Chapter 5. Infinitary combinatorics *

5.1. Club and stationary sets.

Definition. Let λ be a limit ordinal. A set $D \subseteq \lambda$ is *unbounded in* λ if $\forall \alpha < \lambda \, \exists \beta \in D \, \alpha < \beta$. A set $C \subseteq \lambda$ is *closed in* λ if for all limit ordinals $\delta < \lambda$, $C \cap \delta$ is unbounded in δ implies that $\sup(C \cap \delta) \in C$. A set that is closed and unbounded in λ is often called "*club*"

Examples. $\{\gamma < \lambda \,|\, \gamma \text{ is a limit ordinal}\}$ and $\{\gamma < \lambda \,|\, \gamma \text{ is a limit of limit ordinals}\}$ are closed in λ. And they are unbounded if λ is a cardinal.

You should think of club sets as being "large" in some sense.

Proposition. Let κ be an uncountable regular cardinal and $\mathcal{F}$ a set of finitary funtions on κ with $\overline{\overline{\mathcal{F}}} < \kappa$ (*i.e.*, if $f \in \mathcal{F}$ we have that $f : {}^{n(f)}\kappa \longrightarrow \kappa$ for some $n(f) < \omega$.) Then, $C = \{\gamma < \kappa \,|\, \forall f \in \mathcal{F} \; f``\,{}^{n(f)}\gamma \subseteq \gamma\}$ is closed and unbounded.

Proof. It is clear that C is closed: if $\lambda < \kappa$ is a limit ordinal and $C \cap \lambda$ is unbounded, $f \in \mathcal{F}$, $f : {}^{n(f)}\kappa \longrightarrow \kappa$ and $(\alpha_0, \ldots, \alpha_{n-1}) \in {}^{n(f)}\lambda$ then there is some $\gamma \in (C \cap \lambda) \setminus \max(\{\alpha_0, \ldots, \alpha_{n-1}\})$, and we have $f(\alpha_0, \ldots, \alpha_{n-1}) < \gamma < \lambda$.

To see C is unbounded, define $G_0(X) = X$, $G_{n+1}(X) = G_n(X) \cup \bigcup\{\beta \,|\, \exists \alpha_0, \ldots, \alpha_{m-1} \in G_n(X) \; \exists f \in \mathcal{F} \; f(\alpha_0, \ldots, \alpha_{m-1}) = \beta\}$, for each, $n < \omega$ and $G(X) = \bigcup_{n \in \omega} G_n(X)$ for each subset X of κ. If $\xi < \kappa$ it is clear that $\xi \subseteq G(\xi) \subseteq \kappa$ and $\overline{\overline{G(\xi)}} < \kappa$. Because κ is regular we can choose $g(\xi) = \min(\kappa \setminus \sup(G(\xi)))$. Let $g_{n+1}(\xi) = g(g_n(\xi))$ some each $\xi < \kappa$, and let $g_\omega(\xi) = \sup\{g_n(\xi) \,|\, n \in \omega\}$. Then we have that $g_\omega(\xi) \in C$ and $\xi < g_\omega(\xi)$.

Definition. Let $g : \kappa \longrightarrow C$ enumerate a club set C in increasing order. Let $C' = g``\{\omega\gamma \,|\, \gamma < \kappa\}$. C', $\subseteq C$, is the *set of limit points of* C.

Lemma. Let C be a club subset of a regular cardinal κ and define $f : \kappa \longrightarrow \kappa$ by $f(\gamma) = \min(C \setminus (\gamma+1))$. Then $C' = \{\gamma < \kappa \,|\, f``\gamma \subseteq \gamma\}$, and hence C' is club.

Proof. It is clear that $C' = \{\gamma < \kappa \,|\, f``\gamma \subseteq \gamma\}$, and thus by the previous proposition, is closed and unbounded.

Proposition. Let κ be an uncountable regular cardinal, $\mu < \kappa$, and let C_α be closed and unbounded in κ for each $\alpha < \mu$. Then $D = \bigcap\{C_\alpha \,|\, \alpha < \mu\}$ is closed and unbounded in κ.

Proof. It is easy to see that D is closed. Now let $f_\alpha(\gamma) = \min(C_\alpha \setminus \gamma + 1)$ for $\gamma < \kappa$ and for all $\alpha < \mu$. D is unbounded because, by the previous proposition, $\{\gamma < \kappa \,|\, \forall \alpha < \mu \, f_\alpha``\gamma \subseteq \gamma\}$ is unbounded (and closed), and is a subset of D by the lemma.

In fact it is easy to see that one can prove similar results for singular cardinals κ with $\omega < \mathrm{cf}\,(\kappa)$. For example, if $\mu < \mathrm{cf}\,(\kappa)$ and C_α be closed and unbounded in κ for each $\alpha < \mu$ then $D = \bigcap\{C_\alpha \,|\, \alpha < \mu\}$ is closed and unbounded in κ.

