EIGENFUNCTION SCARRING AND IMPROVEMENTS IN L^∞ BOUNDS

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Abstract. We study the relationship between L^∞ growth of eigenfunctions and their L^2 concentration as measured by defect measures. In particular, we show that scarring in the sense of concentration of defect measure on certain submanifolds is incompatible with maximal L^∞ growth. In addition, we show that a defect measure which is too diffuse, such as the Liouville measure, is also incompatible with maximal eigenfunction growth.

1. Introduction

Let (M, g) be a C^∞ compact manifold of dimension n without boundary. Consider the eigenfunctions

$$(-\Delta_g - \lambda_j^2)u_{\lambda_j} = 0, \quad \|u_{\lambda_j}\|_{L^2} = 1 \tag{1.1}$$

as $\lambda_j \to \infty$. It is well known [Ava56, Lev52, Hör68] (see also [Zwo12, Chapter 7]) that solutions to (1.1) satisfy

$$\|u_{\lambda_j}\|_{L^\infty(M)} \le C\lambda_j^{\frac{n-1}{2}} \tag{1.2}$$

and that this bound is saturated e.g. on the sphere. It is natural to consider the situations which produce sharp examples for (1.2). In many cases, one expects polynomial improvements to (1.2), but rigorous results along these lines are few and far between [IS95]. At present, under general dynamical assumptions, known results involve o-improvements to (1.2) [TZ02, SZ02, TZ03, STZ11, SZ16a, SZ16b]. These papers all study the connections between the growth of L^∞ norms of eigenfunctions and the global geometry of the manifold (M, g). In this note, we examine the relationship between L^∞ growth and L^2 concentration of eigenfunctions. We measure L^2 concentration using the concept of a *defect measure* - a sequence $\{u_{\lambda_j}\}$ has defect measure μ if for any $a \in S^0_{\text{hom}}(T^*M \setminus \{0\})$,

$$\langle a(x, D)u_{\lambda_j}, u_{\lambda_j}\rangle \to \int_{S^*M} a(x, \xi)d\mu. \tag{1.3}$$

By an elementary compactness/diagonalization argument it follows that any sequence of eigenfunctions u_{λ_j} solving (1.1) possesses a further subsequence that has a defect measure in the sense of (1.3) ([Zwo12, Chapter 5],[Gér91]). Moreover, a standard commutator argument shows that if $\{u_{\lambda_j}\}$ is any sequence of L^2-normalized Laplace eigenfunctions, the associated defect measure μ is invariant under the geodesic flow; that is, if $G_t : S^*M \to S^*M$ is the geodesic flow, $(G_t)_*\mu = \mu$, $\forall t \in \mathbb{R}$.

Definition **1.1.** *We say that an eigenfunction subsequence is* strongly scarring *provided* $\operatorname{supp}\mu$ *is a finite union of periodic geodesics.*

Theorem **1.** *Let* $\{u_{\lambda_j}\}$ *be a strongly scarring sequence of solutions to* (1.1)*. Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{\frac{n-1}{2}}).$$

We also have improved L^∞ bounds when eigenfunctions are *quantum ergodic*, that is, their defect measure is the Liouville measure on S^*M, μ_L.

Theorem **2.** *Let $\{u_{\lambda_j}\}$ be a quantum ergodic sequence of solutions to* (1.1). *Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{\frac{n-1}{2}}).$$

Theorems 1 and 2 are corollaries of our next theorem where we relax the assumptions on μ and make the following definitions. Define respectively the *flow out* and *time T flow out* by

$$\Lambda_x := \bigcup_{T=0}^{\infty} \Lambda_{x,T}, \qquad \Lambda_{x,T} := \bigcup_{t=-T}^{T} G_t(S_x^*M).$$

Definition **1.2.** *Let $\mathcal{H}^n$ be n-dimensional Hausdorff measure on S^*M induced by the Sasaki metric on T^*M (see for example* [Bla10, Chapter 9] *for a treatment of the Sasaki metric). We say that the subsequence $u_{\lambda_j}; j = 1, 2, ...$ is* admissible at x *if*

$$\mathcal{H}^n(\text{supp } \mu|_{\Lambda_x}) = 0. \tag{1.4}$$

We say that the subsequence is admissible *if it is admissible at x for every $x \in M$.*

We note that in (1.4) $\mu|_{\Lambda_x}$ denotes the defect measure restricted to the flow out Λ_x; for any A that is μ-measurable,

$$\mu|_{\Lambda_x}(A) := \mu(A \cap \Lambda_x).$$

In particular, $\mu|_{\Lambda_x}$ should not be confused with the pushforward measure $(r_x)_*\mu$ where $r_x : S^*M \to \Lambda_x$ is restriction.

