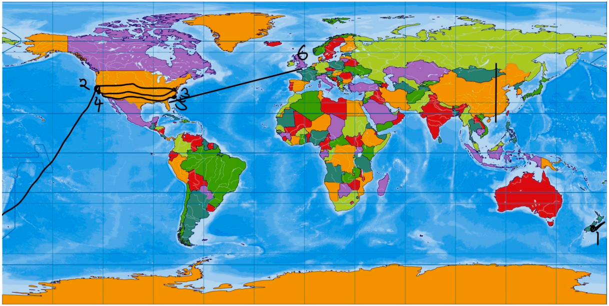
Tropical Geometry

Diane Maclagan

WINGS 2023

Career history

Map credit: http://vecteezy.com



Undergraduate: Christchurch, New Zealand

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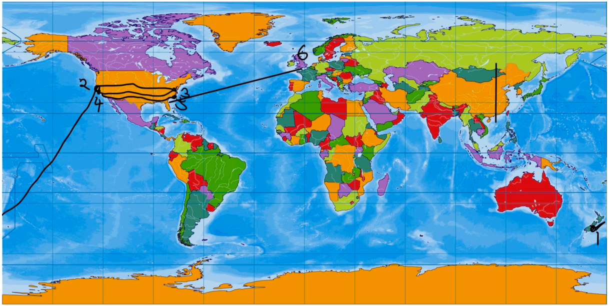

PhD: UC Berkeley 2000

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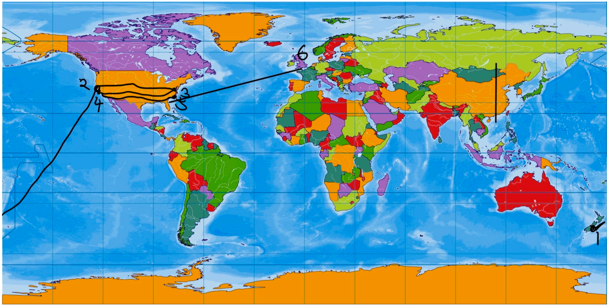

Postdoc 1: IAS, Princeton 2000–2001

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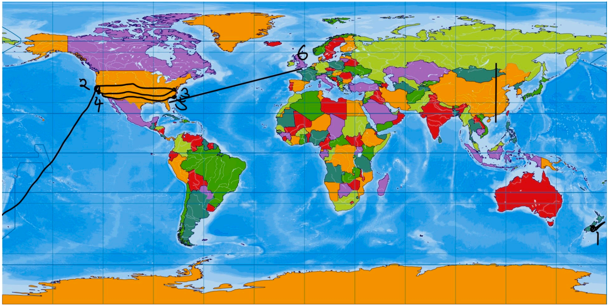

Postdoc 2: Stanford 2001–2004

Career history

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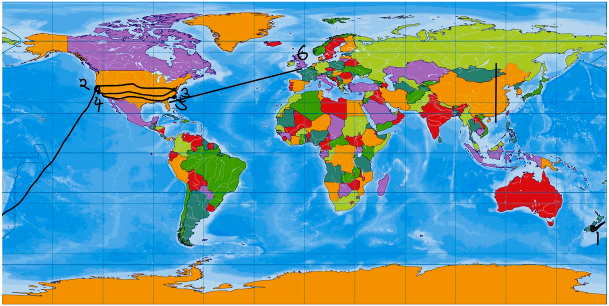

Tenure-track position: Rutgers University, NJ 2004–2007

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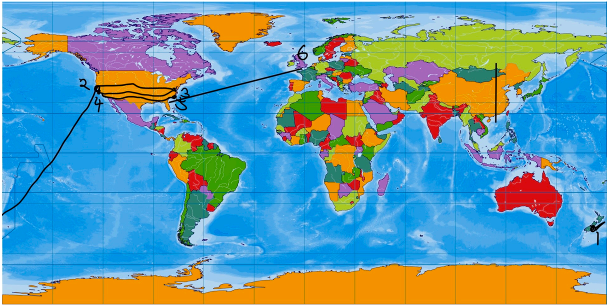


Current position: University of Warwick, 2007–

Research history

- PhD: Combinatorics/Commutative algebra
- Postdocs: Moving towards algebraic geometry
- Tenure track: Work in algebraic geometry. Start hearing about tropical geometry . . .
- Summer after first year at Warwick: teach 4 week summer school on tropical geometry. Over next 7 (!) years turn notes into first text book on the subject.
- Currently: Tropical geometry, with sidelines in other parts of combinatorial algebraic geometry.