First of all prove (by hand) that the intersection of any two club sets is club. Next, take a cofinal function $f : \mathrm{cf}\,(\kappa) \longrightarrow \kappa$ which is closed at limits, that is such that $f(\lambda) = \bigcup\{f(\alpha) \,|\, \alpha < \lambda\}$ for

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limit $\lambda < \mathrm{cf}(\kappa)$, and observe that $f``\mathrm{cf}(\kappa)$ is a club, say $C$, in $\kappa$. Finally, if $D \subseteq \kappa$ let $D' = D \cap C$ and note that $f^{-1}``D'$ is club in $\mathrm{cf}(\kappa)$. So to prove results about, for example, intersections of club subsets of κ (or the diagonal intersection lemma below) simply 'slide' the club sets down to $\mathrm{cf}(\kappa)$ using f^{-1}, use the results for regular cardinals applied to the resulting sets and then use f to slide the result back up to κ

Proposition. *(Diagonal intersections)* Let κ be a regular cardinal and let C_α be closed and unbounded in κ for each $\alpha < \kappa$. Then $D = \{\beta \mid \forall\alpha < \beta\,(\beta \in C_\alpha)\}$ $(= \{\beta \mid \beta \in \bigcap_{\alpha<\beta} C_\alpha)\})$ is closed and unbounded in κ.

Proof. It is easy to see that D is closed: if λ is a limit ordinal and $D \cap \lambda$ is unbounded in λ we have that $\forall\alpha < \lambda\,\exists\beta\,(\alpha \le \beta < \lambda\ \&\ \forall\gamma < \beta\,\beta \in C_\gamma)$. Hence $\forall\alpha < \lambda\,\exists\beta\,(\alpha \le \beta < \lambda\ \&\ \beta \in C_\alpha)$. Thus $C_\alpha \cap \lambda$ is unbounded in λ for all $\alpha < \lambda$. Thus $\lambda \in \bigcap\{C_\alpha \mid \alpha < \lambda\}$.

In order to see that D is unbounded, define $g(\gamma) = \min(\bigcap_{\alpha<\gamma} C_\alpha \setminus \gamma)$ for all $\gamma < \kappa$. (Note that this intersection is unbounded by the proposition that the intersection of fewer than κ club sets is club, thus $g(\gamma)$ is well-defined.) Now let $\beta < \kappa$, $\beta_0 = \beta$, $\beta_{n+1} = g(\beta_n)$ for $n < \omega$, and $\beta_\omega = \sup\{\beta_n \mid n < \omega\}$. Then $\beta_\omega \in D$.

Definition. Let κ be a cardinal with $\omega < \mathrm{cf}(\kappa)$. A set $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for all C which are closed and unbounded in κ. A set $X \subseteq \kappa$ is *non-stationary* if it is not stationary.

Stationary sets are "medium-sized" as subsets of κ and non-stationary sets are "small" in this sense – even though they can have cardinality κ. The proposition that any intersection of fewer than $\mathrm{cf}(\kappa)$ many club sets is club in κ says that any union of fewer than $\mathrm{cf}(\kappa)$ non-stationary sets is non-stationary.

Examples. Let $\mu < \mathrm{cf}(\kappa)$ be a regular cardinal. Then $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\}$ is stationary, because if C is closed and unbounded in κ the μ^{th} element in the increasing enumeration of C has cofinality μ. For example, both $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ and $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\}$ are both stationary in ω_2.

Proposition. *(Fodor's Lemma)* Let κ be a regular cardinal, $\omega < \kappa$, let S be a subset stationary of κ and let $f : S \longrightarrow \kappa$ be such that $f(\alpha) \in \alpha$ for each $\alpha \in S$. Then there is some $\gamma < \kappa$ such that $\{\alpha \in S \mid f(\alpha) = \gamma\}$ is stationary in κ.

Proof. If the conclusion fails, then choose for each $\gamma \in \kappa$ a set C_γ which is closed and unbounded in κ and such that $C_\gamma \cap f^{-1}``\{\gamma\} = \emptyset$. Let $D = \{\beta \mid \forall\gamma < \beta\,(\beta \in C_\gamma)\}$. The diagonal intersections proposition says that D is closed and unbounded in κ. But $D \cap S = \emptyset$, because if $\beta \in D$ we have $f(\beta) \neq \alpha$ for each $\alpha < \beta$. This is a contradition to the supposition that S is stationary.

An illustration that stationary sets are not as "fat" as club sets are is given by the following proposition.

Proposition. Let $\kappa = \mu^+$ and let S be stationary in κ. Then there is some S_α for $\alpha < \kappa$ with $S = \bigcup\{S_\alpha \mid \alpha < \kappa\}$ and $S_\alpha \cap S_\beta = \emptyset$ for each pair α, $\beta < \kappa$ with $\alpha \neq \beta$.

