WAVENUMBER-EXPLICIT COERCIVITY ESTIMATES IN SCATTERING BY SCREENS

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Abstract

We consider time-harmonic acoustic scattering by flat sound soft and sound hard screens occupying an arbitrary bounded open set in the plane. We propose mathematical models for such problems, and show that these are well-posed, by proving the coercivity of the single-layer and hypersingular integral operators arising in the boundary integral equation reformulations of the problems. We also tease out the explicit wavenumber dependence of the norms and coercivity constants of these integral operators, this in part extending previous results of Ha Duong.

Introduction

This paper is concerned with the mathematical analysis of classical time-harmonic acoustic scattering problems, modelled by the Helmholtz equation

$$\Delta u + k^2 u = 0, \tag{1}$$

where $k > 0$ is the *wavenumber*. The scatterer is assumed to be a thin flat screen, occupying some bounded and relatively open set $\Gamma \subset \Gamma_\infty := \{x = (x_1, ..., x_d) \in \mathbb{R}^d : x_d = 0\}$ ($d = 2$ or 3), with (1) assumed to hold in $D := \mathbb{R}^d \setminus \overline{\Gamma}$. We suppose the screen is *sound soft*, in which case $u=0$ on Γ, or *sound hard*, when the normal derivative $\partial u/\partial \mathbf{n} = 0$ on Γ.

This is a well-studied problem, both theoretically and in applications. However, all previous studies assume that $\Gamma \subset \Gamma_\infty$ is at least a Lipschitz relatively open set (in the sense of [2]), and most that Γ is substantially smoother. The focus of the present paper is: (i) to formulate these problems correctly when Γ is an arbitrary bounded relatively open set; (ii) to get wavenumber-explicit estimates on the associated boundary integral operators. For full details see [1].

1 Preliminaries

Our analysis is in the context of Sobolev spaces, for which we follow the notation in [2], except we use wavenumber dependent norms (equivalent to the usual norms). Explicitly, on the Bessel potential space $H^s(\mathbb{R}^{d-1})$, $s \in \mathbb{R}$, we define

$$\|u\|^2_{H^s_k(\mathbb{R}^{d-1})} := \int_{\mathbb{R}^{d-1}} (k^2 + |\boldsymbol{\xi}|^2)^s \, |\hat{u}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi},$$

where $\hat{}$ represents the Fourier transform in $\mathbb{R}^{d-1}$.

Let $H^s(\Gamma) := \{U|_\Gamma : U \in H^s(\mathbb{R}^{d-1})\}$, where $|_\Omega$ denotes the restriction to Γ, and let $\tilde{H}^s(\Gamma)$ denote the closure of $C_0^\infty(\Gamma)$ in the space $H^s(\mathbb{R}^{d-1})$. Then $H^s(\Gamma)$ is the dual space of $\tilde{H}^{-s}(\Gamma)$; we denote by $\langle \cdot, \cdot \rangle_{\Gamma,s}$ the duality pairing on $H^s(\Gamma) \times \tilde{H}^{-s}(\Gamma)$. Let $H^s_{\overline{\Gamma}} := \{u \in H^s(\mathbb{R}^{d-1}) : \operatorname{supp} u \subset \overline{\Gamma}\}$. Clearly $\tilde{H}^s(\Gamma) \subset H^s_{\overline{\Gamma}}$, and when Γ is C^0 (so certainly if Γ is Lipschitz), it holds that $\tilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ [2, Theorem 3.29], but in general these spaces are not equal.

2 Boundary value problems

Definition 2.1 (Problem D). *Given $g_\mathsf{D} \in H^{1/2}(\Gamma)$, find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that*

$$\Delta u + k^2 u = 0, \quad \text{in } D, \tag{2a}$$

$$u = g_\mathsf{D}, \quad \text{on } \Gamma, \tag{2b}$$

$$[u] = 0, \tag{2c}$$

$$[\partial u/\partial \mathbf{n}] \in \tilde{H}^{-1/2}(\Gamma), \tag{2d}$$

and u satisfies the Sommerfeld radiation condition. Here $[f] = f^+ - f^-$ represents the jump of f across Γ (interpreted in the sense of traces).