EDI work

1. LMS Committee for Women and Diversity in Mathematics 2016–2022.
2. Currently Convenor for European Women in Mathematics (EWM)

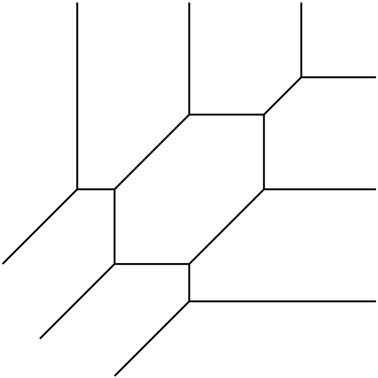
European Women in Mathematics

EWM

- Founded in 1986
- Members in 34 countries in Europe
- Over 400 individual members and 36 institutional members
- Activities:
  - General meeting (every two years)
  - EWM/EMS summer schools at Institut Mittag Leffler
  - Network of country coordinators
  - Mentoring programme
  - Job ads
  - Travel grants
  - Activism

https://www.europeanwomeninmaths.org

Tropical Geometry



The tropical semiring

$$\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot),$$

where $\oplus = \min$ and $\odot = +$.

Examples: $5 \oplus 8 = 5$
$3 \odot 8 = 11$
$(6 \odot 5) \oplus 10 = ?$
$3 \odot (5 \oplus 8) = 8 = 3 \odot 5 \oplus 3 \odot 8$.

This is commutative, associative, ∞ is the additive identity, and 0 is the multiplicative identity.

Warning: No subtraction (so we have a semiring).

Why?

Algebraic geometry “over” the tropical semiring $\overline{\mathbb{R}}$ is a combinatorial shadow of usual algebraic geometry*

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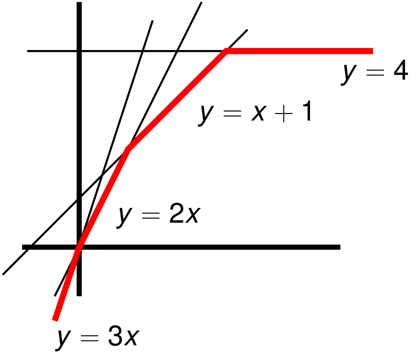
Goal: Use combinatorial shadow to solve the original problem

Tropical polynomials

Tropical polynomials are piecewise linear functions:

Example:

$$F = x^3 \oplus x^2 \oplus 1 \odot x \oplus 4$$
$$= \min(3x, 2x, x+1, 4)$$



Tropical polynomials

Problem: What are the roots of $3 \odot x \oplus -2$?

Tropical polynomials

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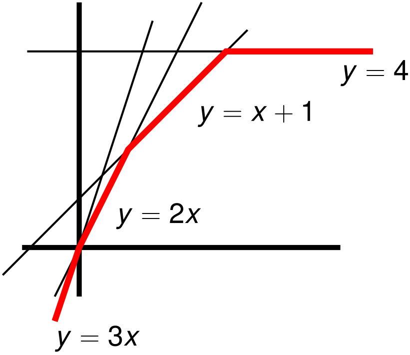
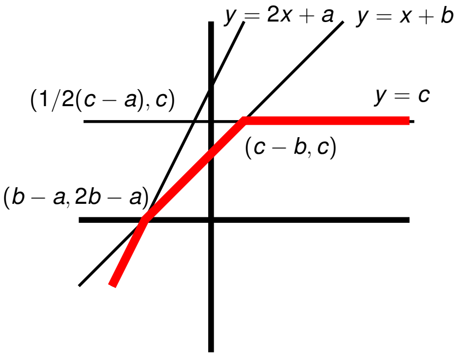
Answer: “Roots” of such equations are points where the graph of F is not differentiable.
Example: 0, 1, and 3 are roots of $F = x^3 \oplus x^2 \oplus 1 \odot x \oplus 4$



The tropical quadratic formula

$F = a \odot x^2 \oplus b \odot x \oplus c = \min(2x + a, x + b, c)$.