Proof. Let $f_\gamma : \gamma \longrightarrow \mu$ be an injection for each $\gamma < \kappa$. Define $S^\xi_\alpha = \{\gamma \mid \alpha < \gamma < \kappa\ \&\ f_\gamma(\alpha) = \xi\} \subseteq \kappa$. Then $S^\xi_\alpha \cap S^\xi_\beta = \emptyset$ if $\alpha \neq \beta$ for each $\xi < \mu$ because each f_γ is a injection. Also we have that $\bigcup_{\xi<\mu} S^\xi_\alpha = \{\gamma \mid \alpha < \gamma < \kappa\}$ is stationary in κ. Because the union of fewer than κ non-stationary subsets of κ is non-stationary, it must be that for each α there is some $\xi_\alpha < \mu$ such that $S^{\xi_\alpha}_\alpha$ is stationary. Since $\mu < \kappa$ there is some $\xi < \kappa$ such that $\overline{\overline{\{\alpha < \kappa \mid \xi = \xi_\alpha\}}} = \kappa$. Let $A = \{\alpha < \kappa \mid \xi = \xi_\alpha\}$ and $\alpha_0 = \min(A)$. Define $S_\alpha = S^\xi_\alpha$ for $\alpha \in A \setminus \{\alpha_0\}$, and define $S_{\alpha_0} = S \setminus \bigcup\{S^\xi_\alpha \mid \alpha \in A \setminus \{\alpha_0\}\}$ (since $S^\xi_{\alpha_0} \subseteq S_{\alpha_0}$). We have that $S_\alpha \subseteq S$ is stationary for each $\alpha \in A$ and $S_\alpha \neq S_\beta$ where α, $\beta \in A$ and $\alpha \neq \beta$.

Fact. With a little more work one can show the proposition for all regular cardinals and not only for successor cardinals as is done above.

5.2 Silver's Theorem.

Now we are going to use the concepts of closed and unbounded and of stationary sets to prove Silver theorem about exponentiation of singular cardinals of uncountable cofinality that was mentioned at the end of §4.4. It is useful to have one more definition.

Definition. $\mathcal{F} \subseteq {}^{\omega_1}\omega_{\omega_1}$ is called *(an) eventually different (family)* if $\forall f,\ g \in \mathcal{F}\,(f \neq g \longrightarrow \overline{\overline{\{\,\alpha \mid f(\alpha) = g(\alpha)\,\}}} \leq \omega$.

Theorem. (Silver, 1975) If $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

Proof. By the hypotheses of the theorem, supposethat$\mathcal{P}(\omega_\alpha) = \{\,X^\alpha_\beta \mid \beta < \omega_{\alpha+1}\,\}$ for all $\alpha < \omega_1$. Then for every $X \subseteq \omega_{\omega_1}$ and $\alpha < \omega_1$ there is some $\beta < \omega_{\alpha+1}$ such that $X \cap \omega_\alpha = X^\alpha_\beta$. So for each $X \subseteq \omega_{\omega_1}$ we can define a function $f_X : \omega_1 \longrightarrow \omega_{\omega_1}$ by $f_X(\alpha) = \beta(< \omega_{\alpha+1})$ if $X \cap \omega_\alpha = X^\alpha_\beta$. Note that $\mathcal{F} = \{\,f_X \mid X \in \mathcal{P}(\omega_{\omega_1})\,\}$ is eventually different because $X \neq Y$ implies that there is some $\alpha < \omega_1$ such that $f_X(\alpha) \neq f_Y(\alpha)$, and $f_X(\alpha) \neq f_Y(\alpha)$ implies $f_X(\gamma) \neq f_Y(\gamma)$ for every $\gamma \geq \alpha$.

Now partially order $\mathcal{F}$ by $f < g$ if and only if $\{\,\alpha < \omega_1 \mid g(\alpha) \leq f(\alpha)\,\}$ is a non-stationary subset. of ω_1. By Zorn's Lemma let $\prec$ be any total order which extends $<$. (See Exercise 3.2.a.) So if $f \prec g$ we have that $\{\,\alpha < \omega_1 \mid f(\alpha) < g(\alpha)\,\}$ is stationary in ω_1. Let $\mathcal{F}_g = \{\,f \in \mathcal{F} \mid f \prec g\,\}$. We shall show that $\overline{\overline{\mathcal{F}_g}} \leq \omega_{\omega_1}$ for each $g \in \mathcal{F}$, which will give us the theorem.

In order to do this we use the following two lemmas.

Lemma. If $\mathcal{G}$ is eventually different and $\forall g \in \mathcal{G}\,\{\,\alpha \mid g(\alpha) < \omega_\alpha\,\}$ is stationary, then $\overline{\overline{\mathcal{G}}} \leq \omega_{\omega_1}$.

