$\mathbb{Q}$-Rational torsion of generalized modular Jacobians

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The modular curve $X_0(N)$

Given a positive integer N, let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

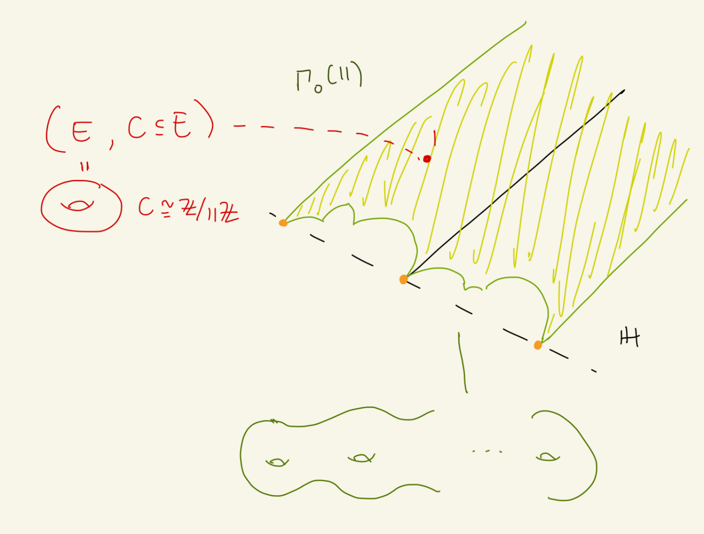
and $X_0(N)/\mathbb{Q}$ the projective non-singular **modular curve** associated to $\Gamma_0(N)$.

$$X_0(N)/\mathbb{C} \quad \simeq \quad \mathcal{H}/\ \Gamma_0(N) \quad \cup \quad \mathbb{P}^1(\mathbb{Q})/\Gamma_0(N)$$

$$\mathcal{H}/\Gamma_0(N) \downarrow Y_0(N) \qquad \mathbb{P}^1(\mathbb{Q})/\Gamma_0(N) \downarrow \mathrm{Cusps}(X_0(N))$$

$X_0(N)$ is the moduli space of $\{(E, C) : E/\mathbb{C} \text{ elliptic curve }, \mathbb{Z}/N\mathbb{Z} \simeq C \subset E\}/\simeq$.

The modular curve $X_0(N)$



The Jacobian $J_0(N)$

Let $J_0(N) := \mathrm{Jac}(X_0(N)) = Pic^0(X_0(N)) = Div^0(X_0(N))/\sim$. By considering the (Abel-Jacobi) injection

$$\iota : X_0(N) \to J_0(N)$$

we can study questions about elliptic curve by studying $J_0(N)$ and its properties as an abelian variety.

By Mordell-Weil, we have

$$J_0(N)(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus J_0(N)(\mathbb{Q})_{\mathrm{tor}}$$

Ogg's Conjecture

Andrew Ogg: All the torsion in $J_0(p)$ comes from the cusps.

For $N = p$ prime, $\text{Cusps}(X_0(N)) = \{0, \infty\} \subseteq J_0(N)(\mathbb{Q})$.

Ogg's Conjecture

Let N be a prime number. Then, the group

$$J_0(N)(\mathbb{Q})_{\text{tor}} = \langle [0 - \infty] \rangle \simeq \mathbb{Z}/\text{num}\left(\frac{N-1}{12}\right)\mathbb{Z}$$

Proofs of Ogg's conjecture: Mazur (1977); Ribet, Wake (2022).

Rational divisor class subgroup

Taking $C_N := \mathbf{im}(\mathbf{Div}^0_{\mathbf{cusp}}(X_0(N))\ \mathbf{in}\ J_0(N)))$, then any $[D] \in C_N$ is torsion! (Manin-Drinfeld).
However, in general $\mathrm{Cusps}(X_0(N)) \not\subseteq J_0(N)(\mathbb{Q})$.
Taking $C_N(\mathbb{Q}) := C_N \cap J_0(N)(\mathbb{Q})$, a natural generalization of Ogg's Conjecture is

Generalized Ogg's Conjecture

For any positive integer N we have

$$J_0(N)(\mathbb{Q})_{\mathrm{tor}} = C_N(\mathbb{Q}).$$

Large evidence in favour of the conjecture! (Go to Elvira's talk :))

Rational divisor class subgroup

Theorem. (Yoo)

For any positive integer N and any odd prime l such that l^2 does not divide $3N$ we have

$$J_0(N)(\mathbb{Q})_{\text{tor}}[l^\infty] \simeq \left(\bigoplus_{d\in D_1(N)} \langle [Z_l(d)] \rangle\right)[l^\infty]$$

for certain divisors $\{[Z_l(d)] \in C_N(\mathbb{Q}) : d \in D_1(N)\}$, where $D_1(N) := \{\text{divisors of N}\} \setminus \{1\}$.

The divisors $Z(d)$ and their orders can be explicitly computed.

Example

Example. Take $N = pq$ and l s.t. $\text{val}_l(p-1) \leq \text{val}_l(q-1)$, then $D_1(N) = \{p, q, pq\}$ and

- $Z_l(p) = (q-p)0 - (q-1)P_p + (p-1)P_q$ has order $\text{num}\left(\frac{(p-1)(q^2-1)}{24}\right)$.
- $Z_l(q) = 0 - P_q$ has order $\text{num}\left(\frac{(p^2-1)(q-1)}{24}\right)$.
- $Z_l(pq) = q0 - qP_p + P_q - \infty$ has order $\text{num}\left(\frac{(p-1)(q-1)\gcd(p+1,q+1)}{24\gcd(p-1,q-1)}\right)$.