"Roots":

$$x = \begin{cases} b - a, c - b & \text{if } 2b \leq a + c \\ 1/2(c - a) & \text{if } 2b > a + c \end{cases}$$

Connection to usual polynomials

$K^* = K \setminus \{0\}$
Consider coefficients in a field K with a nontrivial valuation

$$\mathrm{val} : K^* \to \mathbb{R}$$

$\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$,
$\mathrm{val}(a + b) \geq \min(\mathrm{val}(a), \mathrm{val}(b))$.

Example: $K = \mathbb{Q}$. $\mathrm{val}(p^n a/b) = n$.
$\mathrm{val}_2(20/3) = \mathrm{val}_2(2^2 5/3) = 2$,
$\mathrm{val}_3(5/12) = \mathrm{val}_3(3^{-1} 5/4) = -1$.

Example: $K = \mathbb{C}((t))$ (Laurent series).
$a = 3t^{-2} + 7t^3 + 8t^9 + \dots$ $\mathrm{val}(a) = -2$.

Proposition Given a polynomial $f = \sum_{i=0}^{d} a_i x^i \in K[x]$, set

$$\mathrm{trop}(f) = \bigoplus_{i=0}^{d} \mathrm{val}(a_i) \odot x^i = \min(\mathrm{val}(a_i) + ix).$$

If $f(b) = 0$ then $\mathrm{val}(b)$ is a root of $\mathrm{trop}(f)$.

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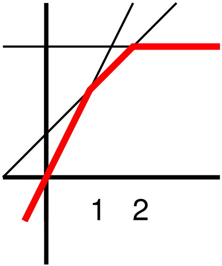
If $f(b) = 0$ then $\mathrm{val}(b)$ is a root of $\mathrm{trop}(f)$.

Example: $f = x^2 + tx + t^3 \in \mathbb{C}((t))[x]$. $f(b) = 0$ for $b = \dfrac{-t \pm \sqrt{t^2 - 4t^3}}{2} = \{-t + t^2 + t^3 + t^4 + \cdots, -t^2 - t^3 - 2t^4 + \dots\}$.

$\mathrm{trop}(f) = x^2 \oplus 1 \odot x \oplus 3$.



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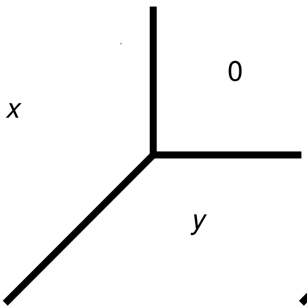
If $f(b) = 0$ then $\mathrm{val}(b)$ is a root of $\mathrm{trop}(f)$.

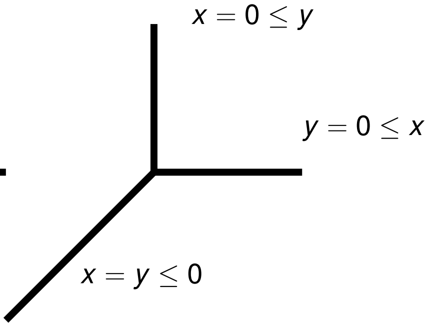
So we can't solve a general quintic, but we can determine the valuations of its roots!

Tropical hypersurfaces

Definition: The tropical hypersurface $V(F)$ defined by the tropical polynomial F is the locus where the graph of F is not differentiable.

Example: $F = x \oplus y \oplus 0$ $\qquad F(x, y) = \min(x, y, 0)$.





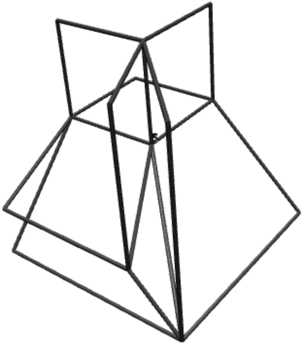
$w \in V(F)$ if and only if the minimum is achieved at least twice in $F(w)$.

Tropical hypersurfaces

Example:

$$f = x^3 \oplus y^3 \oplus x^2z \oplus y^2z \oplus z^2 \oplus x \oplus y \oplus 0$$
$$= \min(3x, 3y, 2x + z, 2y + z, 2z, x, y, 0).$$

Then $V(f)$ is



Tropical hypersurfaces

Notation: For $u \in \mathbb{Z}^n$, $x^u = x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$.
Definition: If $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1, \dots, x_n]$ is a polynomial, then

$$F = \mathrm{trop}(f) := \bigoplus \mathrm{val}(c_u) \odot x^u = \min(\mathrm{val}(c_u) + x \cdot u)$$

The tropical hypersurface $\mathrm{trop}(V(f))$ is the tropical hypersurface of F.

