Chapter 2. Ordinals, well-founded relations.*

2.1. Well-founded Relations.

We start with some definitions and rapidly reach the notion of a well-ordered set.

Definition. For any X and any binary relation $<$ on X, $(X, <)$ is a *partially ordered set* (*p.o. set*) if $<$ is a transitive and anti-symmetric relation on X:

$$\forall x \in X x \not< x \text{ and } \forall x \in X\, \forall y \in X\, \forall z \in X\, (x < y \,\&\, y < z \longrightarrow x < z).$$

Let us say that $<$ *partially orders* X or $<$ is a *partial order* of X.

$(X, <)$ is a *linearly ordered set* (*l.o. set*) ou a *totally ordered set* (*t.o. set*) if $(X, <)$ is a partially ordered set and $\forall x \in X\, \forall y \in X\, (x < y$ or $x = y$ or $y < x)$ as well.

Definition. Now let $<$ be any binary relation on X. If $Y \subseteq X$, then $(Y, <)$ is an *initial segment* of $(X, <)$ if $\forall x \in X\, \forall y \in X\, (x < y \,\&\, y \in Y \longrightarrow x \in Y)$. Often we will just say that Y is an initial segment of X when it is clear about which relation $<$ we are talking. $(Y, <)$ is a *proper* initial segment of $(X, <)$ if $Y \neq X$ as well. For all $s \in X$ and $Y \subseteq X$, we say that s is *a successor* of Y if $s \notin Y$ and $\{y \in X \mid y < s\} \subseteq Y$. And if $Y \subseteq X$ and $y \in X$ we say that y is *$<$-minimal in* Y if $y \in Y$ and $\forall x \in X\, (x < y \longrightarrow x \notin Y)$.

Theorem. The following conditions are equivalent for any X and any binary relation on X.

(1) Each proper initial segment of X has a successor.
(2) Each non-empty subset of X has a $<$-minimal element.
(3) The principle of induction on $<$ is true:

$$\forall x \in X\, \big(\big(\forall y \in X\, (y < x \longrightarrow \phi(y))\big) \longrightarrow \phi(x)\big) \longrightarrow \forall x \in X\, \phi(x).$$

Proof. (2) $\Longrightarrow$ (3). Suppose that $\forall x \in X\, \big(\big(\forall y \in X\, (y < x \longrightarrow \phi(y))\big) \longrightarrow \phi(x)\big)$ but $\neg \forall x \in X\, \phi(x)$. Then $\{x \in X \mid \neg\phi(x)\}$ is non-empty and, by (2), has a member, let us say x_0, which is $<$-minimal. But then we have that $\forall y < x_0\, \phi(y)$ and hence $\phi(x)$, a contradiction.

(3) $\Longrightarrow$ (2). If $Y \subseteq X$ and Y does not have a $<$-minimal element, then we have that

$$\forall x \in X\, (\forall y \in X (y < x \,\&\, y \notin Y)) \longrightarrow x \notin Y.$$

Applying (3), we have that $\forall x \in X\, x \notin Y$. Thus Y is empty.

(2) $\Longrightarrow$ (1). If Y is a proper initial segment of X, then $X \setminus Y$ $(= \{x \in X \mid x \notin Y\})$ is non-empty, and (2) says that there is a member, let us call it s, that is $<$-minimal. Hence s is a successor of Y.

(1) $\Longrightarrow$ (2). For all $x, y \in X$ let $x <^* y$ if there are $x_0, \ldots, x_{n-1} \in X$ and $x < x_0 < \ldots < x_{n-1} < y$. $<^*$ is the transitive closure of $<$: it is transitive, $< \subseteq <^*$ and if R is a transitive binary relation with $< \subseteq R$, then $<^* \subseteq R$.

(In order to see that $<^*$ is transitive take $x <^* y$ and $y <^* z$ and suppose that we have this because $x < x_0 < \ldots < x_{n-1} < y$ and $y < y_0 < \ldots < y_{m-1} < z$ for some $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1} \in X$. Thus, $x < x_0 < \ldots < x_{n-1} < y < y_0 < \ldots < y_{m-1} < z$, and so $x < z$. It is clear that $x < y \Longrightarrow x <^* y$: take $n = 0$ in the definition of $<^*$. And if $x <^* y$ there are $x_0 < \ldots < x_{n-1}$ with

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$x < x_0 < \ldots < x_{n-1} < y$, so if $<\subseteq R$ we have $x\, R\, x_0\, R\, \ldots\, Rx_{n-1}\, Ry$, and hence, if R is transitive, we have $x\, R\, y$.)

Now suppose that $Y \subseteq X$ and Y is non-empty. $\{x \in X \mid x \notin Y \,\&\, \forall y \in X\, (y <^* x \longrightarrow y \notin Y)\}$, $= Z$, let us say, is an initial segment and is proper. If s is a successor of Z we have that $s \in Y$ and s is $<$-minimal in Y.

Definition. A binary relation $<$ on a set X is *well-founded* if satisfies the conditions (1)-(3) of the above theorem. A well-founded total order is called well-ordered or a well-ordering. $(X, <)$ is then a *well-ordered set* (a *w.o. set*).

2.2 Ordinals.

Thus, a well-ordered set is a set along which we can "count" one step at a time. The ordinals are well-ordered sets that will be crucial for us. In fact we will see later that any well-ordered set is isomorphic to some ordinal via an isomorphism which is order-preserving, hence one can say that if we want to "count" we have to use the ordinals.

Definition. An *ordinal* is a transitive set that is well-ordered by $\in$. (Frequently we will write $<$ instead of $\in$ when we are dealing with ordinals.)

Examples. 0, 1, 2, 3, $\ldots$, ω, $\ldots$ are (the first) ordinals. Exercise: if this is not apparent to you then check it.

Proposition. If α is an ordinal and $z \in \alpha$, then z is a transitive set and hence is a ordinal as well.

Proof. Let us suppose that $x \in y \in z \in \alpha$. We have to show that $x \in z$. But α is transitive, so $y \in \alpha$, and thus $x \in \alpha$. α is also well-ordered by $\in$, hence $\in$ is a transitive relation on α. Because we have $x \in y$ and $y \in z$, one has $x \in z$.

Definition. If α is an ordinal define $s(\alpha) = \alpha \cup \{\alpha\}$.

Proposition. If α is an ordinal, then $s(\alpha)$ is an ordinal as well.