Conditions (2c)-(2d) ensure well-posedness for an arbitrary relatively open subset Γ. In interpreting (2c)-(2d) we remark that, a priori, $[u] \in H^{1/2}_{\overline{\Gamma}}$ and $[\partial u/\partial \mathbf{n}] \in H^{-1/2}_{\overline{\Gamma}}$, and, while $[u]|_\Gamma = 0$ (from (2b)), it could hold that $[u] \neq 0$ with $\mathrm{supp}([u]) \subset \partial\Gamma$. We note that (2c)-(2d) are automatically satisfied when Γ is Lipschitz, because then $\partial\Gamma$ cannot support non-zero elements of $H^{1/2}(\mathbb{R}^{d-1})$, and also $\tilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$.

Definition 2.2 (Problem N). *Given $g_\mathsf{N} \in H^{-1/2}(\Gamma)$, find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that*

$$\Delta u + k^2 u = 0, \quad \text{in } D, \tag{3a}$$

$$\frac{\partial u}{\partial \mathbf{n}} = g_\mathsf{N}, \quad \text{on } \Gamma, \tag{3b}$$

$$[\partial u/\partial \mathbf{n}] = 0, \tag{3c}$$

$$[u] \in \tilde{H}^{1/2}(\Gamma), \tag{3d}$$

and u satisfies the Sommerfeld radiation condition.

Again, (3c)-(3d) automatically hold if Γ is Lipschitz.

Example 2.3. *In the scattering by Γ of an incident plane wave $u^i(\mathbf{x}) := \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$, $\mathbf{x} \in \mathbb{R}^d$, where $\mathbf{d} \in \mathbb{R}^d$ is a unit direction vector, a 'sound soft' and a 'sound hard' screen are modelled respectively by problems* D *(with $g_\mathsf{D} = -u^i|_\Gamma$) and* N *(with $g_\mathsf{N} = -\partial u^i/\partial\mathbf{n}|_\Gamma$), with u representing the scattered field.*

3 Boundary integral equations

We introduce the standard single- and double-layer potentials $\mathcal{S}_k : \tilde{H}^{-1/2}(\Gamma) \to C^2(D) \cap W^1_{loc}(D)$ and $\mathcal{D}_k : \tilde{H}^{1/2}(\Gamma) \to C^2(D) \cap W^1_{loc}(D)$, and single-layer and hypersingular operators $S_k : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $T_k : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. For $\phi \in C_0^\infty(\Gamma)$ the latter two have the integral representations

$$S_k\phi(\mathbf{x}) = \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad \mathbf{x} \in \Gamma,$$

$$T_k\phi(\mathbf{x}) = \frac{\partial}{\partial\mathbf{n}(\mathbf{x})}\int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})}\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Gamma,$$

where Φ is the fundamental solution of (1). Problems D and N are equivalent to certain integral equations involving S_k and T_k, as the following theorems show.

Theorem 3.1. *Suppose that u is a solution of problem* D*. Then Green's representation formula*

$$u(\mathbf{x}) = -\mathcal{S}_k\left[\partial u/\partial\mathbf{n}\right](\mathbf{x}), \qquad \mathbf{x} \in D,$$

holds, and $\phi := [\partial u/\partial\mathbf{n}] \in \tilde{H}^{-1/2}(\Gamma)$ satisfies

$$-S_k\phi = g_\mathsf{D}. \tag{4}$$

Conversely, suppose that $\phi \in \tilde{H}^{-1/2}(\Gamma)$ satisfies (4). Then $u := -\mathcal{S}_k\phi$ satisfies problem D*, and $[\partial u/\partial\mathbf{n}] = \phi$.*

Theorem 3.2. *Suppose that u is a solution of problem* N*. Then Green's representation formula*

$$u(\mathbf{x}) = \mathcal{D}_k[u](\mathbf{x}), \qquad \mathbf{x} \in D,$$

holds, and $\psi := [u] \in \tilde{H}^{1/2}(\Gamma)$ satisfies

$$T_k\psi = g_\mathsf{N}. \tag{5}$$

Conversely, suppose that $\psi \in \tilde{H}^{1/2}(\Gamma)$ satisfies (5). Then $u := \mathcal{D}_k\psi$ satisfies problem N*, and $[u] = \psi$.*