Theorem **3.** *Let $\{u_{\lambda_j}\}$ be a sequence of L^2-normalized Laplace eigenfunctions that is admissible in the sense of* (1.4). *Then*

$$\|u_{\lambda_j}\|_{L^\infty} = o(\lambda_j^{\frac{n-1}{2}}).$$

Remark 1.3: We choose to use the Sasaki metric to define $\mathcal{H}^n$ for concreteness, but this is not important and we could replace the Sasaki metric by any other metric on S^*M.

Theorem 3 can be interpreted as saying that eigenfunctions which strongly scar are too concentrated to have maximal L^∞ growth, while diffuse eigenfunctions are too spread out to have maximal growth. However, the reason the adimissiblity assumption is satisfied differs in these cases. In the diffuse case (see Theorem 2), one has $\mu|_{\Lambda_x} = 0$, so that the admissibility assumption is trivially verified. In the case where the eigenfunctions strongly scar (see Theorem 1), $\mu|_{\Lambda_x} \neq 0$ but the Hausdorff dimension of supp $\mu|_{\Lambda_x}$ is $< n$; so again, (1.4) is satisfied. The zonal harmonics on the sphere S^2 lie precisely between being diffuse and strongly scarring (see section 4).

Observe that the condition μ is diffuse is much more general than $\mu = \mu_L$. One example for which there are diffuse eigenfunctions which are not quantum ergodic is the mushroom billiard [Gal14, Gom15] (see also [Riv13] for further examples).

1.1. **Relation with previous results.**

Theorem 2 is related to [STZ11, Theorem 3], where the $o(h^{\frac{1-n}{2}})$ sup bound is proved for all Laplace eigenfunctions on a C^ω surface with ergodic geodesic flow. However, in Theorem 2, we make no analyticity or dynamical assumptions on (M, g) whatsoever, only an assumption on the particular defect measure associated with the eigenfunction sequence. Recently, Hezari [Hez16] gave an independent proof of Theorem 2.

In [SZ02], Sogge–Zelditch prove that any manifold on which (1.2) is sharp must have a self focal point. That is, a point x such that $|\mathcal{L}_x| > 0$ where

$$\mathcal{L}_x := \{\xi \in S_x^*M \mid \text{ there exists } T \text{ such that } \exp_x T\xi = x\}$$

and $|\cdot|$ denotes the normalized surface measure on the sphere. Subsequently, in [STZ11] the authors showed that one can replace $\mathcal{L}_x$ by the set of recurrent directions $\mathcal{R}_x \subset \mathcal{L}_x$ and the assumption $|\mathcal{R}_x| > 0$ for some $x \in M$ is necessary to saturate the maximal bound in (1.2). The example of the triaxial ellipsoid with x equal to an umbilic point shows that latter assumption is weaker than the former. Indeed, in such a case $|\mathcal{L}_x| = 1$ whereas $|\mathcal{R}_x| = 0$. Most recently, in [SZ16a, SZ16b], it was proved that for real-analytic surfaces, the maximal L^∞ bound can only achieved if there exists a periodic point $x \in M$ for the geodesic flow. At such a point, all geodesics starting at $(x, \xi) \in S^*M$ close up smoothly after some finite time $T > 0$.

Together with our analysis, the results of [STZ11] imply that any sequence of eigenfunctions, $\{u_\lambda\}$ having maximal L^∞ growth and defect measure μ must have $\operatorname{supp}\mu \cap S^*_xM \neq \emptyset$ where $|\mathcal{R}_x| > 0$. However, as far as the authors are aware, the results there and in [SZ16a, SZ16b] do not give additional information about μ. On the other hand, under an additional regularity assumption on the measure μ, Theorem 3 shows that $\mu|_{\Lambda_x}$ is not mutually singular with respect to $\mathcal{H}^n$. This implies that the measure μ resembles the defect measure of a zonal harmonic. In a forthcoming paper, the first author removes the necessity for any additional regularity assumption and gives a full characterization of defect measures for eigenfunctions with maximal L^∞ growth [Gal17]. Finally, we note that unlike [SZ02, STZ11, SZ16a, SZ16b], the analysis here is entirely local.

