Combinatorics of Stammering Tableaux

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Abstract

Stammering tableaux are some kinds of walks on the Young lattice introduced by Josuat-Vergès in 2017. In this thesis, we look at three properties of stammering tableux. Firstly, we study a projection from stammering tableaux to oscillating tableaux and describe its image and cardinalities of the preimages. This gives rise to a sequence of polynomials and we describe the coefficients of these polynomials. Secondly, we define a statistic on stammering tableaux called weight and show that the average weight over all stammering tableaux of a given size and shape is a polynomial in the size and shape of the stammering tableaux. Finally, we describe a bijection between stammering tableaux ending at the empty partition and increasing trees, extend this bijection to the case of Stirling permutations, introduce large Laguerre profile of a Stirling permutation and describe the descent statistic of Stirling permutations.

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I would like to thank Neil Sloane and team for the creation of Online Encyclopaedia of Integer Sequences (OEIS) which has been very helpful in the internship which culminated in this thesis. I would also like to thank the makers of SageMath, the computer algebra system, using which I ran several experiments on which this thesis is based.

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Contents

List of Figures

List of Tables

Introduction

A combinatorial family or combinatorial class of objects is a collection of finite sets S_i for $i \in I$ where I is an index set. Generally, $I = \mathbb{N}$. For example, $\mathfrak{S}_n$, the set of bijections on the set $\{1, \ldots, n\}$ is a combinatorial family where the cardinality of $\mathfrak{S}_n$ is $n!$. Enumerative combinatorics is the branch of combinatorics that deals with exact enumeration of such a combinatorial family of objects in terms of exact formulae, generating series, recursions, etc. Some books that deal with enumerative combinatorics are [Flajolet and Sedgewick, 2009, Stanley, 2009, Stanley and Fomin, 1999]. Bijective combinatorics is the branch of combinatorics where we find bijections between combinatorial families of objects having the same size.

Often, the combinatorial classes are related to some objects from other disciplines such as physics, probability or representation theory. One such object of interest is the PASEP (Partially Asymmetric Simple Exclusion Process) which is a probabilistic model of moving particles in a finite strip of size N (see [Blythe et al., 2000, Blythe and Evans, 2007]). We may denote the state of the strip, *i.e.* position of the particles in the strip as a word in $\{\circ, \bullet\}^N$. For example in Figure 1 the strip can be represented as $\circ \circ \bullet\bullet \circ \bullet$.

The possible transitions are:

- a factor $\bullet\circ$ becomes $\circ\bullet$ (a particle moves to the right),
- a factor $\circ\bullet$ becomes $\bullet\circ$ (a particle moves to the left),
- an initial $\circ$ becomes $\bullet$ (a new particle arrives from the left),
- a final $\bullet$ becomes $\circ$ (a particle exits on the right).

These four events occur with probabilities depending on four real positive parameters p, q, α, β.

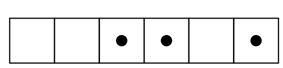


Figure 1: Particles in a strip of size 6.

Thus, the states together with the transition probabilities form a Markov chain with transition matrix M indexed by the 2^N states. Computing the stationary probability distribution *i.e.* a row vector p such that $pM = p$, is of interest. For each state $\tau \in \{\circ, \bullet\}^N$, its probability in the stationary distribution is denoted p_τ.

A recipe to compute the stationary probability p_τ is the *Matrix Ansatz* obtained in [Derrida et al., 1993] which is by taking two operators F and E satisfying a commutation relation:

$$FE - qEF = F + E. \tag{1}$$

A combinatorial class of objects which plays a role in the computing the stationary probabilities is permutation tableaux introduced in [Corteel and Williams, 2007]. Later several other kinds of tableaux were introduced which were in bijection with permutation tableaux such as: alternative tableaux [Viennot, 2008], tree-like tableaux [Aval et al., 2013b] and Dyck tableaux [Aval et al., 2013a]. Here tableaux refer to some filling of some Young diagram or skew diagram by each tableaux having its own filling rules.

The combinatorics of these tableaux is interesting in themselves and often the connection to the PASEP is not immediate. One such tableaux is the stammering tableaux which was introduced in [Josuat-Vergès, 2017]. They are motivated from the commutation relation (1) and from some kinds of tableaux such as oscillating tableaux, vacillating tableaux and hesitating tableaux. Here tableaux refer to some walks on the Young lattice.

Josuat-Vergès showed that stammering tableaux are also counted by factorials similar to the other tableaux related to the PASEP and provided bijections with Dyck tableaux and restricted Laguerre histories.

In this thesis we explore some more properties of stammering tableaux. In Chapter 1 we recall some of the notions that we need for our work. In Chapter 2 we study a projection from stammering tableaux to oscillating tableaux and see some surprising connection to the γ-vectors of Eulerian numbers. In Chapter 3 we define a statistic called weight on stammering tableaux and show that the average weight is a polynomial depending on the size of the tableaux and shape of the tableaux. Finally in Chapter 4, we construct a bijection between stammering tableaux and increasing trees and extend it for Stirling permutations and see some connections with higher order Eulerian numbers.

For Chapter 2 and Chapter 3, we first programmed in Sagemath [Stein et al., 2018] and ran experiments to guess the results we have proved in this thesis. The experiments yielded some sequences which we put on the Online Encyclopedia of Integer Sequences (OEIS) [Inc., 2019] to see if they corresponded to any known sequence. In most cases they did and the others we needed to make a simple multiplicative change in the sequences. Checking OEIS helped in guessing our results and then we used various techniques to prove some of them. For Chapter 4, we composed the known bijections and found a simple new bijection. It was a coincidence that at the same time we were looking at m-Stirling permutations and trying to form a bijection with $(m+1)$-tuple staircases and thus the generalisation to other rook placements from higher arity trees came naturally.

Chapter 1

Preliminaries

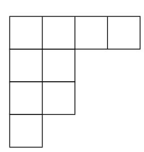
Stammering tableaux were introduced in [Josuat-Vergès, 2017] motivated from PASEP combinatorics and oscillating, vacillating, hesitating tableaux (see [Chen et al., 2006] for more information about these tableaux). In Section 1.1 we introduce stammering Tableaux and fix some notations. In Section 1.2 we state some results about stammering tableaux without proofs.

1.1 Notations and Definitions

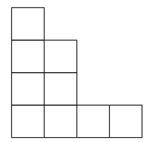
By a partition λ we mean a non-increasing sequence of natural numbers $\lambda_1 \geq \lambda_2 \geq \ldots$ such that there is some $k \in \mathbb{N}_{>0}$ such that $\lambda_n = 0$, $\forall n \geq k$. The λ_i are called the parts of the partition λ. If $\sum_{i=1}^{\infty} \lambda_i = n$ then we say that λ is a partition of n. The size of the partition λ is n. If k is the largest index such that $\lambda_k \neq 0$ then we identify λ with $(\lambda_1, \ldots, \lambda_k)$. For example, $\lambda = (4, 2, 2, 1)$ is a partition of 9. The partition consisting of all zeros is the empty partition $\emptyset$.

We can denote a partition λ diagrammatically in the following representations:

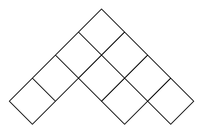
- English: We put a row of λ_1 squared cells, just below it we put a row of λ_2 squared cells beginning just below the first cell of the previous row and so on. For example, for the partition $\lambda = (4, 2, 2, 1)$ the English representation is the following:



- French: We put a row of λ_1 squared cells, just above it we put a row of λ_2 squared cells beginning just above the first cell of the previous row and so on. For example, for the partition $\lambda = (4, 2, 2, 1)$ the French representation is the following:



- Japanese: It is obtained from the French notation by performing a 135 degree clockwise rotation. For example for the partition $\lambda = (4, 2, 2, 1)$ the Japanese representation is the following:



We call these diagrammatic representations of partitions as Young diagrams. Unless, otherwise stated we shall use the French notation.

We can define an order on the set of partitions by $\lambda \leq \mu$ if and only if $\lambda_i \leq \mu_i$ for all i. Thus, $\lambda = (3, 2, 1) \leq \mu = (4, 2, 2, 1)$. The partitions have a unique smallest element which is the empty partition and the partitions form a lattice under this order. We call this lattice the Young lattice and denote it as $\mathcal{Y}$. Further it is a graded lattice, graded by the size of the partitions.

We may also consider $K^{\mathcal{Y}}$, the linear span of all partitions over a field K of characteristic zero. Other than the identity I we define two operators D and U on this vector space:

$$D.\lambda = \sum_{\mu \lessdot \lambda} \mu$$

and

$$U.\lambda = \sum_{\lambda \lessdot \mu} \mu$$

where $\lessdot$ is the covering relation in the poset of $\mathcal{Y}$. The covering relation can be seen in the Young diagrams as $\mu \lessdot \lambda$ if and only if the Young diagram of λ is a cell added to the Young diagram of μ.

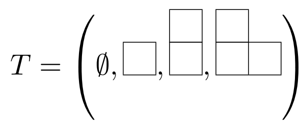
It turns out that

$$DU - UD = I$$

(see Corollary 1.4 in [Stanley, 1988]).

