AST: For week 2.

Have a read through the following page of so which finishes writing down the axioms of ZF in LST and then try the subsequent questions.

Comprehension and classes. By a property of sets I mean some formula $\phi(\vec{x}, \vec{a})$ where $\vec{a} = a_0, \ldots, a_{n-1}$ are names for some specific sets. The property is true of $b_0, \ldots, b_{m-1} = \vec{b}$ if and only if $\phi(\vec{b}, \vec{a})$ is true. The *extension* of a property is the collection of sets of which the property is true. This collection is a subcollection of V, the collection of all sets. We call the extension of a property a *class*. Thus a class is something of the form $\{ x \mid \phi(x, \vec{a}) \}$.

Axiom of Comprehension. For all formulas $\phi(x, \vec{y})$ in LST we have

$$\forall x \forall a_0, \ldots, \forall a_{n-1} \, \exists y \, \forall z (z \in y \longleftrightarrow z \in x \ \& \ \phi(z, \vec{a})).$$

Note, this "axiom" is in fact an axiom schema: we have one axiom for each formula $\phi(x, \vec{y})$ in LTC, and not merely one single axiom.

In effect this schema says that for every x we have that each $x \cap \{ z \mid \phi(z, \vec{a}) \}$ is a set. Thus a class $\{ x \mid \phi(x, \vec{a}) \}$ is a set (in the sense that $\exists z \, \forall x \, (x \in z \longleftrightarrow \phi(x, \vec{a}))$ if and only if $\{ x \mid \phi(x, \vec{a}) \} \subseteq V_\alpha$ for some α. *I.e.*, $\{ x \mid \phi(x, \vec{a}) \}$ is a set if and only if it is bounded in the hierarchy.

It is very useful (as we shall see) to be able to use various classes as informal abbreviations. Everyone does it and so shall we. However, bear in mind that the classes we use are only abbreviations and their use can always be eliminated in favour of talking about the formulas of LST that define them.

For example, if $A = \{ x \mid \phi(x, \vec{a}) \}$ is a class, we can use $b \in A$ as an abbreviation for $\phi(b, \vec{a})$ being true, but one cannot "think about" whether $A \in A$ is true. It is not false, but *doesn't have any meaning*. Similarly, $\mathcal{P}(A)$ does not have any meaning.

Foundation. For all formulas $\phi(x, \vec{z})$ in LTC (with free variables $\vec{z}$)

$$\forall \vec{a} \, \big(\exists y \, \phi(y, \vec{a}) \longrightarrow \exists y \, \phi(y, \vec{a}) \, \& \, \neg \exists z \in y \, \phi(z, \vec{a})\big).$$

In terms of classes this says that if A is a *non-empty* class then A has an $\in$-minimal element.

How high is the hierarchy? Suppose that $\forall x \in u \, \exists y \, \phi(x, y, \vec{a})$. For each x there is a first α_x such that $\exists y \in V_{\alpha_x} \, \phi(x, y, \vec{a})$ (by the axiom of foundation). So we have a function sending u to $\{ \alpha_x \mid x \in u \}$ given by $x \mapsto \alpha_x$. Thus $\{ \alpha_x \mid x \in u \}$ is a "set" of levels, and hence there is a β such that $\alpha_x < \beta$ for all $x \in u$. So we have $V_{\alpha_x} \in V_\beta$ for all $x \in u$. If we write v for V_β we see that $\exists v \, \forall x \in u \, \exists y \in v \, \phi(x, y, \vec{a})$.

But wait! We need an axiom (well, actually a schema of axioma, again) to do the "collecting" that gave us the set of levels we used above.

Axiom of Collection. For each formula $\phi(x, y, \vec{z})$ in LTC (with free variables $\vec{z}$) we have the following:

$$\forall \vec{a} \, \forall u (\forall x \in u \, \exists y \, \phi(x, y, \vec{a}) \longrightarrow \exists v \forall x \in u \, \exists y \in v \, \phi(x, y, \vec{a})).$$

Surprisingly this schema implies that the hierarchy carries on for a very long time.