Proof. Suppose $f \in \mathcal{G}$. Let $g : \{\alpha < \omega_1 \mid \alpha = \bigcup \alpha\} \longrightarrow \omega_1$ be defined by $g(\alpha) =$ the minimum $\gamma < \alpha$ such that $f(\alpha) < \kappa_\gamma$. By Fodor's Lemma there is a stationary subset S_f of $\{\alpha < \omega_1 \mid \alpha = \bigcup \alpha\}$ and some $\gamma_f < \omega_1$ such that $g(\alpha) = \gamma_f$ for all $\alpha \in S_f$. Hence, if $\omega_{\omega_1} < \overline{\overline{\mathcal{G}}}$ there is a subset $\mathcal{G}' \subseteq \mathcal{G}$, a $\gamma < \omega_1$, and a stationary set $S \subseteq \omega_1$ such that $\omega_{\omega_1} < \overline{\overline{\mathcal{G}'}}$ and $\forall f \in \mathcal{G}'\,(\gamma_f, S_f) = (\gamma, S)$. As $\mathcal{G}$ is eventually different, $\forall f,\ g \in \mathcal{G}'\ (f \neq g \longrightarrow f \restriction S \neq g \restriction S)$. But we have

$$\overline{\overline{{}^S\omega_\gamma}} = \omega_\gamma^{\omega_1} \leq \max(\{\,\omega_\gamma^{\omega_\gamma},\ {\omega_1}^{\omega_1}\,\}) = \max(\{\,2^{\omega_\gamma},\ 2^{\omega_1}\,\}) = \max(\{\,\omega_\gamma^+,\ \omega_2\,\}) \leq \kappa.$$

A contradition! Thus $\overline{\overline{\mathcal{G}}} \leq \kappa$, as required.

The other lemma crudely reduces what remains of the proof to a situation in which this last lemma can be applied.

Lemma. If $f : \omega_1 \longrightarrow \omega_{\omega_1}$ is such that $f(\alpha) < \omega_{\alpha+1}$ for each $\alpha \in \omega_1$, $\mathcal{G}$ is eventually different and $\forall g \in \mathcal{G}\,\{\,\alpha \mid g(\alpha) < f(\alpha)\,\}$ is stationary, so $\overline{\overline{\mathcal{G}}} \leq \omega_{\omega_1}$.

Proof. For each $\alpha < \omega$ let $h_\alpha : f(\alpha) \longrightarrow \omega_\alpha$ be an injection. For each $g \in \mathcal{G}$ define $g' : \omega_1 \longrightarrow \omega_{\omega_1}$ by $g'(\alpha) = h_\alpha(g(\alpha))$ and let $\mathcal{G}' = \{\,g' \mid g \in \mathcal{G}\,\}$. Then $\mathcal{G}'$ satisfies the hypothesis of the previous lemma. But because all the functions h_α are injections, $g' = k' \longrightarrow g = k$. Thus $\overline{\overline{\mathcal{G}'}} = \overline{\overline{\mathcal{G}'}} \leq \omega_{\omega_1}$.

Hence $\overline{\overline{\mathcal{F}_g}} \leq \omega_{\omega_1}$ for each $g \in \mathcal{F}$, and, so, $\overline{\overline{\mathcal{F}}} \leq \omega_{\omega_1+1}$. Thus $\overline{\overline{\mathcal{F}}} = 2^{\omega_{\omega_1}} = \omega_{\omega_1+1}$.

Corollary to the Proof. Let κ be a limit cardinal and singular with $\mathrm{cf}\,(\kappa) = \theta < \kappa$ and let $c : \theta \longrightarrow \kappa$ be cofinal. If there is a stationary subset $T \subseteq \theta$ such that $2^{c(\mu)} = c(\mu)^+$ for each $\mu < \theta$, then $2^\kappa = \kappa^+$.

5.3. Ramsey's Theorem.

In the next section we shall see another example of the usefulness of the notions of club and stationary sets. This application will be in graph theory.

Definition. A *graph* is a pair (X, A) where X is a set and $A \subseteq X \times X \setminus \{ (x, x) \mid x \in X \}$. X is the set of *vertices* or *nodes* and A is the set of *edges*). Another way of saying the same thing is to define a graph to be a set X with a binary relation A such that A is symmetric and anti-reflexive.[1] We are interested in graphs only up to isomorphism where $(X, A) \cong (Y, B)$ if there is a bijection $f : X \longrightarrow Y$ such that $(x, y) \in A \iff (f(x), f(y)) \in B$.

There are many examples of graphs, both finite and infinite. Among them are the *complete* graphs: graphs of the form (X, B) where for each pair $\{x, y\}$ of members of X with $x \neq y$ one has $(x, y) \in B$. We write K_κ for the complete graph of size κ. (Clear K_κ is unique up to isomorphism.) For example, we have that K_3 is a triangle and K_4 is a square with its diagonals.

Definition. $H = (Y, B)$ is a subgraph of $G = (X, A)$ if $Y \subseteq X$ and $B = A \cap (Y \times Y)$. Thus H has all of the edges that G does between the vertices in Y.

Definition. A *colouring* of a graph $G = (X, A)$ *with* γ *colours* is a function $f : A \longrightarrow \gamma$.