A standard Young tableaux (SYT) is defined as a finite sequence of partitions $T = (\emptyset = \lambda^{(0)}, \ldots, \lambda^{(n)} = \lambda)$ where $\lambda^{(i)} \lessdot \lambda^{(i+1)}$ for all $i \in \{0, \ldots, n-1\}$. λ is called the shape of the SYT T and n the size of T. It can be considered as applying the operator U on the empty partition n times *i.e.* $U^n.\emptyset$. What we mean here is that we apply the U operator on $\emptyset$ and record one of the summands, apply U to that summand and record one of the summands that appear, and so on.

For example the sequence



(1.1)

is a SYT of the shape $(2, 1)$. However, the standard definition of SYT of shape λ is a filling of the Young diagram λ from numbers in $\{1, \ldots, n\}$ such that the numbers in each cell increase in each column and row. For example, the filling of shape $(2, 1)$ corresponding to the SYT in Equation (1.1) is in Equation (1.2).

2	
1	3

(1.2)

Motivated by this one can define some other kind of tableaux depending on operators different from U:

- Oscillating tableaux for the operator $(U + D)^n.\emptyset$
- Vacillating tableaux for the operator $((U + I)(D + I))^n.\emptyset$
- Hesitating tableaux for the operator $(DU + ID + UI)^n.\emptyset$

Each of these represent certain kinds of walks on $\mathcal{Y}$. One can read more about these tableaux in [Chen et al., 2006].

However, the rest of this thesis is about another kind of walk on $\mathcal{Y}$ called stammering tableaux.

Definition 1.1.0.1. A *stammering tableaux* of size n and shape λ is a finite sequence of partitions, $(\lambda^{(0)}, \ldots, \lambda^{(3n)})$ such that $\lambda^{(0)} = \emptyset$ and $\lambda^{(3n)} = \lambda$ such that

- if $i \equiv 0$ or $1 \pmod 3$ then either $\lambda^{(i)} \lessdot \lambda^{(i+1)}$ or $\lambda^{(i)} = \lambda^{(i+1)}$
- if if $i \equiv 2 \pmod 3$ then $\lambda^{(i)} \gtrdot \lambda^{(i+1)}$.

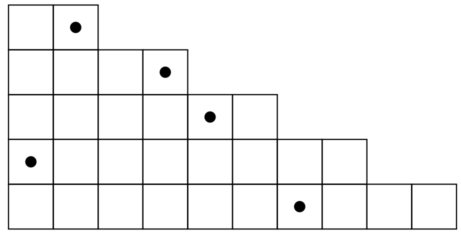
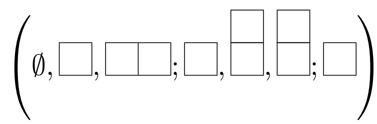


Figure 1.1: Rook placement on $2\delta_5$.

Stammering tableaux of size n can be obtained from the operator $(D(U+I)(U+I))^n$ applied on $\emptyset$.

Example 1.1.0.2.



is a stammering tableaux of size 2 and shape (1).

We denote the set of stammering tableaux of size n of shape λ as $\mathrm{Stam}^{(n)}_{\emptyset,\lambda}$. Let $T^{(n)}_{\emptyset,\lambda}$ denote the cardinality of this set.

Our definition of stammering tableaux is different from the definition provided in [Josuat-Vergès, 2017], where all stammering tableaux have shape $\emptyset$.

In [Roby, 1991] and [Krattenthaler, 2016] it has been shown that oscillating tableaux, via Fomin growth diagrams [Fomin, 1988], are in one-to-one correspondence to some rook placements in Young diagrams. The same can be done with stammering tableaux.

Definition 1.1.0.3. Let $m\delta_n$ denote the partition $(mn, m(n-1), \ldots, m)$. We call this as the m-tuple staircase. A partial filling of the cells of the Young diagram of $m\delta_n$ with rooks is called a *rook placement in* $m\delta_n$ if there is exactly one rook in each row and atmost one rook in each column.

Thus, $2\delta_n$ is the double staircase. An example of a rook placement in $2\delta_5$ is in Figure 1.1.

We now present description of Fomin growth diagrams:

Definition 1.1.0.4. A *growth diagram* is a filling of the corners of the cells in a rook placement of a Young diagram (in French notation) with partitions such that a partition is either equal to or covered by the partition just above it and the partition just to its right. The partition in the North-East corner ρ is determined by the other three partitions λ, μ, ν as shown in Figure 1.2 as per the following rules, (called *local rules*):

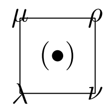


Figure 1.2: Cell of a growth diagram.

- if $\mu \neq \nu$ then $\rho_i = \max(\mu_i, \nu_i)$;
- if $\lambda = \mu = \nu$ and the cell does not have a rook then $\rho = \lambda$;
- if $\lambda = \mu = \nu$ and the cell has a rook then $\rho_1 = \lambda_1 + 1$ and $\rho_i = \lambda_i$ for all $i > 1$;
- if $\lambda \neq \mu = \nu$, then ρ is obtained from μ by adding 1 to the $(k+1)^{st}$ part of μ where k is the smallest index such that $\lambda_k \neq \mu_k$.

Further whenever a partition and its immediate right neighbour are equal there is no rook in the cells below them and whenever a partition and its immediate top neighbour are equal there is no rook in the cells to their left.

Thus, given a labelling of the corners on the left and bottom edge of a rook placement of some Young diagram, the entire growth diagram is determined by the local rules.

Given a rook placement of $2\delta_n$ and a labelling of the bottom corners of the bottom cells with a weakly increasing walk of Young diagrams, *i.e.* walk given up the operator $(U+I)^n$ acting on $\emptyset$, and a filling of the corners on the left edge with $\emptyset$ we get a stammering tableaux by reading the Young diagrams from the other border of $2\delta_n$ from North-West corner to South-East corner. For example, in Figure 1.3 we have the growth diagram corresponding to the stammering tableau in Example 1.1.0.2

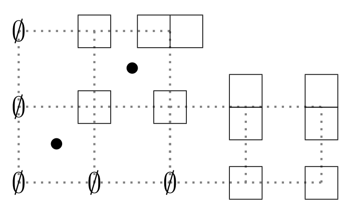


Figure 1.3: Growth diagram for the stammering tableau in Example 1.1.0.2.

If the labels in the corners of the bottom edge of the growth diagram is entirely with $\emptyset$ then the stammering tableau that we obtain is an element of $\mathrm{Stam}^{(n)}_{\emptyset,\emptyset}$.

Given a stammering tableau by following the reverse of the local rules to obtain λ given μ, ν, ρ we obtain can see that a stammering tableau of size n also determines a growth diagram on a rook placement of $2\delta_n$ with the corners of the left edge of the growth diagram entirely labelled with $\emptyset$.

Thus, the study of the stammering tableaux can be done by studying growth diagrams on rook placements on double staircases. Further if we want to study the stammering tableaux in $\mathrm{Stam}_{\emptyset,\emptyset}^{(n)}$ it sufficient to just look at the rook placements on $2\delta_n$ and not at the entire growth diagrams.

1.2 Known Results

We first state the result enumerating the $\mathrm{Stam}_{\emptyset,\lambda}^{(n)}$.

Theorem 1.2.0.1. *Proposition 7.2 in [Josuat-Vergès, 2017] Let λ be a partion of n and f_λ be the number of SYT of shape λ. Then we have that*

$$T_{\emptyset,\lambda}^{(n)} = (n+1)!\binom{n}{k} f_\lambda.$$

Corollary 1.2.0.2. *Thus we have that when $\lambda = \emptyset$,*

$$T_{\emptyset,\emptyset}^{(n)} = (n+1)!.$$

Corollary 1.2.0.2 is the same as counting the number of rook placements on $2\delta_n$ which can be counted by the number of possibilities of inserting a row in each row starting from the top, there are 2 choices for the first row, $(4-1)$ for the second, $(2i - i + 1)$ for the i^{th} row and so on.

Given a rook placement on $2\delta_n$ we can construct a Dyck path of size n.

Definition 1.2.0.3. A Dyck word of size n is a word D of length $2n$ in the alphabet $\{\nearrow, \searrow\}$ such that the number of $\nearrow$ is equal to the number of $\searrow$ in D and for each prefix of D the number of $\nearrow$ is greater than or equal to the number of $\searrow$.

A Dyck path is a diagrammatic representation of a Dyck word. It is a path in the positive quadrant of the real plane. It begins at $(0,0)$ and ends at $(0,2n)$ such for each $\nearrow$ we take a $(1,1)$ step and for each $\searrow$ we take a $(1,-1)$ step.

Example 1.2.0.4. An example of a Dyck word is $\nearrow\nearrow\nearrow\searrow\nearrow\nearrow\searrow\nearrow\searrow\searrow\searrow\searrow$. Its Dyck path is denoted in Figure 1.4.

Given a rook placement R on $2\delta_n$ we define $d(R)$ of length $2n+2$ as

- the first step is $\nearrow$, the last step is $\searrow$,

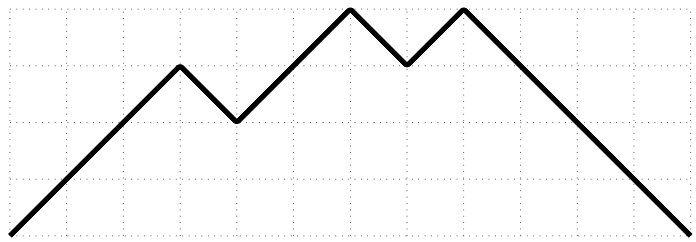


Figure 1.4: Dyck path for the Dyck word $\nearrow\nearrow\nearrow\searrow\nearrow\nearrow\searrow\nearrow\searrow\searrow\searrow\searrow$.