Proof. If $z \in y \in s(\alpha)$, then either $z \in y \in \alpha$ or $z \in y \in \{\alpha\}$. In the first case $z \in \alpha$ because $\in$ is transitive on α, and in the second case $z \in y = \alpha$. Hence, in any caso $z \in s(\alpha)$. Thus $s(\alpha)$ is transitive. And it is clear that $s(\alpha)$ is well-ordered by $\in$.

Proposition. If $x \neq \emptyset$ is a set of ordinals, then $\bigcap x$ is also an ordinal.

Proof. It is clear that any intersection of transitive sets is transitive. And if $\alpha \in x$, then $\bigcap x \subseteq \alpha$ and must be well-ordered by $\in$ α is well-ordered by $\in$.

Proposition. If α, β are ordinals and $\alpha \subseteq \beta$, then (a) α is an initial segment of β, and (b) $\alpha \neq \beta \Longrightarrow \alpha \in \beta$.

Proof. (a) If $\gamma \in \beta$, $\delta \in \alpha$ and $\gamma \in \delta$ (*i.e.*, γ comes before δ in the well-order of β by $\in$), so $\gamma \in \delta \in \alpha$, and thus $\gamma \in \alpha$ by the transitivity of $\in$ on α.

(b) If $\alpha \neq \beta$, then, by the equivalent of being well-founded, there is a sucessor, let us call it s, of α as a subset of β. Note that s is unique because $\in$ is a l.o. of β. Also, because $\in$ is a l.o. of β, $\gamma \in \alpha$ implies that $\gamma \in s$, thus $\alpha \subseteq s$. On the other hand, $S = \alpha \cup \{s\}$ is an initial segment of β and, thus, it is transitive and, so, for any $\gamma \in s (\in S)$ we have $\gamma \in S$. Because $s \notin s$ (by the axiom of foundation), $\gamma \in s$ implies $\gamma \in \alpha$ and $s \subseteq \alpha$. Hence $\alpha = s$, $\in \beta$.

2.3. Trichotomy.

Theorem. If α and β are ordinals one and exactly one of the following holds: $\alpha \in \beta$ or $\alpha = \beta$ or $\beta < \alpha$.

Proof. $\alpha \cap \beta = \alpha$ or $\alpha \cap \beta \subset \alpha$ and $\alpha \cap \beta \neq \alpha$. Similarly $\alpha \cap \beta = \beta$ or $\alpha \cap \beta \subset \beta$ and $\alpha \cap \beta \neq \beta$, giving four cases for us to consider.

(i) $\alpha \cap \beta = \alpha$ and $\alpha \cap \beta = \beta$. Then $\alpha = \beta$.

(ii) $\alpha \cap \beta = \alpha$ and $\alpha \cap \beta \subset \beta$ and $\alpha \cap \beta \neq \beta$. Then the last proposition of §2.2 immediately gives that $\alpha \in \beta$.

(iii) $\alpha \cap \beta = \beta$ and $\alpha \cap \beta \subset \alpha$ and $\alpha \cap \beta \neq \alpha$. Then the last proposition of §2.2 gives that $\beta \in \alpha$.

(iv) $\alpha \cap \beta \subset \alpha, \beta$, $\alpha \cap \beta \neq \alpha$ and $\alpha \cap \beta \neq \beta$. Note that we have already shown that $\alpha \cap \beta$ is an ordinal. Then the last proposition of §2.2 gives that $\alpha \cap \beta \in \alpha$ and that $\alpha \cap \beta \in \beta$. Hence$\alpha \cap \beta \in \alpha \cap \beta$, a contradiction! (To the axiom of foundation.)

Corollary. For all ordinals α and β, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Proof. $\alpha \in \beta \implies \alpha \subseteq \beta$ because β is transitive; $\beta \in \alpha \implies \beta \subseteq \alpha$ because α is transitive; $\alpha = \beta \implies \alpha \subseteq \beta \,\&\, \beta \subseteq \alpha$.

Notation. We write $\alpha \leq \beta$ if $\alpha \in \beta$ or $\alpha = \beta$.

Corollary. The collection of all ordinals is well-ordered by $\in$.

Proof. It is easy to see that the collection is partially ordered by $\in$ by the transitivity of ordinals. The trichotomy theorem gives that $\in$ totally orders the collection. Hence what remains to be shown is that $\in$ is well-founded on the collection.

Suppose that Z is a subset of the collection of all ordinals and $Z \neq \emptyset$. Let $\alpha \in Z$. Either α is $\in$-minimal (when we are finished), or $Y = \alpha \cap Z \neq \emptyset$. Y is a non-empty subset of α, hence there is some $\beta \in Y$ that is $\in$-minimal. But β must be $\in$-minimal in Z as well, because if $\gamma \in \beta$ we have that $\beta \in \alpha$ and, by the transitity of α, $\gamma \in \alpha$. Hence $\gamma \in Z$.

Corollary. The collection of all ordinals is not a set: it is a proper class.

Proof. Suppose the collection was a set, x, let us say. Then we would have that $x \in x$ by the previous corollary and the definition of ordinal. This would contradict the axiom of foundation (again).

This corollary is sometimes called the *Burali-Forti* paradox. As you've already see, as with Cantor's paradox, this is not in fact a true paradox. Again, as with V, it is convenient to have a name for the (proper) class of all ordinals, and so we write *On* for this class.

Corollary. If x is a set of ordinals, then $\bigcup x$ is also an ordinal.

Proof. It is clear that $\bigcup x$ is transitive. And because $\bigcup x \subseteq \mathrm{On}$, trichotomy gives that $\in$ totally orders $\bigcup x$. So, suppose that $\emptyset \neq Z \subseteq \bigcup x$. Take $\alpha \in Z$. As in the corollary two ago, either α already is $\in$-minimal, or $\alpha \cap Z$ has a member that is $\in$-minimal and that is $\in$-minimal in Z as well.

2.4. Successors and limits. Let α be an ordinal. There are two things that can happen.

Either there is a $\in$-maximal element of α, *i.e.*, a $\beta \in \alpha$ such that $\forall \gamma \in \alpha\, \gamma \leq \beta$. In this case $\alpha = \{\beta\} \cup \{\gamma \mid \gamma < \beta\}, = \{\beta\} \cup \beta, = s(\beta)$. If there is some β such that $\alpha = s(\beta)$ we say that α is a *successor ordinal*.