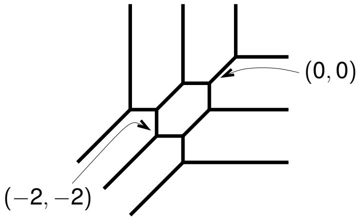
Example: Let $f = x + y + 1$. Then $\mathrm{trop}(f) = x \oplus y \oplus 0 = \min(x, y, 0)$.

$\mathrm{trop}(f)(w) = \min(\mathrm{val}(c_u) + w \cdot u).$

Example:

$f = 7t^4x^3 + 5t^2x^2y + 2txy^2 - t^4y^3 + 2tx^2 - xy + ty^2$
$+3x + 8y + t \in \mathbb{C}((t))[x, y].$

$$\begin{aligned}\mathrm{trop}(f) &= 4 \odot x^3 \oplus 2 \odot x^2y \oplus 2 \odot xy^2 \oplus 4 \odot y^3 \oplus 1 \odot x^2 \oplus xy \oplus 1 \odot y^2 \\ &\quad \oplus x \oplus y \oplus 1 \\ &= \min(3x + 4, 2x + y + 2, x + 2y + 2, 3y + 4, 2x + 1, \\ &\quad x + y, 2y + 1, x, y, 1)\end{aligned}$$



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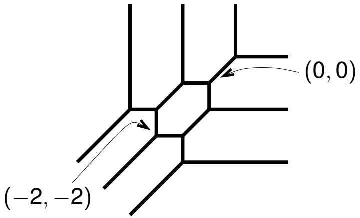


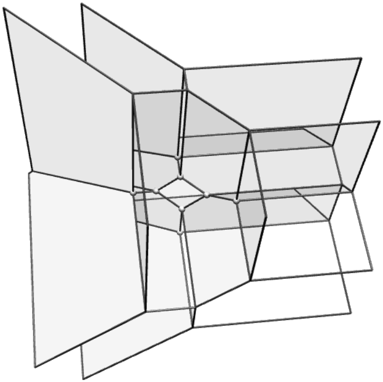
Example:

$f = 12x^2 + 20y^2 + 8z^2 + 7xy + 22xz + 3yz + 5x + 9y + 6z + 4$

with the 2-adic valuation.

$\text{trop}(f) = \min(2x+2, 2y+2, 2z+3, x+y, x+z+1, y+z, x, y, z+1, 2)$

$\text{trop}(V(f))$:



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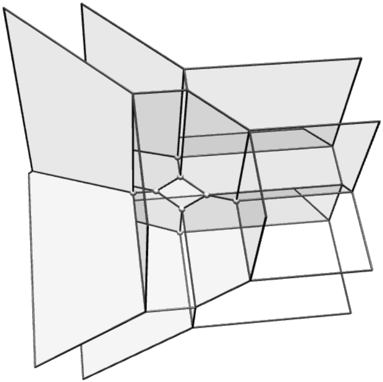

Tropical Varieties

$X = V(f_1, \ldots, f_r) = \{x \in K^n : f_1(x) = \cdots = f_r(x) = 0\}$.

Definition: The tropical variety of X is

$$\mathrm{trop}(X) = \bigcap_{f = \sum g_i f_i} V(\mathrm{trop}(f))$$

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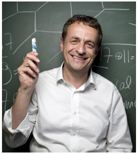
Theorem [Bieri/Groves, Kapranov, Speyer, Sturmfels ...] The tropical variety of X is the support of a balanced polyhedral complex of the same dimension as X. It equals $\overline{\mathrm{val}(X(L))}$ for any algebraically closed $L \supset K$ with a nontrivial valuation.





Why?

Tropical varieties are combinatorial shadows of classical varieties.

Many invariants of the variety can be determined from the combinatorics of the tropical variety.

Turns Algebraic Geometry (hard) into Polyhedral Geometry/Combinatorics (somewhat easier)

Enumerative Geometry

A curve of degree d is a set $\{[x : y : z] \in \mathbb{P}^2 : f(x, y, z) = 0\}$ where f a homogeneous polynomial of degree d.

Question: How many rational curves of degree d pass through $3d - 1$ general points in the plane?

