Axiomatic Set Theory for 16th February.

Fewer exercises this week but more reading. You should look at sections 2.7 and 2.8. We'll use the concept of transitive closure (from 2.8) extensively. Then look at section 3 (and anywhere else you want to) to find out about the Axiom of Choice. I don't intend to go over this material in class. After that have a go at (1)-(6) below.

1. Give a direct proof that (AC) implies (WO).

2. Show that the following are equivalents of the axiom of choice.
   (a) If $f : y \longrightarrow x$ is surjection, then there is a function $g : n \longrightarrow y$ such that $g(f(z)) = z$ for all $z \in y$.
   (b) For each set x of non-empty sets there is a transversal set t, *i.e.*, a set $t \subseteq \bigcup x$ such that $\forall y \in x\ \overline{\overline{y \cap t}} = 1$.
   (c) For any x, y either there is an injection from x to y, or there is an injection from y to x.
   (d) $\mathbf{m}\,.\,\mathbf{m} = \mathbf{m}$ for every infinite cardinal $\mathbf{m}$. (Hint: if x is a set consider the cardinalities of x and of the set given by applying Hartogs Lemma to x, *i.e.*, $\{\alpha \in \mathrm{On} \mid \alpha \prec x\}$.)

3. Use equivalents of the axiom of choice to show the following.
   (a) Any partial order can be extended to a total order.
   (b) Any vector space has a base.
   (c) Any two bases for a vector space have the same cardinality.
   (d) Any field can be embedded in an algebraically closed field.
   (e) Any product of compact topological spaces is compact.
   (e) There is a set of reals which is not Lebesgue measurable.

4. Is the axiom of choice used in showing:
   (a) Ary non finite set has a countable subset?
   (b) the union of a countable set of countable sets is countable ?
   If so, explain how.

5. For any x write $\mathcal{P}_{<\omega}(x)$ for the set of finite subsets of x. Show that if x is infinite then $x \sim \mathcal{P}_{<\omega}$. Are there any sets x for which you do not need the axiom of choice to show this?

6. Let I be a set and let $\{X_i \mid i \in I\}$ and $\{Y_i \mid i \in I\}$ be sets such that there is surjection from X_i to Y_i for any $i \in I$. Show that there is no surjection from $\bigcup\{X_i \mid i \in I\}$ to $\Pi\{Y_i \mid i \in I\}$. Deduce that ifκ is a cardinal and $\{\mu_\alpha \mid \alpha < \kappa\}$ and $\{\nu_\alpha \mid \alpha < \kappa\}$ are sets of cardinals such that $\mu_\alpha < \nu_\alpha$ for all $\alpha < \kappa$, then $\Sigma\{\mu_\alpha \mid \alpha < \kappa\} < \Pi\{\mu_\alpha \mid \alpha < \kappa\}$.