A HIGH FREQUENCY BEM FOR SCATTERING BY NON-CONVEX OBSTACLES

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Abstract

Traditional numerical methods for time-harmonic acoustic scattering problems become prohibitively expensive in the high-frequency regime where the scatterer is large compared to the wavelength of the incident wave. In this paper we propose and analyse a hybrid boundary element method (BEM) for a class of non-convex polygonal scatterers. In this method the approximation space is enriched with oscillatory functions which efficiently capture the high-frequency asymptotics of the solution, so that only $O(\log f)$ degrees of freedom are required as the frequency $f \to \infty$. This appears to be the first effective hybrid BEM for a class of non-convex obstacles.

Introduction

There has been great interest in recent years in the possibility of constructing boundary or finite element methods for time harmonic wave problems which incorporate the oscillatory behaviour of the wave field in the basis functions used to approximate the solution, achieving schemes which, in the high frequency limit, require a number of degrees of freedom which is negligible compared to conventional methods. This work is reviewed in [1], [2], and see [3], [4], [5]. The results to date have been for convex scatterers: exceptions are [6] in which numerics for multiple scattering by two circles are shown, [1] where an algorithm is sketched for mildly non-convex smooth obstacles and some preliminary numerics presented, and the preliminary work in [7]. In this paper, we show convincingly, by numerics and rigorous numerical analysis, that hybrid asymptotic-numeric methods can be made to work for a class of non-convex polygons as effectively as for convex polygons [8], [5].

We consider the two-dimensional problem of scattering of the time harmonic incident plane wave

$$u^i(x) = e^{ikx\cdot d}, \tag{1}$$

by a sound soft non-convex polygon. In (1) $k > 0$ is the wavenumber, $x = (x_1, x_2) \in \mathbb{R}^2$, and d is a unit direction vector. Let Ω denote the interior of the scatterer and $D := \mathbb{R}^2 \backslash \bar{\Omega}$ denote the unbounded exterior domain. The boundary value problem (BVP) we wish to solve is: given the incident field u^i, determine the total field $u \in C^2(D) \cap C(\bar{D})$ such that

$$\Delta u + k^2 u = 0 \text{ in } D, \quad u = 0 \text{ on } \Gamma := \partial\Omega,$$

and $u^s := u - u^i$ satisfies the Sommerfeld radiation condition. This BVP can be reformulated as a boundary integral equation (BIE) for $\frac{\partial u}{\partial n} \in L^2(\Gamma)$ (here n is the unit normal directed into D), taking the form

$$A\frac{\partial u}{\partial n} = f, \tag{2}$$

where $f \in L^2(\Gamma)$ and $A : L^2(\Gamma) \to L^2(\Gamma)$. In the usual combined-potential formulation $A := \frac{1}{2}\mathcal{I} + \mathcal{D}' - i\eta\mathcal{S}$, and $f = \frac{\partial u^i}{\partial n} - i\eta u^i$. Here $\mathcal{I}$ is the identity operator, $\mathcal{S}\psi(x) = \int_\Gamma \Phi(x,y)\psi(y)\,ds(y)$ is the single-layer potential, $\mathcal{D}'\psi(x) = \int_\Gamma \frac{\partial\Phi(x,y)}{\partial n(x)}\psi(y)\,ds(y)$ is the adjoint of the double-layer potential and $\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|)$. From the results in [8] for general Lipschitz domains we know that A is invertible for all k provided $\eta \in \mathbb{R}\backslash\{0\}$, with a standard choice being $\eta = k$.

Recently [9] a new formulation has been derived for the case when Ω is star-like with respect to the origin. This takes the form (2) with $A := (x \cdot n)\left(\frac{1}{2}\mathcal{I} + \mathcal{D}'\right) + x \cdot \nabla_\Gamma \mathcal{S} - i\eta\mathcal{S}$ and $f = x \cdot \nabla u^i - i\eta u^i$. Here ∇_Γ is the surface gradient operator. From [9] we know that (for Ω star-like) A is invertible for all k provided the choice $\eta = k|x| + \frac{i}{2}$ is made.

For both formulations the following lemma holds provided $|\eta| \leq Ck$. Here and for the remainder of this paper $C > 0$ is a constant independent of k, but (possibly) dependent on the geometry of Γ.

