Axiomatic Set Theory for 23th February.

1. For any x write $\mathcal{P}_{<\omega}(x)$ for the set of finite subsets of x. Show that if x is infinite then $x \sim \mathcal{P}_{<\omega}$. Are there any sets x for which you do not need the axiom of choice to show this?

2. Let I be a set and let $\{X_i \mid i \in I\}$ and $\{Y_i \mid i \in I\}$ be sets such that there is surjection from X_i to Y_i for any $i \in I$. Show that there is no surjection from $\bigcup\{X_i \mid i \in I\}$ to $\Pi\{Y_i \mid i \in I\}$. Deduce that ifκ is a cardinal and $\{\mu_\alpha \mid \alpha < \kappa\}$ and $\{\nu_\alpha \mid \alpha < \kappa\}$ are sets of cardinals such that $\mu_\alpha < \nu_\alpha$ for all $\alpha < \kappa$, then $\Sigma\{\mu_\alpha \mid \alpha < \kappa\} < \Pi\{\mu_\alpha \mid \alpha < \kappa\}$.

3. (a) Show that $(\aleph_{\alpha+1})^{\aleph_\beta} = (\aleph_\alpha)^{\aleph_\beta}.\aleph_{\alpha+1}$ for any α, $\beta \in \mathrm{On}$. (Hint: Consider the cases $\alpha \leq \beta$ and $\beta < \alpha$ separately.)
   (b) Deduce that $(\aleph_n)^{\aleph_\beta} = 2^{\aleph_\beta}.\aleph_n$ for every $n \in \omega$.
   (c) When is $(\aleph_\omega)^{\aleph_\beta} = 2^{\aleph_\beta}.\aleph_\omega$?

4. Show that if κ is regular and $2^\mu \leq \kappa$ for all $\mu < \kappa$, then $\Sigma\{\kappa^\mu \mid \mu < \kappa\} = \kappa$.

5. Show that $2^{\aleph_1} = \aleph_2$ implies that $\aleph_\omega^{\aleph_0} \neq \aleph_{\omega_1}$.

6. Show that a cardinal κ is the sum of cf (κ) cardinals each of cardinality less than κ, and hence that $\kappa < \kappa^{\mathrm{cf}\,(\kappa)}$.

7. Do the exercises (H1)-(H6) at the end of Chapter VIII of Kunen's book.

[What Kunen calls the "pressing down" lemma is what I've called "Fodor's Lemma." For reference, his definitions of '$f(\lambda)$', '(CL)' and '(*)' are $f(\lambda) = \sup(\{2^\kappa \mid \kappa < \lambda\})$, (*) is $\forall\lambda\ (\lambda$ is singular and cf $(f(\lambda)$ and (CL) is $\forall X \subseteq \mathrm{On}\ \big(\overline{\overline{X}} \geq \omega_1 \longrightarrow \exists Y \subset \mathrm{On}(X \subseteq Y \,\&\, \overline{\overline{X}} = \overline{\overline{Y}} \,\&\, Y \in L)\big)$.]