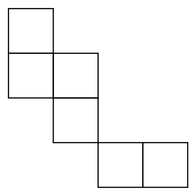


Figure 1.5: Skew diagram for the skew shape $(4, 2, 2, 1)/(2, 1)$.

- if $2 \leq i \leq 2n + 1$, the i^{th} step is $\nearrow$ if the $(i - 1)^{st}$ column of R contains a dot and $\searrow$ otherwise.

Lemma 1.2.0.5 (Lemma 3.2 in [Josuat-Vergès, 2017])**.** *The path given by $d(R)$ is a Dyck path of size $(n + 1)$.*

For example, the Dyck path in Figure 1.4 is the Dyck path obtained by applying $d(\cdot)$ to the rook placement in Figure 1.1.

We consider the shape δ_n the single staircase and have the following definition:

Definition 1.2.0.6. A *skew shape* is a pair of partitions (μ, λ) such that $\mu \leq \lambda$. It is often represented as λ/μ.

A skew shape λ/μ is represented by a skew Young diagram or a skew diagram which is the Young diagram of λ from which the cells of the Young diagram of μ are removed.

For example, the skew shape $(4, 2, 2, 1)/(2, 1)$ is represented by the skew Young diagram in Figure 1.5.

For convenience we shall use the following notion:

Definition 1.2.0.7. A *Dyck shape* is a skew shape δ_n/λ such that $\lambda \leq \delta_{n-1}$.

Dyck shapes are in bijection with Dyck paths which is clearer when they are drawn in the Japanese notation. The Dyck Path in Figure 1.4 is given by the Dyck shape $\delta_5/(3, 1)$

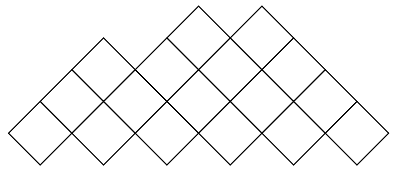


Figure 1.6: Dyck shape $\delta_6/(3,1)$.

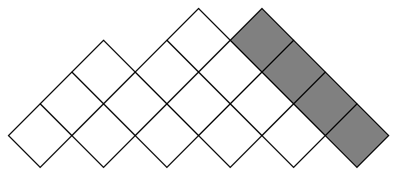


Figure 1.7: Here $D \sqsubset E$ where $D = \delta_5/(2)$ and $E = \delta_6/(3,1)$. Here the gray cells denote the ribbon.

as illustrated in Figure 1.6.

Definition 1.2.0.8. A skew shape is called a *ribbon* if its Young diagram is connected and contains no 2×2 square. Let D and E be two Dyck shapes of respective length $2n$ and $2n+2$, then we denot $D \sqsubset E$ and say that E is obtained from D by addition of a ribbon if on drawing the skew diagrams of E and D in Japanese notation such that the left most cell of D is placed on top of the left most cell of E, D is contained inside the diagram of E; and the difference E/D is a ribbon.

For example, the Dyck shape $D = \delta_5/(2)$ and the Dyck shape $E = \delta_6/(3,1)$ are such that $D \sqsubset E$ as illustrated in Figure 1.7.

Remark 1.2.0.9. *This definition is easily translated in terms of binary words over $\nearrow$ and $\searrow$. Let D a dyck word, then $D \sqsubset E$ if and only if E is obtained from $D \searrow\searrow$ by changing a $\searrow$ into a $\nearrow$ (and each step $\searrow$ of $D \searrow\searrow$ can be changed except the last one, so that there are $n+1$ possibilities for a path of length $2n$).*

Definition 1.2.0.10. An n-chain of Dyck shapes is a sequence $D_1 \sqsubset D_2 \sqsubset \cdots \sqsubset D_n$ where D_i is a Dyck shape of length $2i$. The biggest path D_n is called the shape of the chain, and we say the chains ends at D_n.

Lemma 1.2.0.11. *The number of n-chains of Dyck shapes is $n!$.*

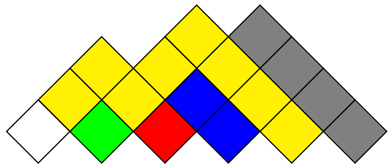


Figure 1.8: The rook placement R in Figure 1.1 corresponds to this chain of Dyck shapes Here the coloured cells denote the different ribbons.

We have the following theorem:

Theorem 1.2.0.12 (Proposition 3.8 in [Josuat-Vergès, 2017])**.** *Let R be a rook placement in $2\delta_{n-1}$. Let R_i be the rook placement in $2\delta_i$ obtained by keeping only the i top rows of R (by convention, R_0 is empty so that $d(R_0) = \delta_1/\emptyset$, the unique Dyck shape of length 2, and $R_{n-1} = R$). Then $d(R_0), \ldots, d(R_{n-1})$ is an n-chain of Dyck shapes. This defines a bijection between rook placements in $2\delta_{n-1}$ and n-chains of Dyck shapes.*

The rook placement R in Figure 1.1 corresponds to the chain of Dyck shapes in Figure 1.8.

Chapter 2

Projection from Stammering Tableaux to Oscillating Tableaux

In this chapter the stammering tableaux have shape $\emptyset$, unless mentioned otherwise. Given a stammering tableaux T of size n, we can obtain an oscillating tableaux of length $2n$, $\Phi_n(T)$ from it by removing the consecutive repetitions of the same shape from T. For example for

$$T = \left(\emptyset, \ydiagram{1}, \ydiagram{2}; \ydiagram{1}, \ydiagram{1}, \ydiagram{2}; \ydiagram{1}, \ydiagram{2}, \ydiagram{2}; \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}; \ydiagram{1}, \ydiagram{1}, \ydiagram{1}; \emptyset\right)$$

we have

$$\Phi_5(T) = \left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{2}, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1}, \emptyset\right).$$

We may ask the following two questions:

1. What is the image of Φ_n?
2. Given an oscillating tableaux S in the image of Φ_n, $\mathrm{Im}(\Phi_n)$, what is $\Phi_n^{-1}(S)$?

Via Fomin local rules and growth diagrams, the map Φ_n is equivalent to taking a rook placement on the double staircase $2\delta_n$ with exactly one rook in each row and atmost one rook in each column, and removing the columns with no rooks and joining the remaining columns to make a Young diagram of n rows and n columns. Thus for T above the rook placement in $2\delta_5$ is in Figure 2.1 and its projection $\Phi_5(T)$ is given by the rook placement in Figure 2.2.

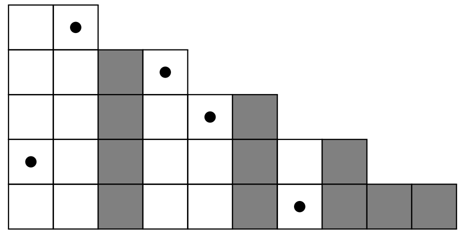


Figure 2.1: Rook placement on $2\delta_5$. The columns with no rooks are shaded in gray.

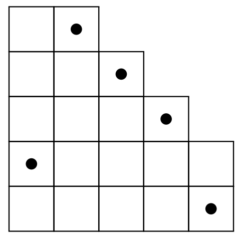


Figure 2.2: Rook placement corresponding to the tableau $\Phi_5(T)$.

We shall see that with this setting the two questions are not difficult to answer. However, while answering these two questions we come across a new sequence of polynomials which yield $(n+1)!$ when substituted with 2 and the tangent numbers when the even polynomials are substituted with 0 and many more properties.

Understanding the coefficients of these polynomials took us to the γ-vectors of Eulerian polynomials which were studied by [Foata and Schützenberger, 1970] and looking at stammering tableaux via Laguerre histories, an object developed by [Françon and Viennot, 1979].

2.1 Results

We shall first answer the two questions in page 8. For that we need to introduce the notion of a block.

Definition 2.1.0.1 (Block)**.** Given a rook placement on a Young diagram, with at most one rook in each column and row, a *block* is a maximal sequence of adjacent columns of the same height. The *size* of a block is the number of columns it has. For a rook placement T we denote the the number of blocks of size 1 of T as $\mathrm{bso}(T)$.

All the blocks in rook placements corresponding to stammering tableaux are of size 2. In rook placement T in Figure 2.2 there is one block of size 2 and three blocks of size 1. Thus, $\mathrm{bso}(T) = 3$.

We state the following lemma which is immediate from the definition of Φ_n, without proof.

Lemma 2.1.0.2. *The size of a block of a rook placement $T \in \mathrm{Im}(\Phi_n)$ can be either 1 or 2.*

Lemma 2.1.0.3. *For $T \in \mathrm{Im}(\Phi_n)$, the cardinality of $\Phi_n^{-1}(T)$ is $2^{\mathrm{bso}(T)}$.*

Proof. We shall try to reconstruct an $S \in \Phi_n^{-1}(T)$ from T. By the number of rooks in T we can find n and thus we can form $2\delta_n$. For each block in T, we look at its size and then place rooks accordingly in the block of that size in $2\delta_n$. For each block of size 2 in T we have a unique way of putting the rooks in S. For each block of size 1 in T we have two choices for putting that block in $2\delta_n$, either as a left column or a right column. The remaining blocks in $2\delta_n$ are kept empty. Thus, the number of S is $2^{\mathrm{bso}(T)}$. □

Thus, we get the following identity:

$$(n+1)! = \sum_{T \in \mathrm{Im}(\Phi_n)} 2^{\mathrm{bso}(T)}.$$

If we replace 2 with an indeterminate x we have the following definition of the sequences of polynomials:

$$p_n(x) = \sum_{T \in \mathrm{Im}(\Phi_n)} x^{\mathrm{bso}(T)}. \tag{2.1}$$

Thus, clearly $p_n(2) = (n+1)!$.