Ramsey's theorem in its simplest form says that for any finite colouring f of the complete countably infinte graph K_ω there is a subgraph $G = (X, A)$ of K_ω such that $G \cong K_\omega$ and all of the edges of G have the same colour: $\overline{\overline{\{ f(e) \mid e \in A \}}} = 1$. We say that G is *monochromatic.* There is also a finite version of the theorem which is deducible from this infinite version: for every $n < \omega$ and $r < \omega$ there is $m < \omega$ such that for any colouring f of K_m there is a subgraph G of K_m such that $G \cong K_n$. Actually it is worthwhile proving a slight extension to Ramsey's theorem that has more or less the same proof.

Definition. For any set x and any cardinal κ let $[x]^\kappa = \{ y \subseteq x \mid \overline{\overline{y}} = \kappa \}$.

Ramsey's Theorem. (1930) Let n, $r < \omega$. If $f : [\omega]^n \longrightarrow r$ there is some $X \in [\omega]^\omega$ (that is, an infinite subset of ω) and there is some $p < r$ such that $f``[X]^n = \{p\}$. (*I.e.*, for any colouring of the set of subsets of size n of ω with r colours there is an infinite subset X of ω such that all the subsets of X of size r have the same colour.)

Firstly, observe the following lemma, a pigeonhole principle, which we will use several times during the proof.

Lemma. Let n, $r < \omega$ and $f : [\omega]^{n+1} \longrightarrow r$. Suppose that $Z \in [\omega]^\omega$, $x \in [Z]^n$, and for each $p < r$ let $x_p = \{ s \in Z \mid s \notin x \ \& \ f(x \cup \{ s \}) = p \}$. Then at least one of the sets x_p is infinite.

Proof of the Lemma. If each x_p was finite we would have that $Z \setminus x = \bigcup \{ x_p \mid p < r \}$ would be a finite union of finite sets, hence finite. A contradiction!

Proof of Ramsey's theorem. The proof is by induction on n. The first case is $n = 1$, and in this case we have some $f : [\omega]^1 \longrightarrow r$. Let $y_p = \{ m \in \omega \mid f(\{ m \}) = p \}$ for each $p < r$. So by the lemma at least one of the sets y_p is infinite, as required.

[1] For more information about graph theory look at, for example, Bela Bollobas, *Combinatorics*, Cambridge University Press, 1986? or Bela Bollobas's *Graph Theory: An introductory course*, Springer, 1979.

Now suppose we have shown the theorem for n and any $r < \omega$. Let $r < \omega$ and $f : [\omega]^{n+1} \longrightarrow r$. We define a colouring $g : [\omega]^n \longrightarrow r$ using f: if $x \in [\omega]^n$, using the Lemma, let $g(x) = p$ if p is minimal such that there is an infinite set of $s < \omega$ with $f(x \cup \{ s \}) = p$.

Next we define a sequence of infinite sets X_i for $i < \omega$ with $X_0 = \omega$ and $X_{i+1} \subseteq X_i$ and write a_i for $\min(X_i)$. For $i < n$ let $X_{i+1} = X_i \setminus \{ a_i \}$. Otherwise suppose that we have $\langle a_i \,|\, i \leq j \rangle$ and $\langle X_i \,|\, i \leq j \rangle$ for some $n - 1 \leq j < \omega$. Let $\langle y_k \,|\, k < l \rangle$ be an enumeration of $[\{ a_i \,|\, i < j \}]^n$ for some $l < \omega$. We shall define infinite sets Y_k for $k \leq l$ with $Y_0 = X_j \setminus (a_j + 1)$ and $Y_{k+1} \subseteq Y_k$ and we will set $X_{j+1} = Y_k$. By the Lemma again, let $q_k < r$ be minimal such that there is an infinite subset Y of Y_k for which for all $b \in Y$ we have $f(y_k \cup \{ b \}) = q_k$, and let $Y_{k+1} = \{ b \in Y_k \,|\, f(y_k \cup \{ b \}) = q_k \}$.

Let $X = \{ a_i \,|\, i < \omega \}$. By the theorem for n we have that there is some $Z \in [X]^\omega$ and a colour $q < r$ such that $g``[Z]^n = \{ q \}$. We show that $\overline{\overline{f``[Z]^{n+1}}} = 1$ as well. Let $x_0, x_1 \in [Z]^{n+1}$ with $b_0 = \max(x_0)$ and $b_1 = \max(x_1)$, and let $y_0 = x_0 \setminus \{ b_0 \}$ and $y_1 = x_1 \setminus \{ b_1 \}$. Then $f(x_0) = g(y_0)$ and $f(x_1) = g(y_1)$. But $g(y_0) = g(y_1)$, because $y_0, y_1 \in [Z]^n$, hence $f(x_0) = f(x_1)$.

5.4. Todorcevic's Theorem.

In contrast to Ramsey's theorem, however, Sierpinski already showed in the 1930s that the analogous theorem for successor cardinals is false, at least assuming GCH. First we prove a useful lemma.

Definition. Let κ be a cardinal. The *lexicographic order on* $\mathcal{P}(\kappa)$ is given by $x < y$ if when z is minimal such that $z \in (x \setminus y) \cup (y \setminus x)$ we have that $z \in y$.

