Chapter 3. The Axiom of Choice.

Now we are going to deal with the last axiom in our system of set theory, the axiom of choice. We will show the equivalence between the axiom of choice and two other very useful principles.

Axiom of Choice. (AC) If x is a set of non-empty sets, then there is a function which picks a member of each of the sets which are elements of x: $f : x \longrightarrow \bigcup x \, \forall y \in x \, f(x) \in x$.

The Well-Ordering Principle. (WO) Let x be a set. Then there is a well-ordering of x.

Zorn's Lemma. (ZL) Let $(X, \leq)$ be a partially ordered set such that any subset C which is totally ordered by $\leq$ has an upper bound. Then X has a maximal element.

Definition. If $(X, \leq)$ is a partially ordered set a *chain* in $(X, \leq))$ is a subset of X which is totally ordered by $\leq$.

So Zorn's Lemma is often stated as "every poset in which every chain has an upper bound has a maximal element."

Theorem. (AE) $\Longleftrightarrow$ (WO) $\Longleftrightarrow$ (ZL).

Proof. (ZL) $\Longrightarrow$ (WO). Let X be a set and consider the set of well-orderings of subsets of X. This is a partially orderd set under the relation of being an initial segment: $(Y, \leq) \prec (Y', \leq')$ if and only if $(Y, \leq)$ is an initial segment of $(Y', \leq')$. It is clear that the union of any chain is an upper bound for the chain. So Zorn's Lemma says that there is a maximal element, say $(Z, \leq)$. If $Z \neq X$ we can choose $x \in X \setminus Z$ and define a well-ordering of $Z \cup \{x\}$ by putting $z < x$ for all $z \in Z$. So $Z = X$ and $\leq$ is a well-ordering of X.

(WO) $\Longrightarrow$ (AE). Let x be a set of non-empty sets. Let $<$ be a well-ordering of x. We can take $f(y) =$ the $<$-minimal element of y for each $y \in x$.

(AE) $\longrightarrow$ (ZL). Let $(X, \preceq)$ be a partially ordered set such that any chain has an upper bound. By the axiom of choice we can find a function $f : \mathcal{P}_{\neq\emptyset}(X) \longrightarrow X$ such that $\forall A \subseteq X \, (A \neq \emptyset \longrightarrow f(A) \in A)$. We will use f to help us choose by recursion a chain in $(X, \prec)$ whose upper bound will be maximal in x.

For any $B \subseteq X$ define $\uparrow(B) = \{x \in X \mid \forall b \in B \, b \prec x\}$. Let $\star$ be a set which is not a member of x. By the strong recursion theorem define a Function $F : \mathrm{On} \longrightarrow X \cup \{\star\}$ such that $F(\alpha) = f(\uparrow(\{F(\beta) \mid \beta < \alpha\}))$ if $\uparrow(\{F(\beta) \mid \beta < \alpha\}) \neq \emptyset$, and $F(\alpha) = \star$ otherwise.

If $\beta < \alpha$ and $F(\alpha) \neq \star$, then $F(\beta) \prec F(\alpha)$. Thus, if $\star \notin \mathrm{rge}(F)$ we would have that F would be an injection, and so $F^{-1} : \mathrm{im}(F) \longrightarrow \mathrm{On}$ would be a surjection. But $\mathrm{im}(F) \subseteq X(\cup\{\star\})$ is in any case a set. So by Replacement we would have that On would also be a set. A contradiction! So $\star \in \mathrm{im}(F)$.

Thus let α_0 be minimal such that $F(\alpha_0) = \star$. We have that $\{F(\beta) \mid \beta < \alpha_0\}$ is a chain in $(X, \prec)$ and $\uparrow(\{F(\beta) \mid \beta < \alpha_0\}) = \emptyset$. If x_0 is an upper bound of $\{F(\beta) \mid \beta < \alpha_0\}$ then x_0 is maximal in X. (x_0 is thus also the maximal element of $\{F(\beta) \mid \beta < \alpha_0\}$ and α_0 is a successor ordinal.)

Note. ZFC, ZF + AC, is the most important set theory for us, giving us the things that expect from set theory, such as that infinite cartesian products are non-empty and that every vector space has a basis. However studying ZF without AC but with some other axiom can be worthwhile. See, for example, Alexander Kechris, *Classical Descriptive Set Theory*, Springer, 1995, or Yiannis Moschovakis, *Descriptive Set Theory*, North Holland, 1980.