We list the first few polynomials p_n in this sequence in Table 2.1.

Remark 2.1.0.4. *It is not difficult to see that if $n - i \equiv 1 \pmod 2$, then $[x^i]p_n(x) = 0$ as can also be seen in Table 2.1 for the first few polynomials. Also, degree of p_n is n. Thus, $x^{n/2}p_n(\frac{x+1}{\sqrt{x}})$ is a polynomial in x.*

Checking the sequence of non-zero coefficients of p_n on OEIS [Inc., 2019], we see that it is given by the triangle of numbers A101280 which are the γ-vectors of the Eulerian numbers.

We can define the Eulerian polynomials as follows:

Table 2.1: First few p_n.

n	$p_n(x)$
1	x
2	$x^2 + 2$
3	$x^3 + 8x$
4	$x^4 + 22x^2 + 16$
5	$x^5 + 52x^3 + 136x$
6	$x^6 + 114x^4 + 720x^2 + 272$
7	$x^7 + 240x^5 + 3072x^3 + 3968x$
8	$x^8 + 494x^6 + 11616x^4 + 34304x^2 + 7936$
9	$x^9 + 1004x^7 + 40776x^5 + 230144x^3 + 176896x$
10	$x^{10} + 2026x^8 + 136384x^6 + 1328336x^4 + 2265344x^2 + 353792$
11	$x^{11} + 4072x^9 + 441568x^7 + 6949952x^5 + 21953408x^3 + 11184128$

Definition 2.1.0.5. The *descent* of a permutation of $\sigma \in \mathfrak{S}_n$ is an $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. Let $\mathrm{des}(\sigma)$ denote the number of descents of a permutation σ.

Definition 2.1.0.6. The n^{th} Eulerian polynomial is defined as

$$S_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)}.$$

This leads to the following theorem:

Theorem 2.1.0.7. *For any $n \geq 1$, we have the following identity:*

$$x^{n/2} p_n \left(\frac{x+1}{\sqrt{x}} \right) = S_{n+1}(x) \tag{2.2}$$

For the proof of the theorem we introduce the notion of large Laguerre profile of a permutation.

Definition 2.1.0.8. Let $\sigma \in \mathfrak{S}_n$. We will take the convention that $\sigma_0 = 0$ and $\sigma_{n+1} = 0$. The integer $\sigma_i \in \{1, \ldots, n\}$ is called:

- a *peak* if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$,

- a *valley* if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$,
- a *double ascent* if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$,
- a *double descent* if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Definition 2.1.0.9. For $\sigma \in \mathfrak{S}_n$ we take $\sigma_0 = \sigma_{n+1} = 0$. We define a function f from $[n-1]$ to $\{\searrow, \nearrow\}^2$. For $i \in [1, n]$, if

- if i is a valley then $f(i) = \nearrow\nearrow$,
- if i is a peak then $f(i) = \searrow\searrow$,
- if i is a double ascent then $f(i) = \nearrow\searrow$,
- if i is a double descent then $f(i) = \searrow\nearrow$.

Now consider the path given by $\nearrow f(1)f(2)\dots f(n-1) \searrow$. This is a Dyck word as shown in [Françon and Viennot, 1979]. We call this Dyck word the *large profile* or the *large Laguerre profile* of the permutation σ and we shall denote it by $\nabla(\sigma)$.

The large profile of the permutation $\sigma = 943127685$ is drawn in Figure 2.3. Here the valleys are 1, 6, the peaks are 7, 8, 9, the double descents are 3, 4, 5 and the double ascents are 2. Thus, the large profile of this permutation is shown in Figure 2.3.

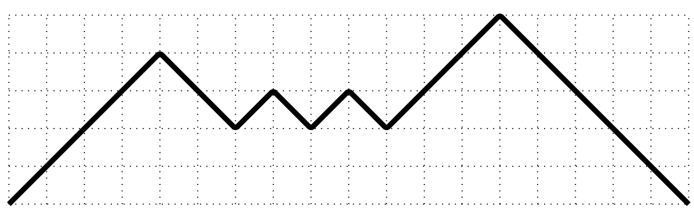


Figure 2.3: $\nabla(\sigma)$ where $\sigma = 943127685$.

Let $\sigma \in \mathfrak{S}_n$. Let σ' be the permutation obtained after removing n from the one line notation of σ. Let $\sigma^{(0)} = \sigma$ and $\sigma^{(i+1)} = (\sigma^{(i)})'$.

For $\sigma = 513462$, the large profile of the permutation is in Figure 2.4.

Lemma 2.1.0.10. *For $\sigma \in \mathfrak{S}_n$, $\nabla(\sigma') \sqsubset \nabla(\sigma)$. Thus, the tuple of Dyck paths $(\nabla(\sigma^{(n-1)}), \dots, \nabla(\sigma^{(0)}))$ forms a chain of Dyck shapes.*

Theorem 2.1.0.11. *For $\sigma \in \mathfrak{S}_n$, the map $\sigma \mapsto (\nabla(\sigma^{(n-1)}), \dots, \nabla(\sigma^{(0)}))$ forms a bijection between permutations and n-chains of Dyck shapes.*

The chain of Dyck paths for the permutation $\sigma = 513462$ is given in Figure 2.5.

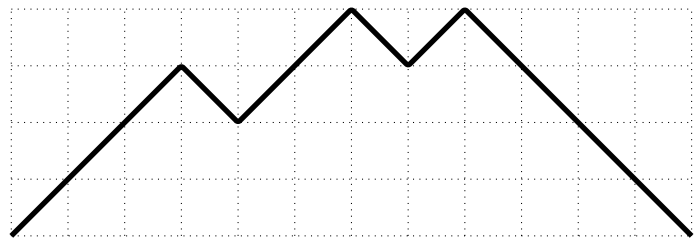


Figure 2.4: Large profile for the permutation $\sigma = 513462$.

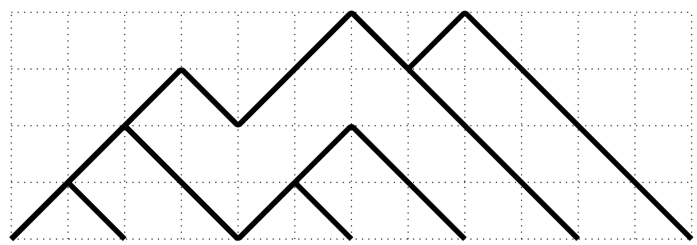


Figure 2.5: Chain of Dyck paths for the permutation $\sigma = 513462$. To get the path $\nabla(\sigma^{(n-i)})$ we take the unique Dyck path from $(0,0)$ to $(2i,0)$.

We shall use Theorem 2.1.0.11 to prove Theorem 2.1.0.7.

We also have the following corollaries from Theorem 2.1.0.7 and 2.1.0.11:

Theorem 2.1.0.12. *We have that $p_{2n}(0)$ are the tangent numbers.*

Theorem 2.1.0.13. *We have that $p_n(1)$ is the number of permutations with no double ascents.*

2.2 Proofs

To prove Theorem 2.1.0.7 we first need to prove Lemma 2.1.0.10 and Theorem 2.1.0.11.

Proof of Lemma 2.1.0.10. In σ', $n-1$ is a peak. Thus in σ',

$$\nearrow f(1)\dots f(n-2)f(n-1)\searrow = \nearrow f(1)\dots f(n-2)\searrow\searrow\searrow = \nabla(\sigma')\searrow\searrow .$$

We have three cases while inserting n in σ,

Case 1: n is the first letter of σ. If i is the first letter of σ' then i is either a double ascent or a peak. Thus the second letter of $f(i)$ is $\searrow$. In σ, the i is a double descent if it was a peak in σ' and a valley if it was a double ascent in σ'. Thus in σ, the second letter of $f(i) \nearrow$. Thus, $\nabla(\sigma)$ is obtained from $\nabla(\sigma') \searrow\searrow$ by changing the $(1+2i)^{th}$ letter from $\searrow$ to $\nearrow$. Thus, from Remark 1.2.0.9, $\nabla(\sigma') \sqsubset \nabla(\sigma)$.

Case 2: n is the last letter of σ. If i is the last letter of σ' then i is either a double descent or a peak. Thus the first letter of $f(i)$ is $\nearrow$. In σ, the i is a double ascent if it was a peak in σ' and a valley if it was a double descent in σ'. Thus in σ, the first letter of $f(i) \nearrow$. Thus, $\nabla(\sigma)$ is obtained from $\nabla(\sigma') \searrow\searrow$ by changing the $(2i)^{th}$ letter from $\searrow$ to $\nearrow$. Thus, from Remark 1.2.0.9, $\nabla(\sigma') \sqsubset \nabla(\sigma)$.