Or there is no $\in$-maximal element (at the top) of α, *i.e.*, $\forall \beta \in \alpha\, \exists \gamma \in \alpha\, \beta < \gamma$. In this case $\alpha = \{\beta \mid \beta < \alpha\} = \{\beta \mid \exists \gamma \in \alpha\, \beta \in \gamma\} = \bigcup \alpha$. If $\alpha = \bigcup \alpha$ we say that α is a *limit ordinal*.

We must check that if $\alpha = s(\beta)$, then $\beta = \bigcup \alpha$.

Examples. 1, 2, 3, ... are successor ordinals. ω is a limit ordinal. (And by our definition, 0 is a limit ordinal as well, albeit that this is a special caso and is typically treated separately.)

In general, if x is a set of ordinals, $\bigcup x$ is the supremum of the ordinals in x, and if x is non-empty, $\bigcap x$ is the infimum of the ordinals in x, and also the minimal element de x. (This latter is true because we have already seen that x has $\in$-minimal member when $\emptyset \neq x \subseteq \mathrm{On}$, α, let us say, and so $\forall \beta \in x\, \alpha \leq \beta$ – *i.e.*, $\forall \beta \in x\, \alpha \in \beta$ or $\alpha = \beta$, whence $\alpha = \bigcap x$.)

2.5. Recursion Theorems. Now we want to define + and . on the ordinals, in a similar manner to that in which we did on the natural numbers. But before we can do this we need to know that we can make definitions using recursion and how to make these defintions. So this section is devoted to that theme. Perhaps at first glance it will seem very technical, but it will be highly useful in the end.

Weak recursion theorem. Let X, Y be two sets. Let $<$ be a well-founded relation on X, and let $g : X \times \mathcal{P}(Y) \longrightarrow Y$ be a function. Then there is a unique function $f : X \longrightarrow Y$ such that $f(x) = g(x, \{f(z) \mid z < x\})$ for all $x \in X$.

Proof. (Part 1.) We say that $\phi : Z \longrightarrow Y$ is an *attempt* if Z is an initial segment of X and $\phi(x) = g(x, \{\phi(z) \mid z < x\})$ for all $x \in Z$. If $\phi_0 : Z_0 \longrightarrow Y$ and $\phi_1 : Z_1 \longrightarrow Y$ are two attempts, they agree on $Z_1 \cap Z_2$: $\phi_1 \restriction Z_1 \cap Z_2 = \phi_2 \restriction Z_1 \cap Z_2$. Because $X_1 \cap X_2$ is clearly an initial segment of X and if as attempts they do not agree there is some x_0 which is $<$-minimal in $\{x \in Z_1 \cap Z_2 \mid \phi_1(x) \neq \phi_2(x)\}$. But in that case we have $\phi_1(x_0) = g(x_0, \{\phi_0(x) \mid x < x_0\}) = g(x_0, \{\phi_1(x) \mid x < x_0\}) = \phi_2(x)$, a contradiction!

(Part 2.) Now let $f = \bigcup\{\phi \mid \phi \text{ is an attempt}\}$, so $f(x) = \phi(x)$ if there is an attempt ϕ with $x \in \mathrm{dom}(\phi)$. It is clear that $\mathrm{dom}(f) = \bigcup\{\mathrm{dom}(\phi) \mid \phi \text{ is an attempt}\}$ is an initial segment of X, and that f satisfies $f(x) = g(x, \{f(z) \mid z < x\})$ for all$x \in \mathrm{dom}(f)$. So all that remains to be shown is that $\mathrm{dom}(f) = X$.

But if $\mathrm{dom}(f) \neq X$ there is some x_0 that is $<$-minimal in $\{x \in X \mid x \notin \mathrm{dom}(f)\}$. For such an x define $\bar{f} : \mathrm{dom}(f) \cup \{x_0\} \longrightarrow Y$ by $\bar{f}(x) = f(x)$ if $x \in \mathrm{dom}(f)$ and $\bar{f}(x_0) = g(x_0, \{f(x) \mid x < x_0\})$. Then one would clearly have that $\bar{f}$ is an attempt, but is not a subset of f. contradiction! Hence $f : X \longrightarrow Y$ as we want.

Now we shall see a stronger recursion theorem which deals with classes.

Definition. Let A and B be classes. Thus $A \times B = \{(x, y) \mid x \in A \,\&\, y \in B\}$ is a class as well, because if $A = \{x \mid \phi(x, \bar{a})\}$ and $B = \{x \mid \psi(x, \bar{b})\}$ for two formulas $\phi(x, \bar{a})$ and $\psi(x, \bar{b})$ in LTC, then $A \times B = \{z \mid z = \{\{x\}, \{x, y\}\} \,\&\, \phi(x, \bar{a}) \,\&\, \phi(y, \bar{b})\}$.

Let A be a class. $<$ is a *well-founded Relation* (note the capital "R") on A if $<$ is a subclass do $A \times A$, $\forall a \in A\, \{b \in A \mid b < a\}$ is a set, and any B with $\emptyset \neq B \subseteq A$ has $<$-minimal elements.

Let A, B be classes. $G : A \longrightarrow B$ is a *Function* (note the capital "F") if G is a subclass of $A \times B$

and for each $x \in A$ there is a unique $y \in B$ such that $(x, y) \in G$. If $x \in A$, $y \in B$ and $(x, y) \in G$ we shall write (naturally) $G(x) = y$. (Note that to say that G is a (sub)class means that there is a formula $\chi(z, \bar{c})$ in LTC such that $x \in G$ if and only if $\chi(x, \bar{c})$ is true.)

Remember that sets are classes as well and if A and B are sets, then $A \times B$ is a set, $<$ is a well-founded relation and G is a function, respectively.

Strong recursion theorem. Let $<$ be a well-founded Relation on a class A and $G : A \times V \longrightarrow V$ a Function. Then there is a unique Function $F : A \longrightarrow V$ such that for all $x \in A$ we have that $F(x) = G(x, \{ F(y) \mid y < x \})$.

Proof. (Part 1.) The first part is just as in the weak recursion theorem. Define ϕ to be an attempt if $\phi : B \longrightarrow V$, where B is a *set* that is an initial segment of A and $\phi(x) = G(x, \{ \phi(y) \mid y < x \})$. Then any two attempts agree on the intersection of their initial segments.