Lemma 1 *[10], [9] For all $k_0 > 0$, there exists $C > 0$ such that*

$$\|A\| \leq Ck^{\frac{1}{2}}, \qquad k \geq k_0.$$

In certain cases A also satisfies the following assumptions:

Assumption 1 *There exist constants $C > 0$, $\beta \geq 0$, and $k_0 > 0$, independent of k, such that*

$$\|A^{-1}\| \leq Ck^{\beta}, \qquad k \geq k_0.$$

Assumption 2 (Coercivity) *There exist constants $\gamma > 0$, $\beta \geq 0$ and $k_0 > 0$, independent of k, such that*

$$\left|\langle A\phi, \phi\rangle_{L^2(\Gamma)}\right| \geq \gamma k^{-\beta} \|\phi\|^2, \quad \phi \in L^2(\Gamma), \, k \geq k_0.$$

We note that, if Assumption 2 holds, then so does Assumption 1 with $C = \gamma^{-1}$. For the standard formulation Assumption 1 is known to hold for all $k_0 > 0$ if Ω is star-like, with $\beta = 0$ and $\eta = k$ [11]. For the star-combined formulation Assumption 2 (and hence Assumption 1) holds (for Ω star-like) for all $k_0 > 0$ if $\eta = k|x| + \frac{i}{2}$, with $\gamma = 1/2 \operatorname{ess\,inf}_{x\in\Gamma}(x \cdot n(x)) > 0$ [9]. By contrast, Assumption 2 has not been proved for the standard formulation except in the special case when the scatterer is circular, although numerical evidence suggests it holds more generally (e.g. for the examples in Figure 2) with $\beta = 0$ [12, Conjecture 6.2].

Regularity of Solutions

Our numerical method for solving (2) uses an approximation space which is adapted to the high-frequency asymptotic behaviour of the solution $\frac{\partial u}{\partial n}$ on each of the sides of the polygon, which we now consider.

At present our analysis applies only to a particular class of non-convex polygons, defined as follows. We denote the corners of the polygon by P_j, $j = 1, \ldots, n+1$, where $P_{n+1} = P_1$, and the sides of the polygon by $\Gamma_j = \left\{P_j + \frac{s}{L_j}(P_{j+1} - P_j) : s \in (0, L_j)\right\}$, where $L_j = |P_{j+1} - P_j|$ is the side length and s denotes arc length along the side. The class of non-convex polygons we consider is defined by two constraints:

1. Each external angle in the polygon is either equal to $\frac{\pi}{2}$ or greater than π.
2. For each external angle equal to $\frac{\pi}{2}$, if local coordinates (x_1, x_2) are defined as in Figure 1, then the polygon lies entirely in the quarter-plane $x_1 \geq -L_2$, $x_2 \leq 0$.

In Figure 2 we show two examples of members of this class; one star-like and one non-star-like.

For a polygon in this class we define two types of side: if the external angles at the endpoints of the side are both greater than π then we say that it is a "convex" side; if one is equal to $\frac{\pi}{2}$ then we say that it is a "non-convex" side. Arguing as in [8], on a typical convex side Γ_j we have

$$\frac{\partial u}{\partial n}(x) = \Psi(x) + e^{iks}v_j^+(s) + e^{-iks}v_j^-(L_j - s), \quad (3)$$

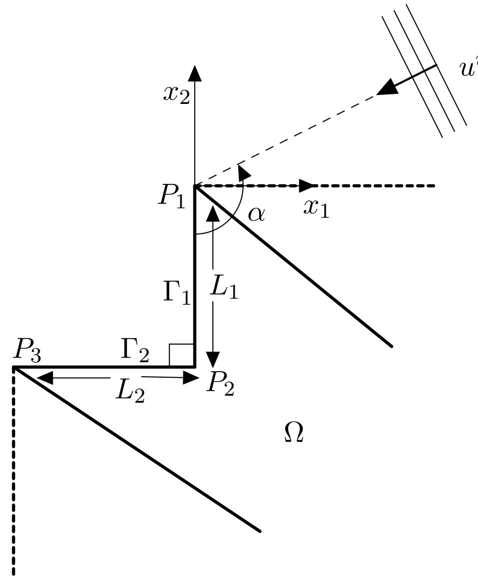
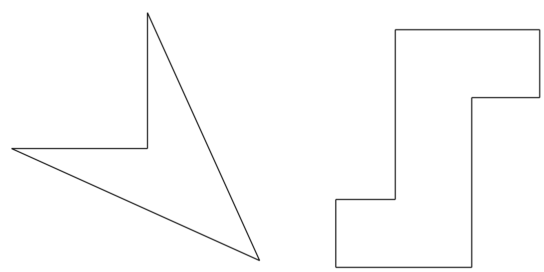


Figure 1: Coordinates near a nonconvex side. The scatterer Ω must lie within the thick dashed lines.