Case 3: n is inserted in between i, j such that $i > j$. Thus, $f(j)$ is the same in both σ and σ'. If i is to the right of n then i is either a double ascent or a peak in σ'. Thus, we can proceed as in Case 1 now. Similarly, if i is to the left of n then i is either a double descent or a peak in σ'. Thus we can proceed as in Case 2 now.

Thus, we get that $\sigma' \sqsubset \sigma$.

□

Proof of Theorem 2.1.0.11. In Lemma 2.1.0.10 we showed that $(\nabla(\sigma^{(n-1)}), \ldots, \nabla(\sigma^{(0)}))$ is a n-chain of Dyck shapes. Thus, from Lemma 1.2.0.11 we only need to show that the map is surjective.

We can use induction. The base case is trivial to check. By induction hypothesis the permutation σ' is given by the chain of Dyck shapes $(\nabla(\sigma'^{(n-2)}), \ldots, \nabla(\sigma'^{(0)}))$. If we want to change the $k^{th} \searrow$ to a $\nearrow$ in $\nearrow f(1) \ldots f(n-1) \searrow\searrow$, then by the proof of Lemma 2.1.0.10 we should do the following:

- If $k-1$ is even then we insert n in the position before $(k-1)/2$ in σ'.
- If $k-1$ is even then we insert n in the position before $k/2$ in σ'.

Thus, it shows that the map is surjective.

□

Proof of Theorem 2.1.0.7. Let $p_n(x) = x^n + a_{n+1,1}x^{n-2} + a_{n+1,2}x^{n-4} + \ldots$, then

$$x^{n/2} p_n\left(\frac{x+1}{\sqrt{x}}\right) = (x+1)^n + a_{n+1,1}x(x+1)^{n-2} + a_{n+1,2}x^2(x+1)^{n-4} + \ldots. \tag{2.3}$$

For the identity in (2.1) to be true the $a_{n+1,i}$ should be the γ-vectors of $S_{n+1}(x)$, $\gamma_{n+1,i}$. See Chapter 4 of [Petersen, 2015] for more details. Foata and Schützenberger in [Foata and Schützenberger, 1970] showed that $\gamma_{n+1,i}$ is the number of permutations of size $n+1$ with no double ascents and where each ascent is a valley and there are i ascents.

Thus, to show $a_{n+1,i} = \gamma_{n+1,i}$ we need to show that the number of oscillating tableaux $T \in \mathrm{Im}(\Phi_n)$ with $\mathrm{bso}(T) = i$ is the number of permutations with of size $n+1$ with no double ascents and where each ascent is a valley. For a $T \in \mathrm{Im}(\Phi_n)$, we have a unique stammering tableau $U \in \Phi_n^{-1}(T)$ such that for each block of U in which there is only one rook, the rook is at the right column.

By using Theorem 2.1.0.11, we can introduce the bijection from $\mathrm{Stam}_{\emptyset,\emptyset}^{(n)}$ to $\mathfrak{S}_{n+1}$, by reading the chain of Dyck Shapes from a stammering tableaux T, as in Proposition 3.8 of [Josuat-Vergès, 2017] and then using the bijection of Theorem 2.1.0.11, we obtain a new bijection between $\mathrm{Stam}_{\emptyset,\emptyset}^{(n)}$ to $\mathfrak{S}_{n+1}$.

Now, $a_{n+1,i}$ is the number of $T \in \mathrm{Stam}_{\emptyset,\emptyset}^{(n)}$ such that $d(T)$ (see Definition 3.1 of Josuat-Vergès [2017]) has no $\nearrow\searrow$ starting at an even step and has i $\nearrow\nearrow$ starting at an even step, *i.e.* the number of $\nearrow\nearrow$ starting at an even step is $\mathrm{bso}(T)$. Via the bijection above, this is the same as the number of $\sigma \in \mathfrak{S}_{n+1}$ such that $\nabla(\sigma)$ has no $\nearrow\searrow$ starting at an even step and has i $\nearrow\nearrow$ starting at an even step, *i.e.* σ has no double ascents and has i valleys and this is counted by $\gamma_{n+1,i}$.

□

2.3 Some Questions

We have the following questions:

1. The sequence $p_n(3)$ is given by the sequence on OEIS [Inc., 2019] A230008.
2. The sequence $p_n(4)$ is given by the sequence on OEIS [Inc., 2019] A234854.
3. Describe $p_n(k)$ for any $k \in \mathbb{N}$.
4. The non-zero coefficients of $p_n(x)$ form a log concave sequence.

Chapter 3

Polynomiality of Average Weights of Stammering Tableaux

In [Hopkins and Zhang, 2015] a new statistic called weight was defined for oscillating tableaux (see Chapter III Section 8 on Up Down Tableaux in [Sundaram, 1986] for details on oscillating tableaux). The shape of an oscillating tableaux is the last shape that it visits and its length is the number of shapes that it visits. Hopkins and Zhang showed that the average weight over all oscillating tableaux of the same shape and length is a quadratic polynomial depending on the weight and length (see Theorem 1.2 in Hopkins and Zhang [2015]) of the tableau.

Han and Xiong in [Han and Xiong, 2018] generalised the notion of weight and showed polynomiality of the average generalised weights as well.

We shall define the notion of weight for stammering tableaux and will see that the average weights form polynomials. Proving this kind of result seems computationally intensive. We have adapted the proof of Hopkins and Zhang to our needs.

We find that not just the average weights are polynomial but the average steps having the same number modulo 3 have polynomial average weights.

3.1 Results

As per Proposition 7.2 in [Josuat-Vergès, 2017],

$$T^{(n)}_{\emptyset,\lambda} = (n+1)!\binom{n}{|\lambda|} f_\lambda.$$

Definition 3.1.0.1. For $T = (\lambda^{(0)}, \ldots, \lambda^{(3n)}) \in \mathrm{Stam}^{(n)}_{\emptyset,\lambda}$, let $wt(T) = \sum_{i=0}^{3n} |\lambda^{(i)}|$ and for $j \in \{0,1,2\}$ let $inn_j(T) = \sum_{i \equiv j \pmod 3} |\lambda^{(i)}|$.

Thus, for $T \in \mathrm{Stam}^{(n)}_{\emptyset,\lambda}$, $wt(T) = \sum_{j=0}^{2} inn_j(T)$.

We have the following theorem about average weights:

Theorem 3.1.0.2. *Let $|\lambda| = k$. Then for all $n \in \mathbb{N}$ and $n \geq k$ we have that,*

$$\frac{1}{T^n_{\emptyset,\lambda}} \sum_{T \in Stam^{(n)}_{\emptyset,\lambda}} wt(T) = \frac{n^2 + 2(k+1)n + 3k}{2} \tag{3.1}$$

and

$$\frac{1}{T^n_{\emptyset,\lambda}} \sum_{T \in Stam^{(n)}_{\emptyset,\lambda}} inn_0(T) = \frac{n^2 + (2k-1)n + 4k}{6}. \tag{3.2}$$

Thus, for $\lambda = \emptyset$, we have that the average weight is $(n^2 + 2n)/2$.

The proof of Theorem 3.1.0.2 is by mimicking the proof of Lemma 8.6 of [Stanley, 2013] using the extension in proof of Theorem 1.2 in [Hopkins and Zhang, 2015].

3.2 Proofs

Proof. Let $(D(U+I)^2)^n = \sum_{i,j \geq 0} b_{i,j}(n) U^i D^j$. Let $[\lambda](D(U+I))^n.\emptyset$ denote the coefficient of λ in $(D(U+I))^n.\emptyset$. Then clearly, $T^n_{\emptyset,\lambda} = [\lambda](D(U+I)^2)^n.\emptyset = (n+1)!\binom{n}{|\lambda|} f_\lambda$ by definition of stammering tableaux. As $[\lambda]U^i D^j.\emptyset = \delta_{j,0}\delta_{i,|\lambda|} f_\lambda$, where $\delta_{i,j}$ denotes the Kronecker delta, we get that $b_{i,0}(n) = (n+1)!\binom{n}{i}$.

Now we apply the technique in proof of Theorem 1.2 in [Hopkins and Zhang, 2015] to obtain the weights. Consider operator $(Qx^{-1}D(QxU + QI)^2))^n$ where $Qx^i = x^i y^i$ for integers i. Thus, $(Qx^{-1}D(QxU + QI)^2))^n = \sum_{i,j \geq 0} x^{i-j} q_{i,j}(n) U^i D^j$ where $q_{i,j}(n) \in \mathbb{Z}[y, y^{-1}]$. Thus we have that,

$$\sum_{T \in \mathrm{Stam}^{(n)}_{\emptyset,\lambda}} wt(T) = \frac{\partial}{\partial y}[\lambda](Qx^{-1}D(QxU + QI)^2))^n \Big|_{x=1,y=1}.\emptyset.$$

We want to find $q_{i,j}(n+1)$.