Note that, feeding the "$\forall u$" through the $\longrightarrow$ one gets that the axiom can be re-written as: for each $\phi(x, y, \vec{z})$ in LTC and tuple of sets $\vec{a}$ one has

$$\forall x \, \exists y \, \phi(x, y, \vec{a}) \longrightarrow \forall u \, \exists v \, \forall x \in u \, \exists y \in v \, \phi(x, y, \vec{a})).$$

Principle of Replacement For each formulas $\phi(x, y, \vec{z})$ in LTC (with free variables $\vec{z}$) we have the following:

$$\forall \vec{a}\ (\forall u\, \exists! v\, \phi(u, v, \vec{a}) \longrightarrow \forall x\, \exists y\, \forall z\, (z \in y \longleftrightarrow \exists w \in x\, \phi(w, z, \vec{a}))).$$

In terms of classes this says, "If F is a operation definable in V then for every set x we have que $F|x$"x (the image of the restriction of F to x) is also a set."

Some authors prefer to use Replacement in place of Collection in formulating the axioms of ZF. I think that it is better to use Collection (I think it is simpler to understand what is going on in it, in part because one does not have to use classes to formulate it), but it does not make any difference formally which you use because Replacement follows from Collection and Comprehension. (See below.)

The ZF axioms apart from Collection/Replacement were formulated by Zermelo, and so collectively are known as *Zermelo set theory* or Z. The 'F' in ZF is for Fraenkel. See Hallet's "Cantorian set theory and the limitation of size" for a good account of who contributed what in the development of the axioms.

Exercises.

1. Write a proof using the axiom of extensionality that if the empty set exists it is unique.
2. Show that the axiom of comprehension and the axiom of infinity implies the axiom that the empty set exists.
3. Show that if $\bigcup x \subseteq x$ then $\bigcup(\bigcup x) \subseteq x$. Is the converse true?
4. How many elements does V_n for n finite? Show that the function $x \longrightarrow \#x$ is a bijection between V_ω and $\mathbb{N}$, where $\#x$ is defined by induction by "$\#x$ is the number in whose binary expansion the 1s occur in the places i exactly for those i which are equal to $\#y$ for some $y \in x$."
5. Show that the following three statements are equivalent to each other and to x being transitive: (i) $x \subseteq \mathcal{P}(x)$, (ii) $\bigcup x \subseteq x$, (iii) $x = \bigcup\{y \cup \{y\} \mid y \in x\}$.
6. Suppose that $x = \{x\}$. (We'll get to whether there actually are any such sets in the next question.) Is x transitive?
7. Are there any sets x such that $x \in x$? Is there a sequence of sets $\ldots x_3 \in x_2 \in x_1 \in x_0$?
8. Consider the principle that each set is an member of a transitive set.
(i) Show that this principle and the axiom of comprehension imply the axiom of unions
(ii) Show that the principle, comprehension and foundation imply the principle of $\in$-induction.
9. Show that comprehension and collection imply that replacement holds. Prove collection from replacement (and any other Z axioms you need.)
10. Show that collection, the existence of the empty set and the power set axiom show the axiom of pairs: $\forall x\, \forall y\, \exists z (\forall w\, w \in z \longleftrightarrow w = x \text{ or } w = y)$. (Pairs – which informally says that for all x and y there is a set $\{x, y\}$ – is also usually explicitly included in ZF).
11. Show that the axiom of pairs can be proved from the principle that for each x and y there is a set $x \cup \{y\}$.
12. So that we can talk about them next Thursday: (i) Think about how to prove $\in$-induction from induction on the levels of the cummulative hierarchy. (ii) *re* Axiom of Choice. Figure out how to prove that Zorn's Lemma implies the Well Ordering Principle and that WO impies AC,