(Part 2.) Let $F = \bigcup \{ \phi \mid \phi \text{ is an attempt} \}$ (that is $(x, z) \in F$ if there is an attempt ϕ with $\phi(x) = z$). Note that the last time this was a set, but here we merely have that $F \subseteq \mathcal{P}(A \times V)$ and we do not know that F is a set, since it will not be one if A is not. On the other hand we have that $F(x) = G(x, \{ F(y) \mid y < x \})$ for all $x \in \text{dom}(F)$. Suppose that $\text{dom}(F) \neq A$, and choose a_0 that is $<$-minimal in the set $A \setminus \text{dom}(F)$.

Now we have that $\forall b < a_0 \, \exists \phi \, (\phi \text{ is an attempt and } b \in \text{dom}(\phi)$. By this we can apply the Axiom of Collection which gives that $\exists v \, \forall b < a_0 \, \exists \phi \in v \, (\phi \text{ is an attempt and } b \in \text{dom}(f)$. Hence $w = v \cap \{ \phi \mid \phi \text{ is anattempt} \}$ is a *set* by the Axiom of Comprehension. Clearly $f = \bigcup w$ is an attempt and we can consider $\bar{f} = f \cup \{ (a_0, G(a_0, \{ f(b) \mid b < a_0 \})) \}$ which is an attempt but is not a subset do f. Thus, as in the weak recursion theorem we have a contradiction! Hence$\text{dom}(F) = A$.

Why are there differences in the proof from that of the weak recursion theory? Well, in order to imitate that proof you would have to know that $G(a_0, \{ F(a) \mid a < a_0 \})$ makes sense before defining $\bar{F} = F \cup \{ (a_0, G(a_0, \{ F(a) \mid a < a_0 \})) \}$. And in order to know this you would have to know that $\{ F(a) \mid a < a_0 \}$ is a set, which is a consequence of applying Replacement to the set $\{ a \mid a < a_0 \}$. Hence one sees that it is not possible to avoid the use of Collection or Replacement.

I ought to warn that one cannot avoid the use of Collection in the strong recursion theorem even if A is a set if G is not one, and this is also so even if $<$ is a well-order A and rather than just a well-founded relation.

2.6 Comparison theorems.

Definition. A function $f : (X, <_X) \longrightarrow (Y, <_Y)$ is an isomorphism which preserves order if f is a bijetion between X and Y and $\forall x,\, y \in X \, (x <_X y \longleftrightarrow f(x) <_Y f(y))$.

Note. If $f : (X, <_X) \longrightarrow (Y, <_Y)$ is a function that preserves order f is injective, and so you only need to know that f is also a surjection in order to have that f is an isomorphism.

Theorem. For any well-ordered set $(X, <)$ there is a unique ordinal α for which there is an (unique) isomorphism from $(X, <)$ to α which preserves order.

Proof. Define, by the (weak) recursion theorem, a function $f : X \longrightarrow V$ (On) which satisfies $f(x) = \{ f(y) \mid y < x \}$.

(i) $\forall x \in X \; f(x) \in \text{On}$. If not choose some x_0 which is $<$-minimal in the set $\{ x \in X \mid f(x) \notin \text{On} \}$. Suppose that $x \in X$ and for all $y < x$ we have that $f(y) \in \text{On}$. Then $f(x) \subseteq \text{On}$ by the definition of f, and $f(x)$ is well-ordered by $\in$. But also, if $z \in y \in f(x)$, we have that $y = f(v)$ for some

$v < x$. Thus $z \in \{ f(u) \mid u < v \}$ and there is some $u < v < x$ such that $z = f(u)$. Consequently, $z \in f(x)$, and $f(x)$ is transitive. Hence $f(x) \in \mathrm{On}$. So, by the principle of indution on $<$, we have that $\forall x \in X\ f(x) \in \mathrm{On}$.

(ii) $\mathrm{im}(f)$ is a set (by the axioms of collection and comprehension or by replacement), and (i) shows that $\mathrm{im}(f) \subseteq \mathrm{On}$, thus $\mathrm{im}(f)$ is well-ordered by $\in$. Suppose that $z \in y \in \mathrm{im}(f)$, hence there is some $x \in X$ such that $z \in y = f(x)$, and so there is some $v \in X$ such that $v < x$ and $z = f(v)$. This gives that $z \in \mathrm{im}(f)$ and $\mathrm{im}(f) \in \mathrm{On}$.

Finally if $g : X \longrightarrow \beta$ is another isomorphism which preserves order, then $f \cdot g^{-1}$ would be an isomorphism from β to α which preserves the order given by $\in$. Thus $\beta = \alpha$ and $f \cdot g^{-1}$ is equal to the identity function by (the first corollary to) trichotomy.

Comparison Theorem. Let $(X, <_X)$ and $(Y, <_Y)$ be two w.o.sets. Then there is an isomorphism that preserves order between one of them and an initial segment of the other.

Proof. The previous theorem says that there are α, $\beta \in \mathrm{On}$ and isomorphisms $f : (X, <_X) \longrightarrow \alpha$ and $g : (Y, <_Y) \longrightarrow \beta$ which preserve order. Now we apply trichotomy. If $\alpha = \beta$ we have that $g^{-1} \cdot f$ is an order-preserving isomorphism between $(X, <_X)$ and $(Y, <_Y)$. If $\alpha \in \beta$ we have that $g^{-1} \cdot f$ is an order-presrving isomorphism between $(X, <_X)$ and a proper initial segment of $(Y, <_Y)$ (of which $g^{-1}(\alpha)$ is a successor). And if $\beta \in \alpha$ we have that $f^{-1} \cdot g$ is an isomorphism which preserves order between $(Y, <_Y)$ and a proper initial segment of $(X, <_X)$ (of which $f^{-1}(\beta)$ is a successor).

2.7. The V_α hierarchy.

Now we can rigorously define (from the axioms) the V_α hierarchy. Define, by the strong recursion theorem, a Function $V : \mathrm{On} \longrightarrow V$ (the collection of all sets) such that $V_\alpha = \bigcup\{ \mathcal{P}(V_\beta) \mid \beta < \alpha \}$.

Proposition. $\forall \alpha\, V_\alpha$ is transitive.