$$\begin{aligned}\sum_{i,j\geq 0} x^{i-j}q_{i,j}(n+1)U^iD^j &= (Qx^{-1}D(QxU+QI)^2)) \sum_{i,j\geq 0} x^{i-j}q_{i,j}(n)U^iD^j \\ &= \sum_{i,j\geq 0} x^{i-j+1}q_{i,j}(n)y^{3(i-j)+4}DU^{i+2}D^j \\ &\quad + x^{i-j}q_{i,j}(n)y^{3(i-j)}(y+y^2)DU^{i+1}D^j \\ &\quad + x^{i-j-1}q_{i,j}(n)y^{3(i-j)-1}DU^iD^j \\ &= \sum_{i,j\geq 0} x^{i-j+1}q_{i,j}(n)y^{3(i-j)+4}(U^{i+2}D^{j+1}+(i+2)U^{i+1}D^j) \\ &\quad + x^{i-j}q_{i,j}(n)y^{3(i-j)}(y+y^2)(U^{i+1}D^{j+1}+(i+1)U^iD^j) \\ &\quad + x^{i-j-1}q_{i,j}(n)y^{3(i-j)-1}(U^iD^{j+1}+iU^{i-1}D^j) \\ &= \sum_{i,j\geq 0} x^{i-j}U^iD^j(q_{i-2,j-1}(n)y^{3(i-j)+1}+(i+1)q_{i-1,j}(n)y^{3(i-j)+1} \\ &\quad + q_{i-1,j-1}(n)y^{3(i-j)}(y+y^2)+(i+1)q_{i,j}(n)y^{3(i-j)}(y+y^2) \\ &\quad + q_{i,j-1}(n)y^{3(i-j)+2}+(i+1)q_{i+1,j}(n)y^{3(i-j)+2}).\end{aligned}$$

Equating the coefficients of x^iU^i on both sides we get the recurrence relation:

$$q_{i,0}(n+1) = (i+1)(q_{i,0}(n)y^{3i}(y+y^2)+q_{i-1,0}(n)y^{3i+1}+q_{i+1,0}(n)y^{3i+2}). \tag{3.3}$$

We use $q_i(n)$ to denote $q_{i,0}(n)$. Let $b_i(n) = q_i(n)\Big|_{y=1}$ and let $c_i(n) = \dfrac{\partial}{\partial y}q_i(n)\Big|_{y=1}$. Thus, $b_i(n) = b_{i,0}(n) = (n+1)!\binom{n}{i}$ and $\sum_{T\in\text{Stam}_{\emptyset,\lambda}^{(n)}} wt(T) = c_i(n)f_\lambda$. Thus,

$$\frac{1}{T_{\emptyset,\lambda}^{(n)}} \sum_{T\in\text{Stam}_{\emptyset,\lambda}^{(n)}} wt(T) = \frac{c_i(n)}{b_i(n)}.$$

Applying the operator $\dfrac{\partial}{\partial y}\Big|_{y=1}$ to both sides of Equation (3.3) we get that,

$$\begin{aligned} c_i(n+1) = (i+1)(& \\ & 2c_i(n)+c_{i-1}(n)+c_{i+1} \\ & +(6i+3)b_i(n)+(3i+1)b_{i-1}(n)+(3i+2)b_{i+1}(n) \\ &) \end{aligned} \tag{3.4}$$

Let $P(n,i) = \frac{n^2+2(k+1)n+3k}{2}$. Thus we want to show that $c_i(n)/b_i(n) = P(n,i)$.

Applying induction to Equation (3.4), we get that

$$\begin{aligned} c_i(n+1) = (i+1)(& \\ & b_i(n)(2P(n,i)+6i+3) \\ & + b_{i-1}(n)(P(n,i-1)+3i+1) \\ & + b_{i+1}(n)(P(n,i+1)+3i+2) \\ &). \end{aligned} \tag{3.5}$$

Dividing both sides by $b_i(n+1)$ and putting in the values of $b_i(n)$, $b_i(n+1)$, $b_{i-1}(n)$ and $b_{i+1}(n)$ we have that

$$\begin{aligned} \frac{c_i(n+1)}{b_i(n+1)} &= \frac{(i+1)(n+1-i)}{(n+1)(n+2)}(2P(n,i)+6i+3) + \frac{(i)(i+1)}{(n+1)(n+2)}(P(n,i-i)+3i+1) \\ &+ \frac{(n-i)(n-i+1)}{(n+1)(n+2)}(P(n,i+1)+3i+2) \end{aligned}$$

Expanding P and checking on Sagemath ([Stein et al., 2018]) we get what we desire.

The proof for the identity in Equation (3.2) follows the same technique and is omitted here. □

On further investigation we see that not only average over the inn_0, but also over inn_1 and inn_2 form a polynomial and they add up to the polynomial of the overall average weight. We omit the proofs here.

3.3 Some Questions

A natural question that arises after seeing the cases for oscillating tableaux and stammering tableaux is given tableaux defined by a polynomial P in operators U, D, I, is the average weight of the tableaux polynomial for the operators P^n where $n \in \mathbb{N}$?

Chapter 4

Stammering Tableaux, Increasing Trees and Stirling Permutations

As stammering tableaux are in bijection with permutations which are in bijection with increasing binary trees (see Example II.17 in [Flajolet and Sedgewick, 2009] for details about increasing trees), we tried to find a direct bijection between rook placements on $2\delta_n$ and increasing trees of size $(n+1)$. We found a simple bijection and it has a nice property that it commutes with the bijection between permutations and rook placements on the double staircase which is obtained from Theorem 2.1.0.11 and Proposition 3.8 of [Josuat-Vergès, 2017].

We found that this bijection between increasing binary trees and permutations can be extended to a bijection between increasing $(r+1)$-ary trees and r-tuple Stirling permutations which are counted by r-tuple factorials. We saw that this is also the number of rook placements on $(r+1)$-tuple staircases. The bijection very naturally extended to $(r+1)$-ary increasing trees and rook placements on $(r+1)$-tuple staircases.

Inspired by this generalisation of the bijection we built an analog of large Laguerre profiles which are rational Dyck paths and chains of rational Dyck paths would give be the missing link in the commutative diagram.

In the paper [Tewari, 2019], a bijection between m-ary increasing trees of size $n+1$ and rook placements on $m\delta_n$ has been mentioned. But at the moment we do not know if our bijection is the same as the bijection mentioned in this paper.

We have can also see the higher order Eulerian numbers from the rook placements on higher tuple staircases.

4.1 Bijection Between Increasing Binary Trees and Rook Placements on Double Staircases

Consider the following algorithm $\mathfrak{B}$ which takes as input an incomplete binary tree of size $(n+1)$ and gives as output a rook placement on $2\delta_n$ with n rooks. Here rows, in French representation, are enumerated from top to bottom, 1 through n and the blocks are enumerated from left to right, 1 through n.

1. If the letter i is a leaf then keep the i^{th} block empty, *i.e.* without any rooks.

2. If the letter i has a left child a and no right child then we put a rook in the right column of the i^{th} block in the $(a-1)^{st}$ row.

3. If the letter i has a right child a and no left child then we put a rook in the left column of the i^{th} block in the $(a-1)^{st}$ row.

4. If the letter i has a right child a and a left child b, then we put a rook in the left column of the i^{th} block in the $(a-1)^{st}$ row and another rook in the right column of the same block in the $(b-1)^{st}$ row.

Clearly, given a rook placement on $2\delta_n$, we can reverse this algorithm to find an increasing tree. Thus, $\mathfrak{B}$ is a bijection.

Consider the permutation $\sigma = 3157264 \in \mathfrak{S}_7$. Then, its increasing tree is given by the tree in Figure 4.1. The stammering tableaux given by the bijection is drawn in Figure 4.2.

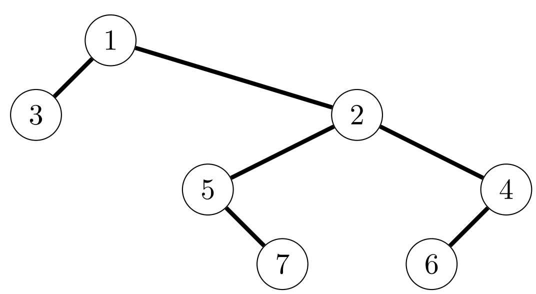


Figure 4.1: Increasing tree for permutation $\sigma = 3157264$.

Theorem 4.1.0.1. *Let us denote the natural bijection between $\mathfrak{S}_{n+1}$ and increasing trees of size $(n+1)$ as ψ, the bijection between chains of Dyck shapes and rook placements on $2\delta_n$ as ζ. Then the commutative diagram in Equation (4.1) holds.*

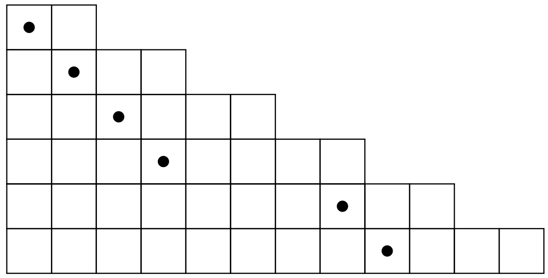


Figure 4.2: Rook placements on $2\delta_6$ for increasing tree in Figure 4.1.

$$\begin{array}{ccc} \mathfrak{S}_{n+1} & \overset{\textit{Chain of Large Profiles}}{\rightleftharpoons} & (n+1)\text{-}\textit{Chain of Dyck Shapes} \\ \Big\Updownarrow \psi & \circlearrowleft & \Big\Updownarrow \zeta \\ \textit{Increasing Trees of size } (n+1) & \overset{\mathfrak{B}}{\rightleftharpoons} & \textit{Rook Placements on } 2\delta_n \end{array} \tag{4.1}$$

Proof. The proof proceeds by induction. We can easily check that the diagram commutes for $n = 1$.

