Axiomatic Set Theory for 9^{th} February.

1. Write a proof that if $Y \subseteq X$ and $<$ is a well-founded relation on X, then the restirction of $<$ to Y is also well-founded. [Easy. But maybe we didn't go through this explicitly on Thursday.]

2. Show that if $<_0$ is well-founded on X_0 and $<_1$ is well-founded on X_1, then $<_0 \times <_1$ is well-founded on $X_0 \times X_1$, where $(a_0, a_1)(<_0 \times <_1)(b_0, b_1)$ if and only if $a_0 <_0 b_0$ and $a_1 <_1 b_1$. Show that if $<_0$ and $<_1$ are well-orders then $<_0 \times <_1$ is as well.

3. Let $<$ be a well-order on X. Construct from $<$ a well-order on $\mathcal{P}_{<\omega}(X)$ – the set of all finite subsets of X.

4. (A glance in the direction of "descriptive set theory.") Let us write ${}^{<\omega}\omega$ for $\bigcup\{{}^{n}\omega \mid n \in \omega\}$, the set of all finite sequences of natural numbers. [Here we are identifying a function from a natural number n to ω with the sequence of elements of ω of length n it takes as values.] Define $<$ by $u < v$ if and only if v is a proper initial subsequence of u. (Note the direction of $<$ here!)

(i) Show that $f : {}^{\omega}\omega \longrightarrow \omega$ is continuous (with refespect to the product topology on ${}^{\omega}\omega$ derived from the discrete topology on ω) if and only if $<$ is a well-founded relation on the set

$$\{u \in {}^{<\omega}\omega \mid f \text{ is } \textit{not} \text{ constant on } \{g \in {}^{<\omega}\omega \mid g \text{ extends the sequence } u\}\}.$$

(ii) Let $T \subseteq {}^{<\omega}\omega$ be such that if $u \in T$ and $u < v$ then $v \in T$. (T is a *tree*.) Write u_i for the i^{th}-member of u, and define a relation $\prec$ on T by: $u \prec v$ if and only if **either** $u < v$, **or** for the least i such that $u_i \neq v_i$ one has $u_i < v_i$ (in the usual order on ω). Show that $<$ is a a well-founded relation on T if and only if $\prec$ is a well-order on T.

5. If $X \subseteq \text{On}$ is a set show that either $\bigcup X \in X$ or $\bigcup X = \bigcap\{\alpha \mid \forall\beta \in X\, \beta < \alpha\}$.

6. Find a subset of $\mathbb{R}$ (with its usual order) which is isomorphic to $\omega + \omega$.

7. Define linear orders on ω that are isomorphic under isomorphisms which preserve order with (i) ω^2, (ii) ω^ω, (iii) $\mathbb{Q}$ with its usual order.

8. Show that $\alpha + \beta = \alpha' + \beta'$ and $\beta' > \beta$ implies $\alpha' < \alpha$, but the converse is false.

9. When is $\alpha + \beta$ a limit ordinal? When is $\alpha.\beta$?

10. Let α, $\beta \in \{1, \omega, \omega.\omega\}$. How many possibilities are there for (i) $\alpha + \beta$, (ii) $\alpha.\beta$?

11. Show that if α, $\beta \in \text{On}$, then $\alpha\beta = \beta\alpha$ if and only if $\alpha^2\beta^2 = \beta^2\alpha^2$.

12. Let $\alpha \in \text{On}$. Show that there is some $k < \omega$ and ordinals $\gamma_0, \ldots, \gamma_k$ and natural numbers $n_0, \ldots, n_k$ such that $\alpha = \omega^{\gamma_k}.n_k + \ldots + \omega^{\gamma_0}.n_0$. (Hint: start by showing that $\alpha = \omega^\beta + \delta$ for some ordinal β and some $\delta < \omega^\beta$ and then analyse δ.)

13. Suppose that $f : \text{On} \longrightarrow \text{On}$ is strictly increasing ($\forall\alpha < \beta\, f(\alpha) < f(\beta)$) and continuous at limits (*i.e.* if λ is a limit ordinal, then $f(\lambda) = \bigcup\{f(\alpha) \mid \alpha < \lambda\}$). Show that there is some $\zeta \in \text{On}$ such that $f(\zeta) = \zeta$. Show that the function g defined by $g(\alpha) = \alpha^\omega$ is stricly increasing and continuous at limits. What is the first ordinal for which $f(\zeta) = \zeta$?