Proof. If not choose α ($<$-)minimal such that V_α not transitive. We have already seen (in the first proposition of §1.5) that $\mathcal{P}(X)$ is transitive if X is transitive. So $\forall \beta < \alpha\ \mathcal{P}(V_\beta)$ is transitive. But $V_\alpha = \bigcup\{ \mathcal{P}(V_\beta) \mid \beta < \alpha \}$ is a union of transitive sets and so is also transitive.

Proposition. If $\alpha \leq \beta$ we have that $V_\alpha \subseteq V_\beta$.

Proof. If $\alpha = \beta$ it is trivial. If $\alpha < \beta$, then $V_\alpha \subseteq \mathcal{P}(V_\alpha)$, because V_α and, so, $\mathcal{P}(V_\alpha)$ are transitive, and $\mathcal{P}(V_\alpha) \subseteq V_\beta$.

Corollary. If α is a successor ordinal, then $V_{s(\alpha)} = V_{\alpha+1} = \mathcal{P}(V_\alpha)$.

Proof. $V_{s(\alpha)} = \bigcup\{ \mathcal{P}(V_\beta) \mid \beta \leq \alpha \} = \bigcup\{ \mathcal{P}(V_\beta) \mid \beta \leq \alpha \} \cup \mathcal{P}(V_\alpha)$. But $V_\beta \subseteq V_\alpha$ if $\beta < \alpha$, and so $\mathcal{P}(V_\beta) \subseteq \mathcal{P}(V_\alpha)$ if $\beta < \alpha$. Hence $\bigcup\{ \mathcal{P}(V_\beta) \mid \beta \leq \alpha \} \subseteq \mathcal{P}(V_\alpha)$.

Corollary. If λ is a limit ordinal then $V_\lambda = \bigcup\{ V_\alpha \mid \alpha < \lambda \}$.

Proof. $\bigcup\{ V_\alpha \mid \alpha < \lambda \} \subseteq V_\lambda = \bigcup\{ \mathcal{P}(V_\alpha) \mid \alpha < \lambda \} = \bigcup\{ V_{s(\alpha)} \mid \alpha < \lambda \} \subseteq \bigcup\{ V_\alpha \mid \alpha < \lambda \}$ because $s(\alpha) < \lambda$ if $\alpha < \lambda$.

Theorem. The axiom of foundation is equivalent to "$\forall x\, \exists \alpha \in \mathrm{On}\, x \in V_\alpha$."

Proof. "$\Longrightarrow$". If not let x_0 be $\in$-minimal such that $\forall \beta \in \mathrm{On}\ x \notin V_\beta$. So $\forall x \in x_0\, \exists \beta\, x \in V_\beta$. If we apply the axiom of Collection we have $\exists v\, \forall x \in x_0\, \exists \beta \in v\, x \in V_\beta$. Let $u = v \cap \mathrm{On}$. Then $\forall x \in x_0\, \exists \beta \leq \sup(u)\, x \in V_\beta$, and this gives that $\forall x \in x_0\, x \in V_{\sup(u)}$. It follows that $x_0 \subseteq V_{\sup(u)}$, and so that $x_0 \in V_{s(\sup(u))}$. contradiction!

"$\Longleftarrow$". Let $A \neq \emptyset$ be a class with $A \subseteq \bigcup\{ V_\alpha \mid \alpha \in \mathrm{On} \}$. Take α minimal such that $A \cap V_\alpha \neq \emptyset$. Then any $a \in A \cap V_\alpha$ is $\in$-minimal in A.

Definition. Define a function $\mathrm{rk} : V \longrightarrow V$ (On) by $\mathrm{rk}(x) = \bigcup\{ s(\mathrm{rk}(y)) \mid y \in x \}$, using the (strong) recursion theorem.

Proposition. (i) $\forall x \in V$ $\mathrm{rk}(x) \in \mathrm{On}$. (ii) $\mathrm{rk}(x) =$ the minimal α such that $x \subseteq V_\alpha$. (iii) $\mathrm{rk}(x)+1 =$ the minimal α such that $x \in V_\alpha$.

Proof. Exercise. (Use induction over $(V, \in)$.)

2.8. Transitive closures.

For each set define, by recursion over $(\omega, \leq)$, a function $t(n, x)$ such that

$$t(0, x) = x \text{ and } t(n+1, x) = t(n, x) \cup \bigcup t(n, x).$$

At the end, define $\mathrm{TC}(x) = \bigcup\{ t(n, x) \mid n \in \omega \}$. $\mathrm{TC}(x)$ is the *transitive closure* of x

Proposition. For every x, $\mathrm{TC}(x)$ is the minimal transitive set z with $x \subseteq z$. (That is, if y is transitive and $x \subseteq y$, then $\mathrm{TC}(x) \subseteq y$.)

Proof. Let $u \in v \in \mathrm{TC}(x)$. Then there is some $n < \omega$ such that $u \in v \in t(n, x)$. So $u \in \bigcup t(n, x) \subseteq t(n+1, x) \subseteq \mathrm{TC}(x)$. Thus $\mathrm{TC}(x)$ is transitive.

Let y be transitive $x \subseteq y$. We shall show by induction on ω that each $t(n, x) \subseteq y$. This is clear when $n = 0$. Suppose that $t(n, x) \subseteq y$. If $z \in t(n+1, x)$ we either have $z \in t(n, x)$, when $z \in y$, or $z \in \bigcup t(n, x)$. So there is some $v \in t(n, x)$ such that $z \in v$. But if $z \in v \in t(n, x) \subseteq y$, we have that $z \in y$ by the transitivity of y. So we have $t(n+1, x) \subseteq y$, and thus $\mathrm{TC}(x) = \bigcup t(n, x) \subseteq y$.

Proposition. The axiom of foundation for sets (each non-empty set has a $\in$-minimal element) implies the axiom of foundation for classes (each non-empty class has a $\in$-minimal element).

Proof. Let A, $\neq \emptyset$, be a class. Let $x \in A$. Either x is $\in$-minimal (when we are done) or $x \cap A \neq \emptyset$. Then $\mathrm{TC}(x) \cap A \neq \emptyset$. Let y be $\in$-minimal in the *set* $\mathrm{TC}(x) \cap A$. (We know that it is a set by the previous proposition.) We have that $y \in A$. Now suppose that $z \in y$. Then $z \notin \mathrm{TC}(x) \cap A$. But $z \in y \in \mathrm{TC}(x)$ and $\mathrm{TC}(x)$ is transitive, so $z \in \mathrm{TC}(x)$. Thus $z \notin A$, and we have shown that y is $\in$-minimal in the class A.

