Asymptotics for Integral Points of Bounded Height on a log Fano Variety

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In the spirit of Manin's conjecture

X a Fano variety, k a number field

Conjecture

Assume that $X(k)$ is Zariski dense in X and consider an anticanonical height function $H : X(k) \to \mathbb{R}_{\geq 0}$. There exists an open subvariety V of X such that

$$\#\{x \in V(k) \ : \ H(x) \leq B\} \sim cB \log(B)^{r-1},$$

where r is the Picard number of X and c is a constant.

The set-up

Consider the blow-up of $\mathbb{P}^3_{\mathbb{Q}} = \mathrm{Proj}(\mathbb{Q}[a, b, c, d])$ along $C = V(a^2 + b^2 + c^2, d)$

$$\pi : X \to \mathbb{P}^3_{\mathbb{Q}}$$

and

$$U = X \setminus \pi^{-1}(V(a)).$$

$$N(B) = \#\{x \in U(\mathbb{Z}) \cap W(\mathbb{Q}) \ : \ H(x) \leq B\}.$$

$\to$ Universal torsors to parametrize integral points

Parametrization I

X the blow-up of $\mathbb{P}^3_{\mathbb{Q}}$ along $C = V(a^2+b^2+c^2, d)$,
$\mathcal{X}$ the blow-up of $\mathbb{P}^3_{\mathbb{Z}}$ along $C = V(a^2+b^2+c^2, d)$

We find the $\mathbb{G}^2_{m,\mathbb{Z}}$-torsor over $\mathcal{X}$

$$\mathcal{T} = \operatorname{Spec}\left(\frac{\mathbb{Z}[a,b,c,x,y,z]}{(a^2+b^2+c^2-yz)}\right) \setminus V((a,b,c,z)(x,y))$$

with the morphism $p : \mathcal{T} \to \mathcal{X}$.

Parametrization II

Lemma

The morphism p induces a 4-to-1 correspondence

$$\mathcal{T}(\mathbb{Z}) = \left\{(a,b,c,x,y,z) \in \mathbb{Z}^6 \;:\; \substack{a^2+b^2+c^2-yz=0,\\ \gcd(a,b,c,z)=\gcd(x,y)=1}\right\} \to \mathcal{X}(\mathbb{Z})$$

$$\#\{x \in U(\mathbb{Z}) \cap W(\mathbb{Q})\} =$$

$$\frac{1}{2}\#\left\{(b,c,x,y,z) \in \mathbb{Z}^5 \;:\; \substack{1+b^2+c^2-yz=0,\ \gcd(x,y)=1,\\ b,c,x,z\neq 0}\right\}$$

Height functions

Write any rational point $P \in \mathbb{P}^n(\mathbb{Q})$ as

$$P = (x_0 : \ldots : x_n)$$

with $x_0, \ldots, x_n \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$.

Definition

Define the **Weil height function** (or the **multiplicative height function**) on $\mathbb{P}^n(\mathbb{Q})$ as

$$H(P) = \max\{|x_0|, \ldots, |x_n|\}.$$

For any point $(a, b, c, x, y, z) \in \mathcal{T}(\mathbb{Z})$,

$$H(a, b, c, x, y, z) = \max \left\{ \begin{matrix} |a^2x|, |b^2x|, |c^2x|, \\ |ay|, |by|, |cy|, |x^3z^2| \end{matrix} \right\}.$$

The counting function

X the blow-up of $\mathbb{P}^3_{\mathbb{Q}}$ *along* $C = V(a^2 + b^2 + c^2, d)$,
$U = X \setminus \pi^{-1}(V(a))$

$$N(B) = \#\{\mathrm{x} \in \mathcal{U}(\mathbb{Z}) \cap W(\mathbb{Q}) \ : \ H(\mathrm{x}) \leq B\}$$

$$= \frac{1}{2}\#\left\{(b, c, x, y, z) \in \mathbb{Z}^5 \ : \ \begin{smallmatrix} 1+b^2+c^2-yz=0, \ \mathrm{gcd}(x,y)=1, \\ H(1,b,c,x,y,z)\leq B, \\ b,c,x,z\neq 0 \end{smallmatrix}\right\},$$

$$H(a, b, c, x, y, z) = \max\{|a^2x|, |b^2x|, |c^2x|, |z^2x^3|, |ay|, |by|, |cy|\}.$$

Are there any questions?