Note. It is clear that $<$ is a linear order. And note that this really is the lexicographic order on ${}^\kappa 2$ where we have the isomorphism $\chi : \mathcal{P}(\kappa) \longrightarrow {}^\kappa 2$ given by $\chi_x(n) = 1$ if and only if $n \in x$ for $x \in \mathcal{P}(\kappa)$.

Lemma. Let κ be an infinite cardinal and let $<$ be the lexicographic order on $\mathcal{P}(\kappa)$. There is no subset of $\mathcal{P}(\kappa)$ which is well-ordered by $<$ and which has cardinality greater than κ.

Proof. Suppose that $Y \subseteq \mathcal{P}(\kappa)$ is well-ordered by $<$ and $\kappa < \overline{\overline{Y}}$. Then there is some ordinal ρ such that $(Y, <) \cong (\rho, \in)$ by an o.p. isomophism g and $\kappa^+ \leq \rho$. Let Y_0 be the initial segment of Y such that $(Y_0, <) \cong (\kappa^+, \in)$ by g. Let γ_α be the least $\gamma < \kappa$ such that $\forall \beta < \alpha\, \gamma_\beta < \gamma$ and there is $x \in Y_\alpha$ such that $\gamma \in x$, let $Y_{\alpha+1} = \{ x \in Y_\alpha \,|\, \gamma_\alpha \in x \}$, and let $Y_\lambda = \bigcap \{ Y_\alpha \,|\, \alpha < \lambda \}$ for limit $\lambda < \kappa$. Further, let $a_\alpha \in Y_\alpha$ if $Y_\alpha \neq \emptyset$ and $\beta = \{ \alpha \,|\, Y_\alpha \neq \emptyset \}$.

Then the function $f : \beta \longrightarrow Y_0$ defined by $f(\alpha) = a_\alpha$ for $\alpha < \beta$ and is cofinal into Y_0 and we have that $g \cdot f$ shows that $\mathrm{cf}\,(\kappa^+) \leq \mathrm{cf}\,(\beta) \leq \kappa$. A contradiction!

Proposition. (Sierpinski) There is a colouring f of $[\mathcal{P}(\kappa)]^2$ with 2 colour such that there is no subset X of $\mathcal{P}(\kappa)$ of cardinality greater than κ with $\overline{\overline{f``[X]^2}} = 1$.

Proof. Let $<$ be the lexicographic order on $\mathcal{P}(\kappa)$. Let $\prec$ be any well-ordering of $\mathcal{P}(\kappa)$. Define $f : [\mathcal{P}(\kappa)]^2 \longrightarrow 2$ by setting $f(\{ x, y \}) = 0$ if $x < y$ and $x \prec y$ or $y < x$ and $y \prec x$, and $f(\{ x, y \}) = 1$ if $x < y$ and $y \prec x$ or $y < x$ and $x \prec y$.

If $H \subseteq \mathcal{P}(\kappa)$ is such that $f``[H]^2 = \{ 0 \}$ then $<$ is a well-ordering of $H$. And if $H \subseteq \mathcal{P}(\kappa)$ is such that $f``[H]^2 = \{ 1 \}$ then $\{ \kappa \setminus x \,|\, x \in H \}$ is well-ordered by $<$. So, in either case $\overline{\overline{H}} \leq \kappa$ by the Lemma.

Definition. A cardinal κ is *weakly compact* if for any colouring f of $[\kappa]^2$ with 2 colours there is some $x \in [\kappa]^\kappa$ such that $\overline{\overline{f\text{“}[x]^\kappa}} = 1$.

Example. ω is weakly compact by Ramsey's theorem.

Proposition. If κ is weakly compact, then κ is strongly inaccessible.

Proof. If κ is not strongly inaccessible then either there is some $\mu < \kappa$ such that $\kappa \leq 2^\mu$ or κ is singular. If $\mu < \kappa$ and $\kappa \leq 2^\mu$ we have that there is a colouring of $[2^\mu]^2$ with 2 colours without a monochromatic subset of 2^μ of size μ^+ by Sierpinski's theorem. But $\mu^+ \leq \kappa$, so the restriction of this colouring to $[\kappa]^2$ has no monochromatic subset of cardinality κ.

If κ is singular, let $g : \mathrm{cf}\,(\kappa) \longrightarrow \kappa$ be cofinal and strictly increasing. Define $f : [\kappa]^2 \longrightarrow 2$ by $f(\gamma, \delta) = 0$ if and only if there is some $\alpha < \mathrm{cf}\,(\kappa)$ such that $g(\alpha) \leq \gamma,\ \delta < g(\alpha + 1)$. Because for each $\alpha < \mathrm{cf}\,(\kappa)$ we have that $\overline{\overline{\{\,\gamma < \kappa \mid g(\alpha) \leq \gamma < g(\alpha+1)\,\}}} < \kappa$, there is no $x \in [\kappa]^\kappa$ such that $f\text{“}[x]^2 = \{\,0\,\}$. But also there is no $x \in [\kappa]^{\mathrm{cf}\,(\kappa)^+}$ such that $f\text{“}[x]^2 = \{\,1\,\}$.