2.9. Ordinal arithmetic: addition of ordinals.

We will use recursion to define addition and mutlipication on On.

Definition. Define, for α fixed and β variable, $\alpha + \beta = \alpha \cup \{ \alpha + \gamma \mid \gamma < \beta \}$

Proposition. Let α, $\beta \in \mathrm{On}$. Then $\alpha + \beta \in \mathrm{On}$.

Proof. If not, let β_0 be minimal with $\alpha + \beta_0 \notin \mathrm{On}$. Then for all $\gamma < \beta_0$ we have that $\alpha + \gamma \in \mathrm{On}$, so $\alpha + \beta_0 = \alpha \cup \{ \alpha + \gamma \mid \gamma < \beta_0 \} \subseteq \mathrm{On}$ and, hence, is well-ordered by $\in$.

Now we will show that $\alpha + \beta_0$ is transitive, and so is an element of On, giving a contradiction to the supposition. Let $z \in y \in \alpha \cup \{ \alpha + \gamma \mid \gamma < \beta_0 \}$. Then, either $y \in \alpha$, when $z \in \alpha$, because α is an ordinal, or there is some $\gamma < \beta$ such that $y = \alpha + \gamma \in \mathrm{On}$, hence, again, $z \in \{\alpha + \delta \mid \delta < \gamma\}$. In each case we have that $z \in \alpha + \{\alpha + \gamma \mid \gamma < \beta_0\}$, giving the desired contradiction. So $\alpha + \beta_0 \in \mathrm{On}$.

Lemma. $\forall \alpha, \beta, \gamma, \delta \in \mathrm{On}$: (i) $\beta < \gamma \Longrightarrow \alpha + \beta < \alpha + \gamma$; (ii) $\alpha \leq \delta \Longrightarrow \alpha + \beta \leq \delta + \beta$.

Proof. (i) If $\beta < \gamma$, then $\alpha + \beta \in \alpha + \gamma$.
(ii) Suppose that the Lemma is false for some pair α and δ, and let β_0 be minimal such that $\alpha + \beta \not\leq \delta + \beta$. Then $\forall \beta < \beta_0$ we have that $\alpha + \beta \leq \delta + \beta$ and so either $\alpha + \beta = \delta + \beta$ or $\alpha + \beta < \delta + \beta$. In the latter case either there is some $\gamma < \beta$ such that $\alpha + \beta = \delta + \gamma$, or $\alpha + \beta \in \delta$. So $\alpha + \beta_0 = \alpha \cup \{\alpha + \beta \mid \beta < \beta_0\} \subseteq \delta \cup \{\delta + \gamma \mid \gamma < \beta_0\} = \delta + \beta_0$.

Proposition. (i) $\alpha + 0 = \alpha$; (ii) $\alpha + s(\beta) = s(\alpha + \beta)$; (iii) if λ is a limit ordinal, then $\alpha + \lambda = \bigcup\{\alpha + \gamma \mid \gamma < \lambda\}$.

Proof. (i) Trivial.
(ii) $\alpha + s(\beta) = \alpha \cup \{\alpha + \gamma \mid \gamma < s(\beta)\} = \alpha \cup \{\alpha + \gamma \mid \gamma < s(\beta)\} = \alpha \cup \{\alpha + \gamma \mid \gamma < \beta\} \cup \{\alpha + \beta\} = (\alpha + \beta) \cup \{\alpha + \beta\} = s(\alpha + \beta)$.
(iii) if $x \in \bigcup\{\alpha + \gamma \mid \gamma < \lambda\}$ then there is some $\gamma < \lambda$ such that $x \in \alpha \cup \{\alpha + \beta \mid \beta < \gamma\}$. Thus $x \in \alpha \cup \{\alpha + \beta \mid \beta < \lambda\}$. But we know that $\lambda = \bigcup \lambda = \bigcup\{\gamma \mid \gamma < \lambda\}$, hence if $x \in \alpha \cup \{\alpha + \beta \mid \beta < \lambda\}$ either $x \in \alpha$ or $x = \alpha + \beta$ for some $\beta < \lambda$ and there is some $\gamma < \lambda$ such that $\beta < \gamma$. Hence $x \in \bigcup\{\alpha + \gamma \mid \gamma < \lambda\}$.

Theorem. Let $\alpha, \beta \in \mathrm{On}$ and let $\prec$ be the anti-lexicografic order on $\alpha \times \{0\} \cup \beta \times \{1\}$ defined by $(\gamma, i) \prec (\delta, j)$ if and only if either $i < j$ or $i = j$ and $\gamma < \delta$. So this order is isomorphic to $(\alpha + \beta, \in)$

Proof. Define a function $f : \alpha \times \{0\} \cup \beta \times \{1\} \longrightarrow \alpha + \beta$ by $f(\gamma, 0) = \gamma$ and $f(\delta, 1) = \alpha + \delta$. It is clear that f is surjective and also that $\prec$ is a total order on $\mathrm{dom}(f)$. We shall show that $(\gamma, i) \prec (\delta, j)$ implies $f(\gamma, i) < f(\delta, j)$.

If $i < j$ then $i = 0$ and $j = 1$, $f(\gamma, i) = \gamma$ and $f(\delta, j) = \alpha + \delta$, and we have that $\gamma < \alpha \leq \alpha + \delta$. If $i = j = 0$, then $f(\gamma, i) = \gamma$ and $f(\delta, j) = \delta$ and we have $\gamma < \delta$. And if $i = j = 1$ we have that $\alpha + \gamma < \alpha + \delta$ by the previous lemma.

Note. $1 + \omega = \omega$, while $\omega + 1 = s(\omega)$, so $+$ is non-comutative in general.

INSERT Pictures of $1 + \omega$ and $\omega + 1$.

Proposition. If $x \subseteq \mathrm{On}$ is a non-empty set then $\alpha + \bigcup x = \bigcup\{\alpha + \beta \mid \beta \in x\}$. (+ is continuous in the second place/variable.)

Proof. If $\beta \in x$ then $\beta \leq \bigcup x$, and so $\forall \beta \in x\, (\alpha + \beta \leq \alpha + \bigcup x)$. Hence $\bigcup\{\alpha + \beta \mid \beta \in x\} \subseteq \alpha + \bigcup x$.

