# Hybrid numerical-asymptotic approximation of high frequency scattering by penetrable convex polygons

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## Abstract

We consider time-harmonic scattering by penetrable convex polygons. Standard numerical methods for such problems become prohibitively expensive in the high frequency regime. High frequency asymptotic methods, on the other hand, are non-convergent and may be insufficiently accurate at low to medium frequencies. Here, we describe a beam tracing algorithm that calculates the leading order high frequency asymptotics and we present an ansatz for the oscillatory remaining terms which represent the diffracted field. We demonstrate that including oscillatory basis functions in the approximation space enables an accurate approximation of the solution on the boundary of the scatterer with a cost independent of frequency.

## Introduction

We consider the two-dimensional problem of scattering of a time-harmonic wave by a penetrable convex polygon, $\Omega$. We wish to determine the total field $u_1$ in the exterior domain $D$ and the total field $u_2$ within the polygon such that

$$\Delta u_1 + k_1^2 u_1 = 0, \quad \text{in } D, \tag{1}$$

$$\Delta u_2 + k_2^2 u_2 = 0, \quad \text{in } \Omega, \tag{2}$$

$$u_1 = u_2 \quad \text{and} \quad \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \quad \text{on } \partial\Omega, \tag{3}$$

where $k_1, k_2$ are the exterior and interior wavenumbers, respectively. An example solution for $\Omega$ an equilateral triangle is shown in Figure 1.

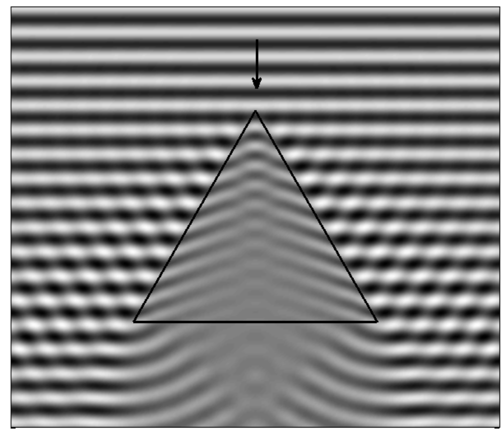


Figure 1: Scattering by a highly absorbing penetrable equilateral triangle.

This problem arises in numerous areas of physical interest in which the relative size of the particle to the wavelength can vary between one and thousands. Conventional numerical methods using piecewise polynomial approximation spaces suffer from the limitation that a fixed number of degrees of freedom is required per wavelength in order to represent the oscillatory solution. This leads to prohibitive computational expense when the size of the scatterer is large relative to the wavelength.

Much work has been done on developing Hybrid Numerical-Asymptotic (HNA) methods (see [1] and the references therein) which overcome this limitation by approximating the solution, $u$, in a Boundary Element Method (BEM) framework using an ansatz of the form

$$u(x,k) \approx u_{go}(x,k) + \sum_{m=1}^{M} v_m(x,k)\exp(\mathrm{i}k\psi_m(x)), \; x \in \partial\Omega. \tag{4}$$

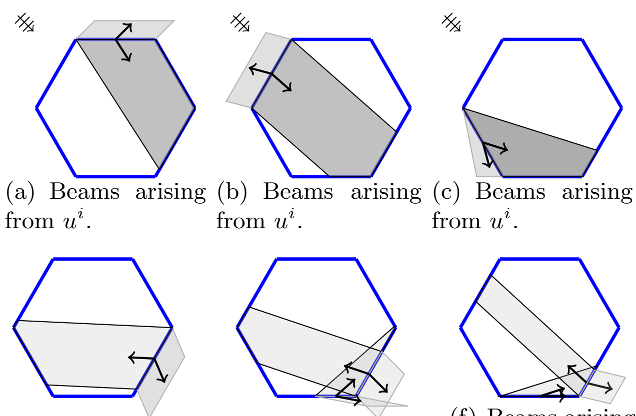
In this representation, $u_{go}$ is the known leading order high frequency asymptotics, namely the Geometrical Optics (GO), the phases $\psi_m$ are chosen *a priori* using knowledge of the high frequency asymptotics and the amplitudes $v_m$ are approximated numerically. The expectation is that if $u_{go}$ is calculated correctly and $\psi_m$ are chosen wisely, the amplitudes $v_m$ will be much less oscillatory than $u$ and so can be efficiently approximated by piecewise polynomials.

To date, the HNA approach has been applied solely to problems of scattering by *impenetrable* scatterers. The main difficulty in the generalisation of the HNA method to the penetrable case is that the high frequency asymptotic behaviour is much more complicated than in the impenetrable case. In particular, the diffracted waves are reflected infinitely many times within the scatterer, so there are infinitely many phases $\psi_m$ in (4). This complicates the development of our ansatz because, to create a viable method, we must choose a finite number of these phases.