Much later Stevo Todorčević proved a beautiful theorem which greatly strengthens Sierpinski result. This theorem serves as another example of how useful the notions of 'closed and unbounded' and of 'stationary' are. (The proof that we give here is a variation by Dan Velleman on Todorčević's original proof.)

Theorem. (Todorčević[2]) There is a colouring f of $[\omega_1]^2$ with ω_1 colours such that $f\text{“}[x]^2 = \omega_1$ for each $x \in [\omega_1]^{\omega_1}$. In other words, there is a colouring of K_{ω_1} with ω_1 colours such that any subgraph isomorphic to K_{ω_1} must contain edges coloured with each of the colours.

Proof. Let $\{\,r_\alpha \mid \alpha < \omega_1\,\} \subseteq {}^\omega 2$. Also let $e_\alpha : \alpha \longrightarrow \omega$ be an injection for each $\alpha < \omega_1$. Define $\sigma(\alpha, \beta) =$ the least $n < \omega$ such that $r_\alpha(n) \neq r_\beta(n)$ for $\alpha < \beta < \omega$, and let $f(\alpha, \beta) =$ the least $\delta < \omega_1$ such that $\alpha \leq \delta < \beta$ and $e_\beta(\delta) \leq \sigma(\alpha, \beta)$ if there is some such δ, and $= 0$ otherwise.

To prove the theorem it is enough to show that if $Z \in [\omega_1]^{\omega_1}$, then there is a club set C such that $C \subseteq f\text{“}[Z]^2$. In order to see this suffices remember that by the proposition immediately following the definition of stationarity there are stationary sets $S_i \subseteq \omega_1$ for each $i < \omega_1$, with $\omega_1 = \bigcup\{\,S_i \mid i < \omega_1\,\}$ and $S_i \cap S_j = \emptyset$ if $i,\ j < \omega_1$ and $i \neq j$. If we set $g(\alpha) = i$ if and only if $\alpha \in S_i$ we have that $g \cdot f\text{“}[Z]^2 = g\text{“}C_Z$ where C_Z is closed and unbounded for each $Z \in [\omega_1]^{\omega_1}$. But for each $i < \omega_1$ we have that there is some $\alpha_i \in S_i \cap C_Z$ because S_i is stationary. So if $i < \omega_1$ there is some $\{\,x,\, y\,\} \in [Z]^2$ such that $f(\{\,x,\, y\,\}) = \alpha$ and $g(\alpha) = i$. Hence $g \cdot f$ instantiates the theorem.

So, let $Z \in [\omega_1]^{\omega_1}$ and for each $g \in \bigcup\{\,{}^n 2 \mid n < \omega\,\} = {}^{<\omega}2$ let $B_g = \{\,\alpha \in Z \mid g \subseteq r_\alpha\,\}$. Also set $C = \{\,\delta < \omega_1 \mid \forall g \in {}^{<\omega}2,\ \text{either } B_g \subseteq \delta \text{ or } (\overline{\overline{B_g}} = \omega_1 \;\&\; \delta = \bigcup(B_g \cap \delta))\,\}$. I claim that C is club in ω_1.

In order to see that C is closed suppose that λ is a limit ordinal and $\bigcup C \cap \lambda = \lambda$. If $g \in {}^{<\omega}2$ and $B_g \not\subseteq \lambda$, we have $B_g \not\subseteq \delta$ for each $\delta < \lambda$, and so we have $\overline{\overline{B_g}} = \omega_1$, because $C \cap \lambda \neq \emptyset$, and $\lambda = \bigcup C \cap \lambda = \bigcup(\bigcup\{\,B_g \cap \delta \mid \delta \in C \cap \lambda\,\}) = \bigcup B_g \cap \lambda$. Thus C is closed.

It is also not difficult to see that C is also unbounded in ω_1. $\overline{\overline{{}^{<\omega}2}} = \omega$, so there is some $\delta < \omega_1$ such that $\bigcup(\bigcup\{\,B_g \mid \overline{\overline{B_g}} < \omega_1\,\}) < \delta$. Let $\{\,B_n \mid n < \omega\,\} = \{\,B_g \mid \overline{\overline{B_g}} = \omega_1\,\}$. So given $\alpha < \omega_1$ one can recursively choose $\beta_{\omega.n+m}$ to be the least member of B_m such that α, δ, and $\beta_{\omega.n'+m'} < \beta_{\omega.n+m}$ if $n' < n$ or $n' = n$ and $m' < m$. It is clear that $\alpha < \gamma = \bigcup\{\,\beta_{\omega.n+m} \mid n,\ m \in \omega\,\} \in C$.