(a) Star-like. (b) Non-star-like.

Figure 2: Examples of scatterers being considered.

where Ψ is the Physical Optics approximation ($\Psi := 2\frac{\partial u^i}{\partial n}$ if Γ_j is lit and $\Psi := 0$ otherwise), and the other two terms in (3) can be thought of as contributions due to diffracted rays travelling along the side. The functions $v_j^{\pm}(s)$ are non-oscillatory and analytic in Re$[s] > 0$ where they satisfy the following bounds:

Theorem 1 *There exists $\delta \in (0, 1/2)$ such that*

$$|v_j^{\pm}(s)| \leq \begin{cases} CMk|ks|^{-\delta}, & |ks| \leq 1, \\ CMk|ks|^{-1/2}, & |ks| > 1, \end{cases} \quad (4)$$

where $M := \sup_{x\in D}|u(x)|$.

We now consider the behaviour on a typical non-convex side, which we denote by Γ_2, with local coordinates as in Figure 1. Let $d = (-\sin\alpha, \cos\alpha)$, so that α, measured anti-clockwise from the negative x_2-axis, describes the direction from which the incident wave arrives.

Theorem 2 *On the non-convex side Γ_2 the representation*

$$\frac{\partial u}{\partial n}(x) = \Upsilon(x) + e^{ikx_1} v_2^-(L_2 + x_1) + e^{-ikx_1} v_2^+(L_2 - x_1) + e^{ik\sqrt{x_1^2+L_1^2}} \tilde{v}_2(x_1), \qquad x \in \Gamma_2, \quad (5)$$

holds, where

(i)

$$\Upsilon(x) := \begin{cases} 2\frac{\partial u^d}{\partial n}(x), & \frac{\pi}{2} \leq \alpha \leq \frac{3\pi}{2}, \\ 0, & otherwise, \end{cases} \quad (6)$$

and u^d is the known exact solution to the problem of diffraction of u^i by the infinite knife edge $\{(0, x_2) : x_2 \leq 0\}$ with Dirichlet boundary conditions;

(ii) the functions $v_2^\pm(s)$ are analytic in $\mathrm{Re}[s] > 0$ where they satisfy the bounds (4) for some $\delta \in (0, 1/2)$;

(iii) the function $\tilde{v}_2(s)$ is analytic and bounded in a k-independent complex neighbourhood of $[-L_2, 0]$. Specifically, let $\epsilon_* := \frac{L_1}{128}\left(\frac{L_1}{L_1+L_2}\right)^3$, and define $D_{\epsilon_*} := \{s \in \mathbb{C} : \mathrm{dist}(s, [-L_2, 0]) < \epsilon_*\}$. Then if Assumption 1 holds, $\tilde{v}_2(s)$ is analytic in D_{ϵ_*} with

$$|\tilde{v}_2(s)| \leq Ck^{1+\beta} \log^{1/2} k, \quad s \in D_{\epsilon_*},\ k \geq k_0. \quad (7)$$

Although we do not give details here, we remark that the proof of Theorem 2 is based on the application of Green's theorem in the quarter-plane Q with boundary $\partial Q := \gamma_1 \cup \Gamma_1 \cup \Gamma_2 \cup \gamma_2$, where $\gamma_1 = \{(0, x_2) : x_2 > 0\}$ and $\gamma_2 = \{(x_1, -L_1) : x_1 < -L_2\}$ are the extensions of Γ_1 and Γ_2, the Dirichlet Green's function of which is given explicitly by the method of images as

$$G(x, y) := \Phi(x, y) - \Phi(x, y^*) - \Phi(x, y') + \Phi(x, y^{*\prime}),$$

where the superscripts $*$ and $\prime$ are operations of reflection in the lines $\gamma_1 \cup \Gamma_1$ and $\gamma_2 \cup \Gamma_2$, respectively.

hp Approximation Space and Approximation Results

Our approximation space $V \subset L^2(\Gamma)$ is the set of functions that on each convex side has the form (3), with v_j^+ and v_j^- replaced by piecewise polynomials supported on geometric meshes, graded towards the singularities at P_j and P_{j+1} respectively. Explicitly, the nodes of the mesh used to approximate v_j^+ are given by $s_0 = 0, s_m = \sigma^{N_j^+ - m} L_j, m = 1, \ldots, N_j^+$, where N_j^+ is the number of layers and $\sigma \in (0, 1)$ is a grading parameter (commonly $\sigma \approx 0.15$); similarly for v_j^- we have a mesh graded towards P_{j+1}. On a non-convex side (say Γ_2) we use the representation (5), with v_2^- replaced by a piecewise polynomial supported on a geometric mesh, graded towards the singularity at P_3, and v_2^+ and $\tilde{v}$ replaced by polynomials supported on the whole of Γ_2.