In this paper, we briefly describe a Beam Tracing Algorithm (BTA) for determining $u_{go}$ and then go on to make a sensible choice of $\psi_m$ in the ansatz (4). This involves truncating the infinite series of diffracted terms. To do this, we begin by examining highly absorbing scatterers for which relatively few terms are required and then investigate how to include additional terms as the absorption is reduced.

## Beam Tracing Algorithm

The GO term, $u_{go}$, in (4) is calculated using a BTA. Consider the hexagon in Figure 2 illuminated from the top left by a plane wave $u^i$. This wave strikes 3 sides, from each side part of the wave is reflected and part is transmitted into the hexagon, as depicted in Figures 2a, 2b, 2c, obeying Snell's laws of reflection and refraction and the Fresnel formulae. The transmitted portions or 'beams' go on to strike further interfaces giving rise to more beams as shown in Figures 2d, 2e, 2f. This process continues indefintely, however we terminate it when the amplitudes or the beams reflected back into the shape are smaller than a user defined tolerance.



(a) Beams arising from $u^i$. (b) Beams arising from $u^i$. (c) Beams arising from $u^i$.

(d) Beams arising from transmitted beam in (a). (e) Beams arising from transmitted beam in (b). (f) Beams arising from transmitted beam in (c).

Figure 2: Beam tracing in a hexagon

## Ansatz for highly absorbing scatterers

For large absorption we anticipate that the influence of diffraction on each side is only due to adjacent corners, so a sensible ansatz for the solution on one side is

$$u \approx u_{go} + v_1^+ \mathrm{e}^{\mathrm{i}k_1 s} + v_2^+ \mathrm{e}^{\mathrm{i}k_2 s} + v_1^- \mathrm{e}^{-\mathrm{i}k_1 s} + v_2^- \mathrm{e}^{-\mathrm{i}k_2 s}, \quad (5)$$

where $v_1^+, v_2^+, v_1^-, v_2^-$ are slowly varying amplitudes to be approximated using piecewise polynomials on overlapping graded meshes as shown in Figure 3. In

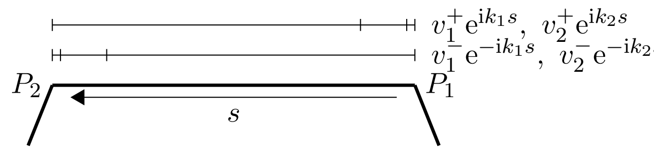



Figure 3: Approximate $v_1^+$, $v_2^+$, $v_1^-$, $v_2^-$ by piecewise polynomials on overlapping meshes, graded towards the corners.

order to demonstrate the suitability of this ansatz, we perform a least squares fit of (5) to a reference solution, $u$, obtained using a standard BEM. This is done at varying frequencies and the error examined. Table 1 shows how the error in the best fit $U$ compares to the error in using the GO alone. We see a significant improvement over GO using a small, fixed number (168) of degrees of freedom.

| $k$ | $\frac{\lVert u-u_{go}\rVert}{\lVert u\rVert}$ | $\frac{\lVert u-U\rVert}{\lVert u\rVert}$ |
|---|---|---|
| 5 | $1.88 \times 10^{-1}$ | $1.66 \times 10^{-2}$ |
| 10 | $1.37 \times 10^{-1}$ | $1.03 \times 10^{-2}$ |
| 20 | $1.00 \times 10^{-1}$ | $8.41 \times 10^{-4}$ |
| 40 | $7.25 \times 10^{-2}$ | $2.23 \times 10^{-4}$ |
| 80 | $5.19 \times 10^{-2}$ | $2.58 \times 10^{-4}$ |
| 160 | $3.69 \times 10^{-2}$ | $2.31 \times 10^{-4}$ |

Table 1: Best fit and GO errors for a highly absorbing triangle.

## Reducing absorption

Reducing the scatterer's absorption causes the influence of diffraction from non-adjacent corners to become significant so we add terms of the form $\mathrm{e}^{\mathrm{i}k_1 r_j}$ to the ansatz (5), where $r_j$ is the distance from the non-adjacent corner $P_j$. With these additions, we show that an accuracy similar to that in Table 1 can be achieved for absorptions down to 0.0125i. In fact, in the far-field, better than 1% accuracy can be achieved for all levels of absorption.

## References


[1] S. N. Chandler-Wilde, *Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering*, Acta Numerica (2012), pp. 89–305.