

Derived Geometry Relative to Monoidal Quasi-abelian Categories

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This categorical approach will allow us to construct and compare lots of different types of geometries.

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Theorem (Grothendieck)

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Quasi-abelian categories have a well-developed theory of homological algebra with notions of exact sequences, derived categories etc. Moreover, any quasi-abelian category admits a fully faithful embedding into an abelian category.

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Proposition (Ben-Bassat, Kremnizer, 2015)

The formal homotopy Zariski topology lines up with the G -topology on affinoid algebras.

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$$\mathbf{D}(\mathbf{CBorn}_k) \simeq \mathbf{D}(\mathbf{IndBan}_k)$$

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Mod \mathbb{Z}

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Mod $_{\mathbb{Z}}$



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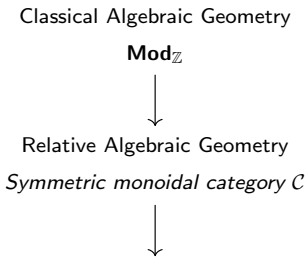
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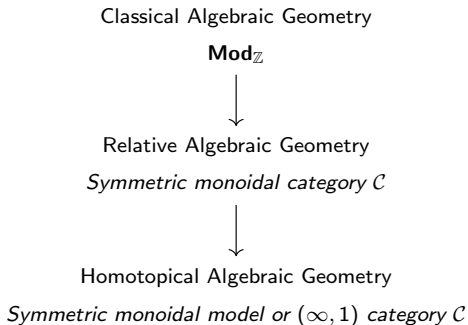
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Symmetric monoidal category \mathcal{C}

Homotopical Algebraic Geometry



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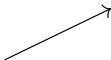
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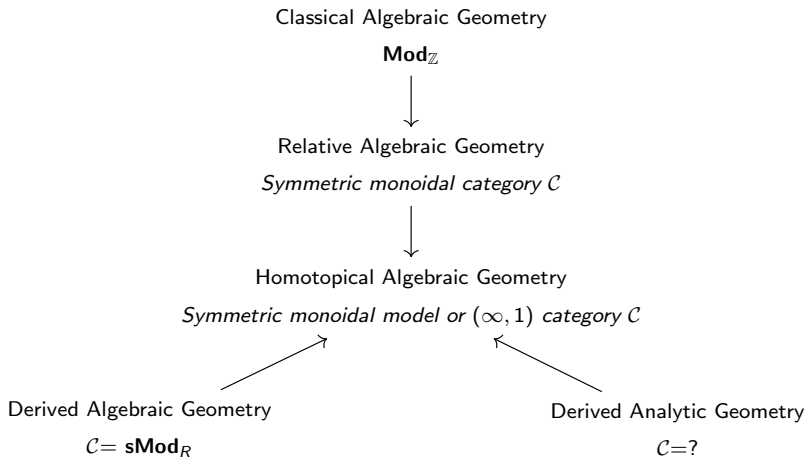
Symmetric monoidal model or $(\infty, 1)$ category \mathcal{C}



Derived Algebraic Geometry

$\mathcal{C} = \mathbf{sMod}_R$

Homotopical Algebraic Geometry



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The missing piece I am providing is a representability theorem for higher stacks in this context.

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Example

Take $\mathcal{C} = \mathrm{Aff}$ considered as an $(\infty, 1)$ -category. When τ is the Zariski topology and \mathbf{P} is the collection of open immersions, 0-geometric stacks are schemes. When we have the étale topology, and smooth maps, we obtain our usual notion of algebraic stacks.

We consider the $(\infty, 1)$ -category $D^- \mathbf{Aff}_A := \mathbf{Comm}(\mathbf{sMod}_A)^{op}$ for $A \in \mathbf{Comm}(\mathbf{IndBan}_k)$. Define some topology τ on it and a class \mathbf{P} of maps in $D^- \mathbf{Aff}_A$.

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- When is \mathcal{F} n -geometric for some n ?
- Do we have nice examples of derived analytic moduli stacks which are n -geometric?

Thanks for listening!