Derived Geometry Relative to Monoidal Quasi-abelian Categories

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This categorical approach will allow us to construct and compare lots of different types of geometries.

Monoidal Categories

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Theorem (Grothendieck)

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Quasi-abelian categories have a well-developed theory of homological algebra with notions of exact sequences, derived categories etc. Moreover, any quasi-abelian category admits a fully faithful embedding into an abelian category.

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Proposition (Ben-Bassat, Kremnizer, 2015)

The formal homotopy Zariski topology lines up with the G -topology on affinoid algebras.

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We note that the category $\mathbf{CBorn}_k$ of complete bornological spaces is a full concrete subcategory of $\mathbf{IndBan}_k$ and there is an equivalence of derived categories

$$\mathbf{D}(\mathbf{CBorn}_k) \simeq \mathbf{D}(\mathbf{IndBan}_k)$$

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Derived Analytic Geometry

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The missing piece I am providing is a representability theorem for higher stacks in this context.

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Example

Take $\mathcal{C} = \mathrm{Aff}$ *considered as an* $(\infty, 1)$*-category.*

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Example

Take $\mathcal{C} = \text{Aff}$ considered as an $(\infty, 1)$-category. When τ is the Zariski topology and $\mathbf{P}$ is the collection of open immersions, 0-geometric stacks are schemes. When we have the étale topology, and smooth maps, we obtain our usual notion of algebraic stacks.

We consider the $(\infty, 1)$-category $D^{-}\mathbf{Aff}_A := \mathbf{Comm}(\mathbf{sMod}_A)^{op}$ for $A \in \mathbf{Comm}(\mathbf{IndBan}_k)$. Define some topology τ on it and a class $\mathbf{P}$ of maps in $D^{-}\mathbf{Aff}_A$.

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Representability

We consider the $(\infty, 1)$-category $D^{-}\mathbf{Aff}_A := \mathbf{Comm}(\mathbf{sMod}_A)^{op}$ for $A \in \mathbf{Comm}(\mathbf{IndBan}_k)$. Define some topology τ on it and a class $\mathbf{P}$ of maps in $D^{-}\mathbf{Aff}_A$. Suppose that $\mathcal{F} \in \mathbf{Stk}(D^{-}\mathbf{Aff}_A, \tau)$.

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- When is $\mathcal{F}$ n-geometric for some n?
- Do we have nice examples of derived analytic moduli stacks which are n-geometric?

Thanks for listening!