Using the regularity results provided by Theorems 1 and 2 we can prove that the best approximation error in approximating $\frac{\partial u}{\partial n}$ by an element of V decays exponentially as the degree of the approximating polynomials increases.

For simplicity we assume the same number of layers N on each graded mesh, and the same degree p of polynomial approximation on each element. We then have the following best approximation result:

Theorem 3 *There exist constants $c_1, c_2, c_3, C, \tau > 0$, independent of k, such that if $N \geq c_1 \log k + c_2 p$ and $p \geq c_3 \log k$, then*

$$\inf_{v \in V} \left\| \frac{\partial u}{\partial n} - v \right\| \leq CM\sqrt{k} e^{-p\tau}, \quad k \geq k_0.$$

Galerkin Method

Our Galerkin method is: find $\varphi \in V \subset L^2(\Gamma)$ such that

$$\langle A\varphi, v \rangle_{L^2(\Gamma)} = \langle f, v \rangle_{L^2(\Gamma)}, \quad \forall v \in V.$$

Lemma 1, Assumption 2 and Céa's lemma together imply the quasi-optimality estimate

$$\left\| \frac{\partial u}{\partial n} - \varphi \right\| \leq \frac{Ck^{\frac{1}{2}+\beta}}{\gamma} \inf_{v \in V} \left\| \frac{\partial u}{\partial n} - v \right\|, \quad k \geq k_0.$$

which, combined with Theorem 3, gives:

Theorem 4 *Suppose that Assumption 2 holds and that N and p satisfy the assumptions of Theorem 3. Then*

$$\left\| \frac{\partial u}{\partial n} - \varphi \right\| \leq \frac{C}{\gamma} M k^{1+\beta} e^{-p\tau}, \quad k \geq k_0.$$

Numerical Results

We now present numerical results for the standard combined-potential formulation. The scatterer is as shown in Figure 2(a) with non-convex sides of length 2π and convex sides of length 4π, and the direction of incidence is $\alpha = \frac{\pi}{4}$. Let $\phi := \frac{1}{k}\frac{\partial u}{\partial n}$ and denote its approximation, computed using the numerical method described above, by ϕ_p. With polynomial degree p on each element and $N = 2(p+1)$ layers on each graded mesh, the method has $2\,(3N+2)\,(p+1) = 12p^2 + 28p + 16$ degrees of freedom in total. In the following tables the "exact" solution ϕ is computed using the same scheme with large p and N values and the L^2 error is approximated by a discrete L^2 norm sampling at 200,000 evenly spaced points around Γ.

Table 1: Fixed $k = 20$, 40, increasing p, $N = 2(p+1)$. Note that $\|\phi\| \to \left\|\frac{1}{k}\Psi\right\| = 2\sqrt{\pi} \approx 3.5449$ as $k \to \infty$.

p	DOF	$\|\phi - \phi_p\|$ $k = 20$	$\|\phi - \phi_p\|$ $k = 40$
0	16	6.4504×10^{-1}	4.5232×10^{-1}
1	56	3.9477×10^{-1}	2.9069×10^{-1}
2	120	8.9357×10^{-2}	6.5900×10^{-2}
3	208	1.0591×10^{-2}	7.3555×10^{-3}
4	320	4.5999×10^{-3}	3.1311×10^{-3}
5	456	2.1713×10^{-3}	1.3410×10^{-3}
6	616	1.2953×10^{-3}	6.3100×10^{-4}

The numerical results in Table 1 support the exponential convergence estimate of Theorem 4.

Table 2: Fixed $p = 4$, $N = 10$, increasing k. DOF/λ is the degrees of freedom per wavelength.

k	DOF	$\frac{\text{DOF}}{\lambda}$	$\|\phi - \phi_p\|$
5	320	10.7	2.0915×10^{-2}
10	320	5.3	1.0671×10^{-2}
20	320	2.7	4.5999×10^{-3}
40	320	1.3	3.1311×10^{-3}
80	320	0.7	1.8617×10^{-3}

In Table 2 the accuracy increases as k increases for a fixed number of degrees of freedom. This is stronger than we have proved in Theorem 4; the estimate there suggests that logarithmic growth in the degrees of freedom is needed to ensure accuracy is maintained.

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