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2. A local version of 3

In the following, we will freely use semiclassical pseudodifferential calculus where the semiclassical parameter is h with $h^{-1} = \lambda \in \operatorname{Spec}\sqrt{-\Delta_g}$. We start with a local result:

Theorem 4. *Let $\{u_h\}$ be sequence of Laplace eigenfunctions that is admissible at x. Then for any $r(h) = o(1)$,*

$$\|u_h\|_{L^\infty(B(x,r(h)))} = o(h^{\frac{1-n}{2}}).$$

Theorem 3 is an easy consequence of Theorem 4.

Proof that Theorem 4 implies Theorem 3. Suppose that u is admissible and

$$\limsup_{h\to 0} h^{\frac{n-1}{2}}\|u_h\|_{L^\infty} \neq 0.$$

Then, there exist $c > 0$, $h_k \to 0$, x_{h_k} so that

$$|u_{h_k}(x_{h_k})| \geq c h_k^{-\frac{n-1}{2}}.$$

Since M is compact, by taking a subsequence, we may assume $x_{h_k} \to x$. But then $d(x, x_{h_k}) = o(1)$ and since u is admissible at x, Theorem 4 implies

$$\limsup_{k\to\infty} h_k^{\frac{n-1}{2}}|u_{h_k}(x_{h_k})| = 0.$$

□

3. Proof of Theorem 4

In view of the above, it suffices to prove the local result: Theorem 4.

Proof. Fix $\delta > 0$ and let $\rho \in \mathcal{S}(\mathbb{R})$ with $\rho(0) = 1$ and $\operatorname{supp} \hat{\rho} \subset (\delta, 2\delta)$. Let

$$S^*M(\varepsilon) := \{(x,\xi); ||\xi|_x - 1| \leq \varepsilon\}$$

and $\chi(x,\xi) \in C_0^\infty(T^*M)$ be a cutoff near the cosphere S^*M with $\chi(x,\xi) = 1$ for $(x,\xi) \in S^*M(\varepsilon)$ and $\chi(x,\xi) = 0$ when $(x,\xi) \in T^*M \setminus S^*M(2\varepsilon)$. Let $\chi(x, hD) \in Op_h(C_0^\infty(T^*M))$ be the corresponding h-pseudodifferential cutoff. Also, in the following, we will use the notation

$$\Gamma_x := \operatorname{supp}\ \mu|_{\Lambda_x}$$

to denote the support of the restricted defect measure corresponding to the eigenfunction sequence $\{u_{h_j}\}$ in Theorem 3.

Then, we have

$$u_h = \rho(h^{-1}[-h^2\Delta - 1])u_h = \int_{\mathbb{R}} \hat{\rho}(t) e^{it[-h^2\Delta-1]/h} \chi(y, hD_y) u_h \, dt + O_\varepsilon(h^\infty). \tag{3.1}$$

3.1. Microlocalization to the flow out Λ_x.

Set

$$V(t,x,y,h) := \Big(\hat{\rho}(t) e^{it[-h^2\Delta-1]/h} \chi(y, hD_y)\Big)(t,x,y).$$

Then, by propagation of singularities,

$$WF_h'(V(t,\cdot,\cdot,h)) \subset \{(x,\xi,y,\eta); (x,\xi) = G_t(y,\eta),\ ||\xi|_x - 1| \leq 2\varepsilon\,, t \in [\delta, 2\delta]\}. \tag{3.2}$$

Let $b_{x,\varepsilon} \in C_0^\infty(T^*M)$ be a family of h-pseudodifferential cutoffs with symbols

$$b_{x,\varepsilon} \in C_0^\infty(\{(y,\eta) \mid (y,\eta) = G_t(x_0,\xi) \text{ for some } (x_0,\xi) \in S^*_{x_0}M(3\varepsilon) \text{ with } r(x,x_0) < 2\varepsilon, \frac{\delta}{2} < t < 3\delta\},$$

with

$$b_{x,\varepsilon} \equiv 1 \text{ on } \{(y,\eta) \mid (y,\eta) = G_t(x_0,\xi) \text{ for some } (x_0,\xi) \in S^*_{x_0}M(2\varepsilon) \text{ with } r(x,x_0) < \varepsilon, \delta < t < 2\delta\}.$$

By wavefront calculus together with (3.2), it follows that for $r(x(h), x) = o(1)$,

$$u_h(x(h)) = \int_M \bar{V}(x(h), y, h)\, b_{x,\varepsilon}(y, hD_y) u_h(y) dy + O_\varepsilon(h^\infty), \tag{3.3}$$

where,

$$\bar{V}(x(h), y, h) := \int_{\mathbb{R}} \hat{\rho}(t) \big(e^{it[-h^2\Delta-1]/h} \chi(y, hD_y)\big)(t, x(h), y)\, dt.$$

By a standard stationary phase argument [Sog93, Chapter 5],

$$\begin{aligned} \bar{V}(x,y,h) &= V_+(x,y,h) + V_-(x,y,h) \\ V_\pm(x,y,h) &= h^{\frac{1-n}{2}} e^{\pm i r(x,y)/h} a(x,y,h)\, \hat{\rho}(r(x,y)) + O_\varepsilon(h^\infty), \end{aligned} \tag{3.4}$$

where $a(x,y,h) \in S^0(1)$.