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2 Stevo Todorčević, *Partitioning pairs of countable ordinals*, Acta Mathematica, **159** (1987), pp. 261-294. This great paper has many other extremely interesting facets. Highly recommended reading!

We shall show that $C \subseteq f``[Z]^2$. Suppose that $\delta \in C$ and $\beta \in Z$ with $\delta < \beta$. We shall find some $\alpha \leq \delta$ such that $f(\alpha, \beta) = \delta$.

Let $n = e_\beta(\delta)$ and $g = r_\beta \restriction n$. Then $\beta \in B_g$, and so $B_g \not\subseteq \delta$ and we have that $\overline{\overline{B_g}} = \omega_1$.

For each $\gamma \in B_g$ such that $\beta < \gamma$ let $m_\gamma = \sigma(\beta, \gamma)$ and $h_\gamma = r_\gamma \restriction (m_\gamma + 1)$. Note that $h_\gamma \restriction m_\gamma = g \restriction m_\gamma$, but $h_\gamma(m_\gamma) \neq g(m_\gamma)$. Because $\overline{\overline{B_g}} = \omega_1$ there is some subset $B \in [B_g]^{\omega_1}$ and some $m < \omega$ and $h : m \longrightarrow \omega$ such that $m = m_\gamma$ and $h = h_\gamma$ for each $\gamma \in B$. Thus $\overline{\overline{B_h}} = \omega_1$ and by the definition of C we have that $\delta = \bigcup \delta \cap B_h$.

Let $F = \{\epsilon < \delta \mid e_\beta(\epsilon) \leq m\}$. F is finite because e_β is an injection. So there is some $\alpha \in B_h \cap \delta$ such that $F \subseteq \alpha$. Now we have that $\alpha \in B_h$, so $h \subseteq r_\alpha$, and we have that $h \restriction m = r_\beta \restriction m$ and $h(m) \neq r_\beta(m)$. Hence $\sigma(\alpha, \beta) = m$. If $\alpha \leq \epsilon < \delta$ we have that $m < e_\beta(\epsilon)$, but $e_\beta(\delta) = m$. So δ is least such that $\alpha \leq \delta < \beta$ and $e_\beta(\delta) \leq m$. Thus $\delta = f(\alpha, \beta)$.

It would be very interesting to have similar results for graphs other than K_{ω_1}. For example András Hajnal and Péter Komjáth's, *Some remarks on the simultaneous chromatic number*, gives some information on this problem, but there is still lots of room for improvement on their results.

To take one example, let us say that $G = (X, A)$ has *chromatic number* ω_1 if ω_1 is the least cardinal κ such that there is some $f : \kappa \longrightarrow X$ such that $f(x) \neq f(y)$ if $\{x, y\} \in A$. It is clear that K_{ω_1} has chromatic number ω_1.

The principle $\diamondsuit^+$ says that there is some collection $\{B_{\alpha,n} \mid \alpha < \omega_1 \ \& \ n < \omega\}$ such that $B_{\alpha,n} \subseteq \alpha$ for each $\alpha < \omega_1$ and $n < \omega$, and if $B \subseteq \omega_1$, then there is some club set, C, such that

$$\forall \gamma \in C \, \exists n < \omega \, (B \cap \gamma = B_{\gamma,n}).$$

Thus $\diamondsuit^+$ says that there is a system $\{B_{\alpha,n} \mid \alpha < \omega_1 \ \& \ n < \omega\}$ which predicts *any* subset B of ω_1 correctly on at large (closed and unbounded) number of places. $\diamondsuit^+$ is a strengthening of (CH), and so is not provable in ZF (although it is true if $V = L$, for example).

[$\diamondsuit^+$ implies (CH) because if $x \subseteq \omega$ there is some C such that $\forall \gamma \in C \setminus \omega$ there is some $n < \omega$ with $x = x \cap \gamma = B_{\gamma,n}$. Thus $2^\omega = \overline{\overline{\mathcal{P}(\omega)}} \leq \overline{\overline{\{B_{\alpha,n} \mid \alpha < \omega_1 \ \& \ n < \omega\}}} = \omega_1$.]

Hajnal and Komjáth show that $\diamondsuit^+$ implies that if $G = (\omega_1, A)$ has chromatic number ω_1, then there is some $f : A \longrightarrow \omega_1$ such that if $X \in [\omega_1]^{\omega_1}$ and $(X, A \cap (X \times X))$ also has chromatic number ω_1 one has that $f``(A \cap (X \times X)) = \omega_1$. But, for example, it is still an open question as to whether one can prove in ZFC (alone) the following natural slightly weaker statement.

Whenever $G = (\omega_1, A)$ has chromatic number ω_1, there is some $f : A \longrightarrow \omega_1$ (with $f``A = \omega_1$) such that if $g : \omega_1 \longrightarrow \omega$ there is some $i \in \omega$ such that $f``A \cap g^{-1}``\{i\} = \omega_1$. (In words, if f is a colouring of G with ω_1 colours and g is a partition of the vertices of G into countably many disjoint sets then f takes all of the colours on one of the sets.)