We note that for $\sigma \in \mathfrak{S}_{n+1}$, $\psi(\sigma')$ is obtained by removing $(n+1)$ which is a leaf of $\psi(\sigma)$. Let $Large(\cdot)$ denote the function that sends a permutation to its chain of large profiles. Also, $Large(\sigma')$ is also obtained from $Large(\sigma)$ by removing the outermost Dyck path. By induction as $\mathfrak{B}(\psi(\sigma')) = \zeta(Large(\sigma'))$, and as the rook placements in the first $(n-1)$ rows of $\mathfrak{B}(\psi(\sigma))$ and $\zeta(Large(\sigma))$ are the same as $\mathfrak{B}(\psi(\sigma')) = \zeta(Large(\sigma'))$, we only need to check if the rook placed in the last row of $\mathfrak{B}(\psi(\sigma))$ and $\zeta(Large(\sigma))$ are in the same position. If the parent of $(n+1)$ in $\psi(\sigma)$ is i then either $(n+1)$ is a right child of i or a left child of i. Thus, we have the following two cases:

Case 1: If $(n+1)$ is a right child of i, then in the last row of $\mathfrak{B}(\psi(\sigma))$, there is a rook in the left column of i^{th} block. Thus, $\sigma(\sigma^{-1}(i)+1) = (n+1)$. As, i is a is either a double ascent or a valley in σ, the first letter of $f(i)$ is $\nearrow$ and thus there is a rook in the left column of i^{th} block of $\zeta(Large(\sigma))$. As, there are no rooks in the first $(n-i)$ rows of the left column of the i^{th} block the rook is in the last row. Thus, $\mathfrak{B}(\psi(\sigma)) = \zeta(Large(\sigma))$.

Case 2: If $(n+1)$ is a left child of i then $\sigma(\sigma^{-1}(i)-1) = (n+1)$. Thus, i is either a double descent or a valley and thus the second letter of $f(i)$ is $\nearrow$. Now, we proceed as in Case 1. $\square$

4.2 m-Stirling Permutations and Rook Placements on $(m+1)$-tuple staircases

For this section we begin with the definition of m-Stirling permutations.

Definition 4.2.0.1 (m-Stirling Permutations of size n)**.** An *m-Stirling permutation of size n* is a permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_{mn}$ of the multiset $\{1^m, \dots, n^m\}$ such that whenever for $i < j$, $\sigma_i = \sigma_j = l \in \{1, \dots, n\}$, $\sigma_k \geq l$ for all $k \in \{i, \dots, j\}$.

Let $\mathcal{Q}_n(m)$ denote the set of m-Stirling permutations of size n. Given $\sigma \in \mathcal{Q}_n(m)$, let $\sigma(i)$ for $i \in \{1, \dots, n\}$ denote the largest subword of σ which consists of all i in σ and all its letters are greater than or equal to i. Thus, $\sigma(1) = \sigma$ and $\sigma(n) = \underbrace{n, \dots, n}_{m}$.

Let σ' denote the permutation σ with $\sigma(n)$ removed. Thus, σ' is an m-Stirling permutations of size $n-1$. To go from σ' to σ we need to insert $\underbrace{n, \dots, n}_{m}$ in σ' and we have $(mn+1)$ choices for this insertion. Thus, the cardinality of $\mathcal{Q}_n(m)$ is $(mn+1) \cdot \ldots (m+1) \cdot 1 = (mn+1)\underbrace{! \ldots !}_{m}$. Thus, the cardinality of $\mathcal{Q}_n(m)$ is given by m-tuple factorials.

There is a bijection between m-Stirling permutations of size n and increasing $(m+1)$-ary trees of size n as first described in [Park, 1994] and explained in [Janson et al., 2011]. Consider the 2-Stirling permutation 126624413355. Then its ternary increasing tree is in Figure 4.3.

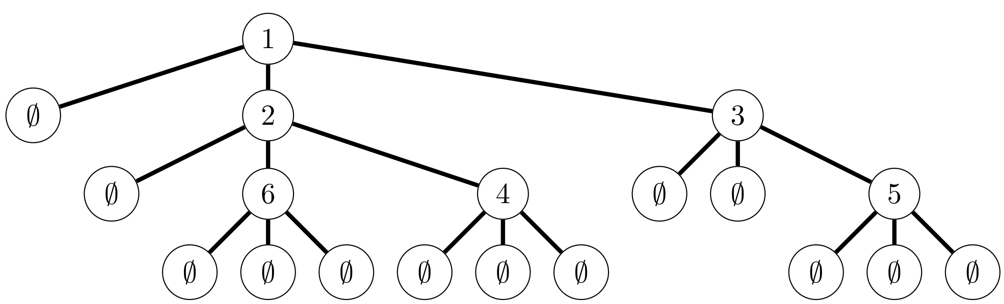


Figure 4.3: Ternary increasing tree for Stirling permutation 126624413355.

We easily can generalise the bijection $\mathfrak{B}$ to a bijection $\mathfrak{B}_m$ between increasing $(m+1)$-ary trees of size n and rook placements on $(m+1)\delta_{n-1}$ as follows:

1. If the letter i has all empty ($\emptyset$) children then keep the i^{th} block empty, *i.e.* without any rooks.

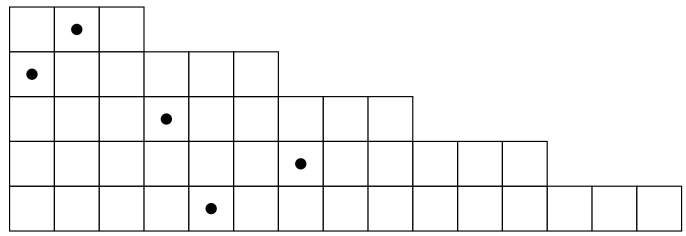


Figure 4.4: Rook placements on $3\delta_5$ for increasing tree in Figure 4.3.

2. If the letter i has a non-empty k^{th} child a then we put a rook in the $(m+1-k+1)^{st}$ column of the i^{th} block in the $(a-1)^{st}$ row.

The corresponding rook placement of the 2-Stirling permutation 126624413355 in $3\delta_5$ is given in Figure 4.4.

4.2.1 Laguerre Profile of m-Stirling Permutation

We first define m-Dyck paths.

Definition 4.2.1.1. An m-Dyck path of size n is defined as a sequence of up $\uparrow$ and right $\rightarrow$ steps from $(0,0)$ to (mn,n) such that at any point the path does not cross the line $x = my$, *i.e.* at any step in the path m times the number of $\uparrow$ in the path is greater than or equal to the number of $\rightarrow$.

We can obtain a path beginning at $(0,0)$ and ending at $(mn+m, n+1)$ from rook placements on $(m+1)\delta_n$ by the following algorithm:

1. Put a $\uparrow$ in the first step and $(\rightarrow)^m$ in the last m steps.
2. If the i^{th} column has a rook then put a $\uparrow$ as the $(i+1)^{st}$ step else put a $\rightarrow$ step as the $(i+1)^{st}$ step.

Theorem 4.2.1.2. *The above algorithm gives a m-Dyck path of size $n+1$ from rook placements on $(m+1)\delta_n$.*

Proof. In the $(m+1)\delta^n$ there are n columns with rooks and mn empty columns. These will correspond to $n+1$ $\uparrow$ and $mn+m$ $\rightarrow$ in the path obtained. We need to show that this path always stays above the $x = my$ line *i.e.* at any step in the path m times the number of $\uparrow$ in the path is greater than or equal to the number of $\rightarrow$.

The $(k+1)^{st}$ step depends whether there are rooks in the first k columns. There

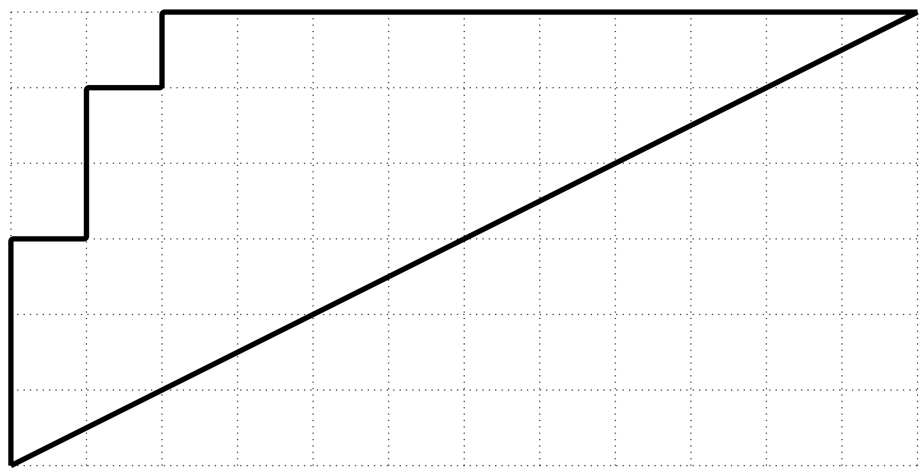


Figure 4.5: 2-Dyck Path obtained from the rook placement in Figure 4.4.

are $\lfloor \frac{k}{m+1} \rfloor$ rows already covered by the first k columns and thus, there are atleast $1 + \lfloor \frac{k}{m+1} \rfloor$ $\uparrow$ and atmost $k - \lfloor \frac{k}{m+1} \rfloor$ $\rightarrow$ in the first $(k+1)$ steps. Thus,

$$m(a_{k+1}) - (b_{k+1}) \geq m\left(1 + \left\lfloor \frac{k}{m+1} \right\rfloor\right) - \left(k - \left\lfloor \frac{k}{m+1} \right\rfloor\right) \geq 0. \tag{4.2}$$

In Equation (4.2), a_{k+1} is the number of $\uparrow$ in the first $k+1$ steps, and b_{k+1} is the number of $\rightarrow$ in the first $k+1$ steps. Thus, the path forms a m-Dyck path.