Then, in view of (3.4) and (3.3),

$$u_h(x(h)) = u_+(x(h)) + u_-(x(h)) + O_\varepsilon(h^\infty)$$

$$u_\pm(x(h)) = (2\pi h)^{\frac{1-n}{2}} \int_{\frac{\delta}{2}<|y-x|<2\delta} e^{\pm ir(x(h),y)/h} a_\pm(x(h),y,h)\hat{\rho}(r(x(h),y))\, b_{x,\varepsilon}(y,hD_y)u_h(y)dy. \tag{3.5}$$

3.2. **Further microlocalization along** $\operatorname{supp}\mu|_{\Lambda_x}$**.** Let $\mathcal{H}^n$ be the n-dimensional Hausdorff measure on the flow out Λ_x. By assumption, $\mathcal{H}^n(\text{supp } \mu|_{\Lambda_x}) = 0$. In view of the microlocalization above, we are only interested in the annular subset

$$A_x(\delta/2, 3\delta) := \Lambda_{x,3\delta} \setminus \Lambda_{x,\delta/2}.$$

By monotonicity of measure, we also have

$$\mathcal{H}^n(\text{supp } \mu|_{A_x(\delta/2,3\delta)}) = 0$$

and so for any $\varepsilon_1 > 0$, there exist n-dimensional balls $B(r_j) \subset A_x(\delta/4, 4\delta); j = 1,2,...$ with radii $r_j > 0, j = 1,2,...$ such that

$$\text{supp } \mu|_{\Lambda_x} \subset \bigcup_{j=1}^\infty B(r_j), \quad \mathcal{H}^n\Big(\bigcup_{j=1}^\infty B(r_j)\Big) < \varepsilon_1.$$

Note that for $\delta > 0$ small enough, the canonical projection $\pi : T^*M \to M$ restricts to a diffeomorphism

$$\pi : A_x(\delta/4, 4\delta) \to \{y \in M; \delta/4 < r(x,y) < 4\delta\}.$$

Consider the closed set

$$K = \pi(\text{supp } \mu|_{A_x(\delta/4,4\delta)}) \subset M$$

with open covering

$$G := \pi\Big(\bigcup_{j=1}^\infty B(r_j)\Big), \quad \text{satisfying} \quad \mathcal{H}^n(G) = O(\varepsilon_1). \tag{3.6}$$

By the C^∞ Urysohn lemma, there exists $\chi_{\Gamma_x} \in C_0^\infty(M;[0,1])$ with

$$\chi_{\Gamma_x}|_K = 1, \qquad \operatorname{supp}\chi_{\Gamma_x} \subset G.$$

(Note that χ_{Γ_x} depends on ε_1, but we suppress this dependence to simplify notation.) Then, starting from (3.5) we make the further decomposition

$$u_+(x(h)) = I_1(x(h),h) + I_2(x(h),h) + O_\varepsilon(h^\infty) \tag{3.7}$$

where

$$I_1 := (2\pi h)^{\frac{1-n}{2}} \int_{\delta<|y-x|<2\delta} e^{ir(x(h),y)/h} a_+(x(h),y,h)\hat{\rho}(r(x(h),y))\, \chi_{\Gamma_x}(y)\, b_{x,\varepsilon}(y,hD_y)u_h(y)dy$$

$$I_2 := (2\pi h)^{\frac{1-n}{2}} \int_{\delta<|y-x|<2\delta} e^{ir(x(h),y)/h} a_+(x(h),y,h)\hat{\rho}(r(x(h),y))\, (1-\chi_{\Gamma_x}(y))\, b_{x,\varepsilon}(y,hD_y)u_h(y)dy.$$

Here, both $I_1(x(h),h)$ and $I_2(x(h),h)$ also depend on the parameters $\varepsilon_1, \varepsilon > 0$ although we suppress this in the notation.