Let $\rho = \bigcup\{\alpha + \beta \mid \beta \in x\}$ If $\rho \in \alpha + \bigcup x$, then either $\rho \in \alpha$ or there is some $\gamma \in \bigcup x$ such that $\rho = \alpha + \gamma$. If $\rho \in \alpha$, then $\forall \beta \in x\, \alpha + \beta \leq \rho < \alpha$, a contradiction. And if $\rho = \alpha + \gamma$ with $\gamma \in \bigcup x$, then there is some $\beta \in x$ such that $\gamma < \beta$ and so $\rho < \alpha + \beta$ and $\rho < \bigcup\{\alpha + \beta \mid \beta \in x\} = \rho$, a contradiction again. So, $\rho \notin \alpha + \bigcup x$ and we have that $\rho = \alpha + \bigcup x$.

We can define subtraction as well, but because addition is not commutativity it is not as useful as addition.

Definition. $\alpha - \beta = \{\gamma \mid \beta + \gamma < \alpha\}$. Alternatively one can define $\alpha - \beta$ by recursion: $\alpha - \beta = 0$

if $\beta > \alpha$ and $\alpha - \beta = \{\alpha - \gamma \mid \gamma < \beta\}$ if $\beta \leq \alpha$.

Proposition. Let α, $\beta \in \mathrm{On}$ and $\beta \leq \alpha$. Then $\beta + (\alpha - \beta) = \alpha$.

Proof. $\beta + (\alpha - \beta) = \beta \cup \{\beta + \gamma \mid \gamma < \alpha - \beta\} = \beta \cup \{\beta + \gamma \mid \beta + \gamma < \alpha\} \subseteq \alpha$. Suppose that $\beta + (\alpha - \beta) < \alpha$. Then $(\alpha - \beta) \in \{\gamma \mid \beta + \gamma < \alpha\} = \alpha - \beta$. A contradiction. So $\beta + (\alpha - \beta) = \alpha$.

Note. $(\alpha - \beta) + \beta \neq \alpha$ in general. For example, $\omega - 1 = \omega$, hence $(\omega - 1) + 1 = s(\omega) \neq \omega$.

2.10 Ordinal aritmetic: multiplication of ordinals.

By the recursion theorem define $\alpha.\beta = \{\alpha.\gamma + \delta \mid \gamma < \beta \,\&\, \delta < \alpha\}$.

Proposition. $\forall \alpha$, $\beta \in \mathrm{On}$ $(\alpha.\beta \in \mathrm{On})$.

Proof. Suppose that $\alpha.\gamma \in \mathrm{On}$ for each $\gamma < \beta$. So $\alpha.\gamma + \delta \in \mathrm{On}$ for each $\gamma < \beta$ by the proposition that $\xi + \zeta \in \mathrm{On}$ if ξ, $\zeta \in \mathrm{On}$, and we have that $\alpha.\beta \subseteq \mathrm{On}$ and hence is well-ordered by $\in$. Also, for each $\gamma < \beta$ the function $f_\gamma : \alpha \longrightarrow \{\alpha.\gamma + \delta \mid \delta < \alpha\}$ given by $f_\gamma(\delta) = \alpha.\gamma + \delta$ has $f_\gamma``\alpha = \{\alpha.\gamma + \delta \mid \delta < \alpha\}$, and hence $\{\alpha.\gamma + \delta \mid \delta < \alpha\}$, $A_\gamma$ let's call it, is a set by Replacement. Let $g$ be the function given by $\gamma \mapsto A_\gamma$ for all $\gamma < \beta$. Then $g``\beta$ is again a set and so $\alpha.\beta = \bigcup\{A_\gamma \mid \gamma < \beta\}$ is the union $\bigcup g``\beta$ of a set and is also a set. Finally, if $x \in y \in \alpha.\beta$, then there is some $\gamma < \beta$ and $\delta < \alpha$ such that $x \in y = \alpha.\gamma + \delta$. Hence either $x \in \alpha.\gamma$ or there is some $\sigma < \delta$ such that $x = \alpha.\gamma + \sigma$. In the first case there are some $\epsilon < \gamma$ and $\tau < \alpha$ such that $x = \alpha.\epsilon + \tau$. So in each case we have that $x \in \alpha.\beta$. Because we now have that $\forall \beta\, (\forall \gamma < \beta\, (\alpha.\gamma \in \mathrm{On}) \longrightarrow \alpha.\beta \in \mathrm{On})$, we have that $\forall \beta \in \mathrm{On}\ \alpha.\beta \in \mathrm{On}$.

Lemma. $\forall \alpha \in \mathrm{On}$ (i) $0.\alpha = 0$; (ii) $1.\alpha = \alpha$; (iii) $\forall \beta$, $\gamma \in \mathrm{On}$ $(\beta < \gamma) \Longrightarrow \alpha.\beta < \alpha.\gamma$; and (iv) $\forall \delta$ $\beta \in \mathrm{On}$, $(\alpha \leq \delta \Longrightarrow \alpha.\beta \leq \delta.\beta)$.

Proof. (i)–(iii) are obvious. (iv) Suppose that α, $\delta \in \mathrm{On}$, $\beta \in \mathrm{On}$ and for all $\gamma < \beta$ we have that $\alpha.\gamma \leq \delta.\gamma$. Using the lemma about addition from §2.8, we have for all $\gamma < \beta$ and all $\epsilon < \alpha$ that $\alpha.\gamma + \epsilon \leq \delta.\gamma + \epsilon$. Hence either $\alpha.\gamma + \epsilon = \delta.\gamma + \epsilon$, when $\alpha.\gamma + \epsilon \in \delta.\beta$, or $\alpha.\gamma + \epsilon \in \delta.\gamma + \epsilon \in \delta.\beta$, and $\alpha.\gamma + \epsilon \in \delta.\beta$, by the transitivity of $\delta.\beta$. In each case we have that $\alpha.\beta \subseteq \delta.\beta$ and so $\alpha.\beta \leq \delta.\beta$. But this shows that $(\forall \gamma < \beta\, \alpha.\beta \leq \delta.\gamma) \longrightarrow \alpha.\beta \leq \delta.\beta$. Hence, because $<$ is well-founded on the ordinals, $\forall \beta \in \mathrm{On}\, \alpha.\beta \leq \delta.\beta$.