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Answers:

d	$3d - 1$	N_d = number of curves
1	2	1
2	5	1
3	8	12 (Steiner, 1848)
4	11	620 (Zeuthen, 1873)

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1	2	1
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4	11	620 (Zeuthen, 1873)

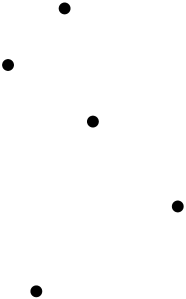
General d (Kontsevich 1994)

$$N_d = \sum_{\substack{d_A+d_B=d,\\ d_A\geq 1, d_B\geq 1}} \left(\binom{3d-4}{3d_A-1} d_A^2 - \binom{3d-4}{3d_A-2} d_A d_B \right) N_{d_A} N_{d_B} d_A d_B$$

(involves integrating on the moduli space of stable maps)

Answers:

(Mikhalkin 2002) You can just count (with multiplicity) the number of tropical rational curves of degree d passing with through $3d - 1$ general points in $\mathbb{R}^2$.



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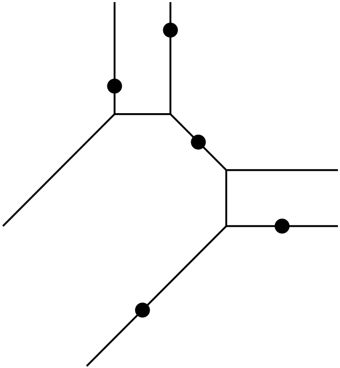
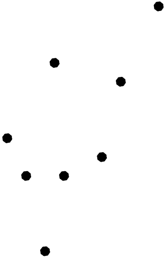


(Gathmann, Markwig 2007) This can be done with tropical intersection theory on the tropical moduli space of stable maps.

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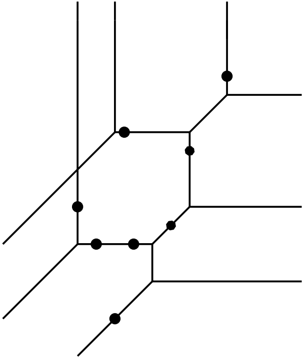

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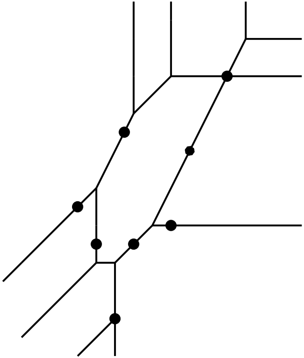

Tropical schemes

Story so far is all about varieties. . .

What about more modern algebraic geometry?

Tropical schemes

Story so far is all about varieties...
What about more modern algebraic geometry?
Start with (affine) schemes.

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Story so far is all about varieties...
What about more modern algebraic geometry?
Start with (affine) schemes.

Problem: The semiring of tropical polynomials $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ is more complicated than the standard polynomial ring.

- Multiplication is not cancellative.
  $(x^2 \oplus 0) \odot (x^2 \oplus x \oplus 0) = (x^2 \oplus x \oplus 0)^2$
- Ideals do not have to be finitely generated.
  $\langle x \oplus y, x^2 \oplus y^2, x^3 \oplus y^3, x^4 \oplus y^4, \ldots \rangle \subset \overline{\mathbb{R}}[x, y]$.
- Varieties can be complicated.

Tropical schemes

Fix: Consider only ideals with extra structure.

(M-Rincón) A tropical ideal is a homogeneous ideal in the semiring of tropical polynomials with the property that each graded piece determines a valuated matroid.

Based on work of Giansiracusa-Giansiracusa on tropicalizing subschemes of $\mathbb{F}_1$ varieties. Gives a definition of a subscheme of a tropical toric variety.



Other applications

- Mirror symmetry (Gross-Siebert program, ...)
- Geometry of curves (Brill Noether theory, ...)
- Analytic geometry (Berkovich spaces, ...)
- Rational points (Chabauty method, ...)
- Real algebraic geometry (Enumeration, Welschinger invariants, ...)
- Integrable systems (Ultradiscrete KdV, ...)
- Optimization (max-plus optimization, complexity of LP, ...)
- Economics (Product-Mix auctions, ...)
- Celestial mechanics (three body problem,...)
- Machine learning (ReLU networks, ...)
- ...