□

The 2-Dyck path for the rook placements in Figure 4.4 is in Figure 4.5.

We call this the *Laguerre profile of the Stirling permutation.*

4.2.2 Enumeration of $(m+1)$-tuple Staircase with a given Laguerre profile

Given an m-Dyck path of size $n+1$ we would like to find how many rook placements on $(m+1)\delta_n$ are possible such that after applying the algorithm $\mathfrak{B}_m$ we obtain the given m-Dyck path of size $n+1$. We can think of the path as a word of length $(m+1)(n+1)$ in the alphabet $\{\uparrow, \rightarrow\}$ such that in any prefix of the word, the m times the number of $\uparrow$ is greater than or equal to the number of $\rightarrow$ and m times the number of $\uparrow$ in the word is equal to the number of $\rightarrow$ in the word.

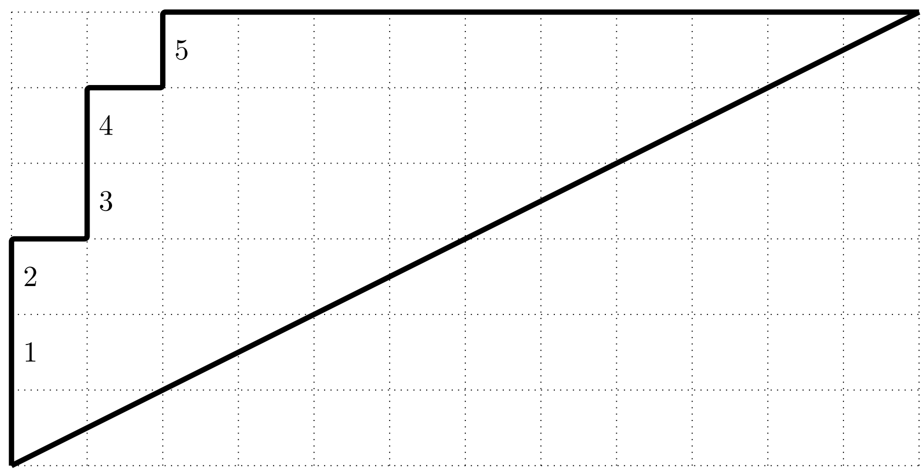


Figure 4.6: pos_P in 2-Dyck Path of Figure 4.5.

Given an m-Dyck path of size $n+1$, P let $P(i)$ denote the i^{th} letter in P. We define a function pos_P on those indexes i of P such that $i > 1$ and $P(i) = \uparrow$. It is the unique increasing function to $[n]$. We have mentioned the values of pos_P of Figure 4.5 in Figure 4.6.

We define $\text{block}_P(i)$ as $\lfloor (i-1)/(m+1) \rfloor + 1$ for all $i > 1$ and $i \leq (m+1)(n+1) - 2$.

Theorem 4.2.2.1. *Given an m-Dyck path of size $n+1$, its word P, the number of rook placements on $(m+1)\delta_n$ such that after applying the algorithm $\mathfrak{B}_m$ we obtain the given m-Dyck path of size $n+1$ is*

$$\prod_{\substack{i \\ P(i)=\uparrow}} (\text{pos}_P(i) - \text{block}_P(i) + 1). \tag{4.3}$$

Proof. We begin putting rooks from the last column of $(m+1)\delta_n$ to the first column of $(m+1)\delta_n$. At the i^{th} column, the last $(n-i)$ would have been filled. If the $P(i+1) = \rightarrow$, then the i^{th} column is empty and there is only one way of doing that. Else, there is a rook in the i^{th} column. It is in block $\text{block}_P(i+1)$, and hence there are $(n - \text{block}_P(i+1) + 1)$ rows in that block. But some of these rows already have a rook in them. There are $(n - \text{pos}_P(i+1))$ columns and hence the same number of rows with a rook. Thus, the number of ways of putting a rook in this column is

$$(n - \text{block}_P(i+1) + 1) - (n - \text{pos}_P(i+1)) = (\text{pos}_P(i+1) - \text{block}_P(i+1) + 1).$$

□

4.2.3 Higher Order Eulerian Numbers

We define descents in a m-Stirling Permutation and using the definition of descents, we define what is the m^{th} order Eulerian polynomials.

Definition 4.2.3.1 (Descents of Stirling Permutations)**.** Given $\sigma \in \mathcal{Q}_n(m)$ a *descent* of σ is a position $i \in \{1, \ldots, mn-1\}$ such that $\sigma_i > \sigma_{i+1}$. Let $\operatorname{des}(\sigma)$ denote the number of descents of σ.

For example, for $\sigma \in \mathcal{Q}_6(2)$ and $\sigma = 126624413355$, the descents are $4, 7$ and thus $\operatorname{des}(\sigma) = 2$.

Notice in Figure 4.4 there are exactly two blocks in which the first column has no rook. This is not a coincidence.

Theorem 4.2.3.2. *Given $\sigma \in \mathcal{Q}_{n+1}(m)$ the number of blocks in the corresponding rook placement in $(m+1)\delta_n$ with the first column empty is* $\operatorname{des}(\sigma)$*.*

Proof. By bijection $\mathfrak{B}_{m+1}$, the first row in the a^{th} block will be empty if and only if in the increasing tree of σ, the node labelled a has an empty $(m+1)^{st}$ child. Thus in $\sigma(a)$, the last letter is a. Now we have two cases,

Case 1: If the last a is not the last letter of σ, then it will be followed by a letter b. If $b > a$ then b will be a letter in $\sigma(a)$ and it will be after the last a in $\sigma(a)$. As this is not possible, $a < b$ and thus, the position of the last a in σ is a descent.

Case 2: If the last a is the last letter of σ then $n+1$ is not the last letter of σ and thus, the position of the last $(n+1)$ in σ is a descent.

Further, if there is a descent in position i then $\sigma_i > \sigma_{i+1}$. Thus, either $\sigma_i = n+1$ in which case there the last letter is a and in the a^{th} block the first column is empty. Else, in the σ_i^{th} block the first column is empty.

Thus the theorem is proved. □

Thus, the study of descents of $\sigma \in \mathcal{Q}_{n+1}(m)$ can be done by its corresponding rook placement in $(m+1)\delta_n$.

Conclusion

In this thesis, we mainly looked at three properties of stammering tableaux: projection from stammering tableaux to oscillating tableaux, polynomiality of average weights of stammering tableaux, and the connection between stammering tableaux and increasing trees, and its generalisation to the case of Stirling permutations. Each of the three directions have their own motivations to pursue them which has been mentioned at the beginning of each of the respective chapters.

Firstly, we looked at projection from stammering tableaux to oscillating tableaux, described the image of this projection and the cardinality of each of the pre-images of the projection. What makes it worthwhile to look at is that we found a surprising connection to γ-vectors of Eulerian numbers. It will be worthwhile to replace the domain of this projection from rook placements on $2\delta_n$ to rook placements on $m\delta_n$, for larger m and see the consequences of such projections in the same manner. We might uncover some properties of higher order Eulerian polynomials. Some other questions and conjectures, as mentioned in Section 2.3 to resolve are determining the numbers $p_n(k)$ for a fixed $k \in \mathbb{N}$.

Secondly, we defined a statistic called weight for stammering tableaux and showed that the average weight of stammering tableaux formed a polynomial. Further, we also noticed that weight can be written as a sum of three statistics which we call inner weights. The average inner weights also form polynomials which leads to their sum, the average weight, to form a polynomial. The proofs of these results depended on mimicking the proof in the case of oscillating tableaux. It will be interesting to see if there is a different proof technique to prove such results. A question that is mentioned in Section 3.3 is if this phenomenon of polynomiality of average weights which is seen in oscillating tableaux and stammering tableaux more general to tableaux defined by a polynomial on the operators U, D and I.

Rook placements on $2\delta_n$ are fundamental in the study of stammering tableaux. They are in the permutation garden and we wanted to describe the bijection with chains of large Laguerre profile and increasing trees such that we get to rook placements on $2\delta_n$. Surprisingly, the bijection was easy to describe and we also got a commutative diagram of bijections. We then extended this bijection to the case of increasing trees of higher arity. This gave a bijection between Stirling permutations and rook placements on higher tuple staircases. We described large Laguerre profiles as rational Dyck paths for Stirling permutations corresponding to this bijection and counted the number of Stirling permutations given by a single rational Dyck path. We also saw a connection to higher order Eulerian numbers. It might be possible to use rook placements and describe the generating series for higher order Eulerian permutations.

One thing that we have not investigated is if stammering tableaux possess some algebraic property such as an existence of a meaningful multiplication or comultiplication. Certainly, the stammering tableaux ending in $\emptyset$ are in bijection with permutations which have a Hopf algebra and operad structure on them. We have also not seen an order relation on stammering tableaux of the same size.

Finally, it will be interesting if we can find new bijections with other objects counted by permutations such as tree-like tableaux, alternative tableaux, permutation tableaux, etc.

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