Because the screen is flat we have Fourier representations for S_k and T_k. For $\phi \in C_0^\infty(\Gamma)$ and $\tilde{\mathbf{x}} \in \tilde{\Gamma} := \{\tilde{\mathbf{x}} \in \mathbb{R}^{d-1} : (\tilde{\mathbf{x}}, 0) \in \Gamma\} \subset \mathbb{R}^{d-1}$,

$$S_k\phi(\tilde{\mathbf{x}}, 0) = \frac{\mathrm{i}}{2(2\pi)^{(d-1)/2}}\int_{\mathbb{R}^{d-1}} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\tilde{\mathbf{x}}}}{Z(\boldsymbol{\xi})}\widehat{\phi}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},$$

$$T_k\phi(\tilde{\mathbf{x}}, 0) = \frac{\mathrm{i}}{2(2\pi)^{(d-1)/2}}\int_{\mathbb{R}^{d-1}} Z(\boldsymbol{\xi})\mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\tilde{\mathbf{x}}}\widehat{\phi}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},$$

where

$$Z(\boldsymbol{\xi}) := \begin{cases} \sqrt{k^2 - |\boldsymbol{\xi}|^2}, & |\boldsymbol{\xi}| \le k \\ \mathrm{i}\sqrt{|\boldsymbol{\xi}|^2 - k^2}, & |\boldsymbol{\xi}| > k, \end{cases} \qquad \boldsymbol{\xi} \in \mathbb{R}^{d-1}.$$

These representations allow us to prove the following k-explicit continuity and coercivity estimates, which improve on those in [3], [4].

Theorem 3.3. *For any $s \in \mathbb{R}$, the single-layer operator $S_k : \tilde{H}^s(\Gamma) \to H^{s+1}(\Gamma)$ is bounded, and $\exists C > 0$, independent of k and Γ, such that, for all $0 \ne \phi \in \tilde{H}^s(\tilde{\Gamma})$ and $k > 0$, and with $A := \operatorname{diam}\Gamma$,*

$$\frac{\|S_k\phi\|_{H_k^{s+1}(\Gamma)}}{\|\phi\|_{\tilde{H}_k^s(\Gamma)}} \le \begin{cases} C(1 + \sqrt{kA}), & d = 3, \\ C\log\left(2 + (kA)^{-1}\right)(1 + \sqrt{kA}), & d = 2. \end{cases}$$

Theorem 3.4. *$S_k : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ satisfies*

$$|\langle S_k\phi, \phi\rangle_{\Gamma,1/2}| \ge \frac{1}{2\sqrt{2}}\|\phi\|^2_{\tilde{H}_k^{-1/2}(\Gamma)}, \quad \phi \in \tilde{H}^{-1/2}(\Gamma),\ k > 0.$$

Theorem 3.5. *For any $s \in \mathbb{R}$, the hypersingular operator $T_k : \tilde{H}^s(\Gamma) \to H^{s-1}(\Gamma)$ is bounded, and*

$$\|T_k\phi\|_{H_k^{s-1}(\Gamma)} \le \frac{1}{2}\|\phi\|_{\tilde{H}_k^s(\Gamma)}, \quad \phi \in \tilde{H}^s(\Gamma),\ k > 0.$$

Theorem 3.6. *$T_k : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ satisfies, for any $k_0 > 0$, and $k \ge k_0$,*

$$|\langle T_k\phi, \phi\rangle_{\Gamma,-1/2}| \ge C(kA)^\beta\|\phi\|^2_{\tilde{H}_k^{1/2}(\Gamma)}, \quad \phi \in \tilde{H}^{1/2}(\Gamma),$$

where $C > 0$ is a constant depending only on k_0A, and $\beta = -2/3$ for $d = 3$ and $\beta = -1/2$ for $d = 2$.

The Lax-Milgram Lemma then implies:

Theorem 3.7. *Equation (4), and hence also problem* D*, has a unique solution for all $g_\mathsf{D} \in H^{1/2}(\Gamma)$.*

Theorem 3.8. *Equation (5), and hence also problem* N*, has a unique solution for all $g_\mathsf{N} \in H^{-1/2}(\Gamma)$.*

References

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