Generalized Jacobians

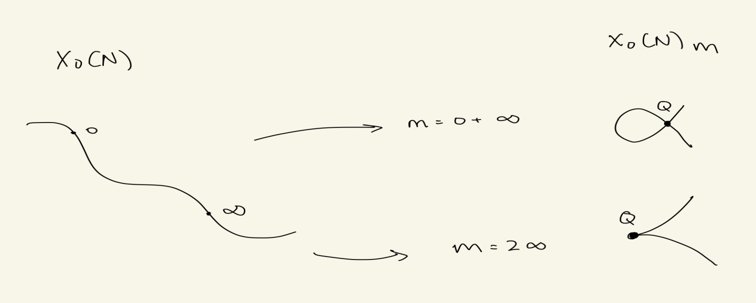
Jacobians help us explore smooth curves.

Generalized Jacobians

Generalised Jacobians help us explore certain singular curves

Let $\mathbf{m} \in \mathrm{Div}(X_0(N))$ an effective rational divisor (call it a **modulus**).
Construct $X_0(N)_{\mathbf{m}}$, the singular curve obtained from $X_0(N)$ by "glueing" the points in $\mathbf{m}$ into a single point Q.

Example (Graphic)



Generalized Jacobians

Definition

The generalised Jacobian $J_0(N)_{\mathbf{m}}$ is the group of isomorphism classes of degree zero line bundles on $X_0(N)_{\mathbf{m}}$.

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Example (Analogies with the "usual" Jacobian)

It has universal property, it can be used to study ramified covers of $X_0(N)$, it can be described in terms of divisors...

Given M a set of points in $X_0(N)$, there exists a modulus $\mathbf{m}$ with $M = \mathrm{Supp}(\mathbf{m})$, and a map

$$\iota_{\mathbf{m}} : X_0(N) \setminus M \to J_0(N)_{\mathbf{m}}$$

with universal property.

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It has universal property, **it can be used to study ramified covers of** $X_0(N)$, it can be described in terms of divisors...

Covers of $X_0(N)$ ramified at $\mathrm{Supp}(\mathbf{m})$ are in 1-to-1 correspondence with maps onto the generalised Jacobian $J(X_0(N))_{\mathbf{m}}$

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It can be used to study ramified covers of $X_0(N)$, it has universal property, **it can be described in terms of divisors**, ...

Let $\mathrm{Div}^0_{\mathbf{m}}(X_0(N))$ be the set of divisors coprime to $\mathbf{m}$. Then the generalized Jacobian $J_0(N)_{\mathbf{m}}$ is

$$J(X)_{\mathbf{m}} = \mathrm{Div}^0_{\mathbf{m}}(X)/\sim_{\mathbf{m}}.$$

where $\sim_{\mathbf{m}}$ denotes the linear equivalence

$$D_1 \sim_{\mathbf{m}} D_2 \text{ iff there is } f \in k(X) \text{ with } \mathrm{div}(f) = D_1 - D_2 \text{ and } f \equiv 1 \pmod{\mathbf{m}}.$$

Cuspidal modulus

This gives

$$0 \to L_{\mathbf{m}} \to J_0(N)_{\mathbf{m}} \to J_0(N) \to 0 \qquad (*)$$

The linear group $L_{\mathbf{m}}$ is copies of $\mathbb{G}_m$'s' and $\mathbb{G}_a$'s.

$J_0(N)_{\mathbf{m}}(\mathbb{Q})_{\mathrm{tor}}$ is not an Abelian variety!

We work with $\mathbf{m} = \sum_{P \in \mathrm{Cusps}(X_0(N))} P$.

Motivation

If $\mathbf{m} \subset \mathrm{Div}_{\mathrm{cusp}}(X_0(N))$, $(*)$ becomes explicit and $J_0(N)_{\mathbf{m}}$ seems to be related to (weakly) modular forms!

What can we say about $J_0(N)_{\mathbf{m}}(\mathbb{Q})_{\mathrm{tor}}$?

Main result

Extend results of Yamazaki-Yang (2016) and Wei-Yamazaki (2019).

Theorem. (C.I.)

Let N be an odd positive integer. For any odd prime l with l^2 not dividing $3N$ we can construct divisors $E_l(d) \in \mathrm{Div}^0_{\mathrm{cusp}}(X_0(N))$ such that

$$J_0(N)_{\mathbf{m}}(\mathbb{Q})_{\mathrm{tor}}[l^\infty] \simeq \left(\bigoplus_{d \in D_2(N)} \langle [E_l(d)] \rangle, \right) [l^\infty]$$

where $D_2(N) = \{d | N, \text{ divisible by at least 2 primes}\}$.

Example

Let $N = p^2 q$. Then $D_2(N) = \{pq, p^2 q\}$ and

- $E(pq) = (q+1)Z(p) - Z(pq)$ and its order is num $\left(\frac{(p-1)(q-1)}{24}\right)$;
- $E(p^2 q) = (q+1)Z(p^2) - Z(p^2 q)$ and its order is num $\left(\frac{(p^2-1)(q-1)}{24}\right)$.

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$$0 \to \bigoplus_{d'|N} (\mathbb{Q}(\zeta_{(N,d')})^\times)_{\mathrm{tor}} \to J_0(N)_{\mathbf{m}}(\mathbb{Q})_{\mathrm{tor}} \to J_0(N)(\mathbb{Q})_{\mathrm{tor}} \xrightarrow{\delta} \bigoplus_{d'|N} \mathbb{Q}(\zeta_{(N,d')})^\times \otimes \mathbb{Q}/\mathbb{Z}.$$

Use Yamazaki-Yang, Yoo's generators and modular units.

Thank you!