e}^{-\mathrm{i}ks}, \tag{2}$$

for $s \in (s_{2j-1}, s_{2j})$, where $\Psi := 2\partial u^i/\partial\mathbf{n}$, and the functions $v_j^\pm(s)$ are analytic in the right half-plane $\mathrm{Re}[s] > 0$, where they satisfy the bound

$$\left|v_j^\pm(s)\right| \leq C(1 + kL)k\,|ks|^{-\frac{1}{2}}.$$

3 Approximation space

Using the representation (2) we can now design an appropriate approximation space $V_{N,k}$ to represent

$$\varphi(s) := \frac{1}{k}\left(\left[\frac{\partial u}{\partial \mathbf{n}}\right](\mathbf{x}(s)) - \Psi(\mathbf{x}(s))\right), \quad s \in \tilde{\Gamma}.$$

The function φ, which we seek to approximate, can be thought of as the scaled difference between $[\partial u/\partial \mathbf{n}]$ and its "Physical Optics" approximation Ψ, with the $1/k$ scaling ensuring that φ is nondimensional. As alluded to earlier, instead of approximating φ directly by conventional piecewise polynomials we instead use the representation (2) with v_j^+ and v_j^- replaced by piecewise polynomials (of order p) supported on overlapping geometric meshes (each with n layers) on each interval (s_{2j-1}, s_{2j}), graded towards the singularities at $s = s_{2j-1}$ and $s = s_{2j}$ respectively. This leads to an identical approximation space on each interval to that used on each side of a convex polygon in [3]. For full details we refer to [2], where the following best approximation result is shown.

Theorem 3.1 *Let n and p satisfy $n \geq cp$ for some constant $c > 0$ and suppose that $k \geq k_0 > 0$. Then there exist constants $C, \tau > 0$, dependent only on n_i, k_0, c and $\min_{m=1,\ldots,2n_i-1}\{s_{m+1} - s_m\}$, such that*

$$\inf_{v \in V_{N,k}} \|\varphi - v\|_{\tilde{H}_k^{-\frac{1}{2}}(\tilde{\Gamma})} \leq C k^{1/2} L^{3/2} \mathrm{e}^{-p\tau}.$$

Whereas our estimates for classes of polygons (see, e.g., [1], [3]) hold in L^2, here we need to work in appropriate Sobolev spaces. For a precise definition of the k-dependent norm $\|\cdot\|_{\tilde{H}_k^{-\frac{1}{2}}(\tilde{\Gamma})}$, we refer to [2].

4 Galerkin method

Having designed an appropriate approximation space $V_{N,k}$, we use a Galerkin method to select an element so as to efficiently approximate φ. That is, we seek $\varphi_N \in V_{N,k}$ such that

$$\langle S_k \varphi_N, v \rangle_\Gamma = \frac{1}{k}\left\langle u^i - S_k \Psi, v \right\rangle_\Gamma, \quad \forall v \in V_{N,k}, \quad (3)$$

where the duality pairings in (3) can be evaluated simply as L^2 inner products. The following error estimate is proved in [2].

Theorem 4.1 *If the assumptions of Theorem 3.1 hold, then there exist constants $C, \tau > 0$, dependent only on n_i, k_0, c and $\min_{m=1,\ldots,2n_i-1}\{s_{m+1} - s_m\}$, such that*

$$\|\varphi - \varphi_N\|_{\tilde{H}_k^{-1/2}(\tilde{\Gamma})} \leq C k L^2 \mathrm{e}^{-p\tau}.$$

5 Numerical results

We now present numerical results for the solution of (3). We take $\mathbf{d} = (1/\sqrt{2}, -1/\sqrt{2})$, $n_i = 1$, and $L = 2\pi$. We take the same number of layers $n = 2(p+1)$ on each graded mesh, giving a total number of degrees of freedom $N = 4(p+1)^2$. Since N depends only on p, we write $\psi_p(s) := \varphi_N(s)$. In Figure 1 we plot on a logarithmic scale the relative L^1 errors $\|\psi_6 - \psi_p\|_{L^1(\tilde{\Gamma})} / \|\psi_6\|_{L^1(\tilde{\Gamma})}$ (used for simplicity of computation) against p for a range of k. The linear plots demonstrate exponential decay as

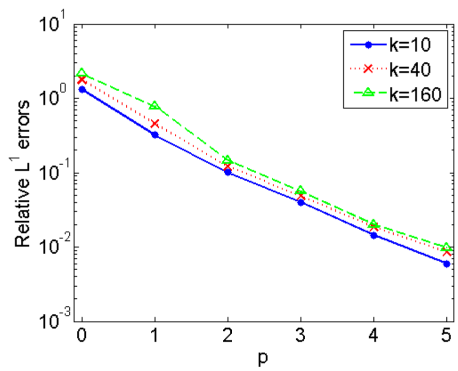

Figure 1: Convergence results

the polynomial degree, p, increases, as we might expect from Theorem 4.1. For fixed p, the relative error increases only very slowly as k increases. For further numerical results, see [2], [4].

References

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