Proposition. (i) $\alpha.0 = 0$; (ii) $\alpha.s(\alpha) = \alpha.\beta + \alpha$; and (iii) if $\lambda \in \mathrm{On}$ is a limit ordinal, then $\alpha.\lambda = \bigcup\{\alpha.\gamma \mid \gamma < \lambda\}$.

Proof. (i) Trivial.
(ii) $\alpha.s(\beta) = \{\alpha.\gamma + \delta \mid \gamma \leq \beta \,\&\, \delta < \alpha\} = \{\alpha.\gamma + \delta \mid \gamma < \beta \,\&\, \delta < \alpha\} \cup \{\alpha.\beta + \delta \mid \delta < \alpha\} = \alpha.\beta + \alpha$.
(iii) If λ is a limit ordinal then

$$\bigcup\{\alpha.\beta \mid \beta < \lambda\} = \{\alpha.\gamma + \delta \mid (\exists \beta < \lambda\, \gamma < \beta) \,\&\, \delta < \alpha\} =$$
$$\{\alpha.\gamma + \delta \mid \gamma \in \bigcup \lambda \,\&\, \delta < \alpha\} = \{\alpha.\gamma + \delta \mid \gamma < \lambda \,\&\, \delta < \alpha,\}$$

with this last step because λ is a limit ordinal.

Theorem. Let α, $\beta \in \mathrm{On}$ and let $\prec$ be the anti-lexicographic order on $\alpha \times \beta$, defined by $(\delta_0, \gamma_0) \prec (\delta_1, \gamma_1)$ if $\gamma_0 < \gamma_1$ or if $\gamma_0 = \gamma_1$ and $\delta_0 < \delta_1$. Hence $(\alpha \times \beta, \prec)$ is isomorphic to $(\alpha.\beta, \in)$ via an order-preserving isomorphism.

Proof. Define $f : \alpha \times \beta \longrightarrow \alpha.\beta$ by $f(\delta, \gamma) = \alpha.\gamma + \delta$. It is clear that f is a surjection. If $(\delta_0, \gamma_0) \prec (\delta_1, \gamma_1)$ *either* $\gamma_0 < \gamma_1$ when $\alpha.\gamma_0 + \delta_0 < \alpha.s(\gamma_0) \leq \alpha.\gamma_1 \leq \alpha.\gamma_1 + \delta_1$, *or* $\gamma_0 = \gamma_1$ and

$\delta_0 < \delta_1$, when $\alpha.\gamma_0 + \delta_0 < \alpha.\gamma_0 + \delta_1$ by the lemma about addition. Hence, f preserves order, and so is an isomorphism.

Note. $2.\omega = \omega$, $3.\omega = \omega$, $\ldots$, but $\omega.\omega \neq \omega$. Note also that $(\omega + 1).\omega = \omega.\omega$

Theorem. Let α, $\beta \in \mathrm{On}$, $\beta > 0$. Then there are unique σ, τ with $\tau < \beta$ such that $\alpha = \beta.\sigma + \tau$.

Proof. Let $\sigma = \{\gamma \mid \beta.s(\gamma) \leq \alpha\}$ and $\tau = \alpha - \beta.\sigma, = \{\delta \mid \beta.\sigma + \delta < \alpha\}$. Hence $\alpha = \beta.\sigma + \tau$. In order to see that $\tau < \beta$, observe that if not either $\beta = \tau$, when $\alpha = \beta.\sigma + \beta = \beta.s(\sigma)$ or $\beta \in \{\delta \mid \beta.\sigma + \delta < \alpha\}$ and, so, $\beta.\sigma + \beta = \beta.s(\sigma) < \alpha$. So in any case $\beta.s(\sigma) \leq \alpha$ and $\sigma \in \sigma$, a contradiction!

Proposition. If $x \subseteq \mathrm{On}$ then $\alpha.\bigcup x = \bigcup\{\alpha.\beta \mid \beta \in x\}$. (Which means that . is continuous in the second place/variable.)

Proof. $\alpha.\bigcup x = \{\alpha.\gamma + \delta \mid \delta < \alpha \,\&\, (\exists\beta \in x\, \gamma < \beta)\} = \{\alpha.\gamma + \delta \mid \exists\beta \in x\, (\gamma < \beta \,\&\, \delta < \alpha)\}$ $= \bigcup\{\alpha.\beta \mid \beta \in x\}$.

Note. In the previous theorem $\sigma = \bigcup\{\gamma \mid \beta.\gamma \leq \alpha\}$.

Ordinals arithmetic: 2.11 Exponentiation of ordinals.

Next we make another definition by recursion and define exponentiation.

Definition. Let α, $\beta \in \mathrm{On}$. Then $\alpha^\beta = \{0\} \cup \{\alpha^\gamma.\delta + \epsilon \mid \gamma < \beta \;\&\; \delta < \alpha \;\&\; \epsilon < \alpha^\gamma\}$.

Proposition. Let α, $\beta \in \mathrm{On}$. Then $\alpha^\beta \in \mathrm{On}$.

Lemma. $\forall\alpha$, β, γ $\delta \in \mathrm{On}$ we have (i) $\alpha^\beta.\alpha^\gamma = \alpha^{\beta+\gamma}$; (ii) $(\alpha^\beta)^\gamma = \alpha^{\beta.\gamma}$; (iii) $\beta < \gamma \Longrightarrow \alpha^\beta < \alpha^\gamma$; and (iv) $\alpha \leq \delta \Longrightarrow \alpha^\beta \leq \delta^\beta$.

Proposition. Let $\alpha \in \mathrm{On}$. (i) $\alpha^0 = 1$; (ii) $\alpha^{s(\beta)} = \alpha^\beta.\alpha$ for each $\beta \in \mathrm{On}$; and (iii) if λ is a limit ordinal, then $\alpha^\lambda = \bigcup\{\alpha^\gamma \mid \gamma < \lambda\}$.

The proofs are similar to the proofs of the equivalents for '.'. You should check these if you desire.

Note. $\forall n \in \omega\, n^\omega = \omega$, but $\omega^\omega \neq \omega$, when $\omega.\omega^\omega = \omega^\omega$. And if we define $\epsilon_0 = \bigcup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$, then $\omega^{\epsilon_0} = \epsilon_0$.

When you come back to this section later you should remember that all of these facts are for exponentiations of *ordinals*, that is very different from the theory of *cardinal exponentiation* that we will see in Chapter (4).