To estimate I_1, we note that since $\mathcal{H}^n(\operatorname{supp}\chi_{\Gamma_x}) \leq C\varepsilon_1$, it follows that $\|\chi_{\Gamma_x}\|^2_{L^2(M)} \leq C\varepsilon_1$ and so by Cauchy-Schwarz,

$$|I_1(x(h),h)| \leq C_\delta (2\pi h)^{\frac{1-n}{2}} \|\chi_{\Gamma_x}\|_{L^2(M)} \|b_{x,\varepsilon}(y,hD_y)u_h\|_{L^2(M)} \leq C'_\delta\, \varepsilon_1^{1/2}\, h^{\frac{1-n}{2}} + o_{\varepsilon,\delta}(1)$$

and in particular,

$$\lim_{\varepsilon_1\to 0}\lim_{\varepsilon\to 0}\limsup_{h\to 0} h^{\frac{n-1}{2}} |I_1(x(h),h)| = 0.$$

Applying Cauchy-Schwarz to I_2 gives

$$|I_2(x(h),h)| \leq C_\delta (2\pi h)^{\frac{1-n}{2}} \|(1-\chi_{\Gamma_x})\, b_{x,\varepsilon}(y,hD_y)u_h\|_{L^2(M)},$$

and so,

$$\limsup_{h\to 0} h^{\frac{n-1}{2}} |I_2(x(h),h))| \leq\ C \limsup_{h\to 0} \|(1-\chi_{\Gamma_x}) b_{x,\varepsilon}(y,hD_y)u_h\|_{L^2} \tag{3.8}$$

Taking $\varepsilon \to 0^+$ on the RHS of (3.8), one gets

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{h\to 0} \|(1-\chi_{\Gamma_x}) b_{x,\varepsilon}(y,hD_y)u\|^2_{L^2} &= \lim_{\varepsilon\to 0}\int_{S^*M} |(1-\chi_{\Gamma_x})(y)\, b_{x,\varepsilon}(y,\xi)|^2 d\mu \\ &\leq C \int_{\Lambda_{x,4\delta}\setminus\Lambda_{x,\delta/4}} |(1-\chi_{\Gamma_x})(y)|^2\, d\mu = 0\end{aligned}$$

since by construction, for all $\varepsilon_1 > 0$,

$$(1-\chi_{\Gamma_x})(y) = 0, \quad \forall y \in \pi(\operatorname{supp}\, \mu|_{\Lambda_{x,4\delta}\setminus\Lambda_{x,\delta/4}}).$$

In particular, since the left hand side of (3.7) is independent of ε and ε_1

$$\lim_{h\to 0} h^{\frac{n-1}{2}} |u_+(x(h))| = 0.$$

The analysis of $u_-(x(h))$ is identical. ☐

4. The example of zonal harmonics

Let (S^2, g_{can}) be the round sphere and (r,θ) be polar variables centered at the north pole $p = (0,0,1) \in \mathbb{R}^3$. The geodesic flow is a completely integrable system with Hamiltonian

$$H = |\xi|^2_g = \xi_r^2 + (\sin r)^{-2}\xi_\theta^2, \quad r \in (0,\pi) \tag{4.1}$$

and Claurault integral $p = \xi_\theta$ satisfying $\{H,p\} = 0$. The associated moment mapping is $\mathcal{P} = (H,p) : T^*S^2 \to \mathbb{R}^2$ and the connected components of the level sets are, by the Liouville-Arnold Theorem, Lagrangian tori Λ_c indexed by the values of the moment map $(1,c) \in \mathcal{P}(T^*S^2)$.

The associated quantum integrable system is given by the Laplacian Δ_g and the rotation operator hD_θ. The corresponding L^2-normalized joint eigenfunctions are the standard spherical harmonics Y^k_m with

$$-\Delta_g Y^k_m = k(k+1)Y^k_m, \quad hD_\theta Y^k_m = mY^k_m.$$

These eigenfunctions can be separated into various sequences (i.e. *ladders*) associated with different values ($\in \mathcal{P}(T^*S^2)$; specifically, the correspondence is given by $c = \lim_{m\to\infty}\frac{m}{k}$). The eigenfunctions with maximal L^∞ blow-up are the sequence of *zonal* harmonics given by

$$u_h(r,\theta) = Y^k_0(r,\theta) = \frac{\sqrt{2k+1}}{2\pi}\int_0^{2\pi} (\cos r + i\sin r\cos\tau)^k d\tau; \quad h = k^{-1},\ k = 1,2,3,... \tag{4.2}$$

It is obvious from (4.2) that
$$|Y_0^k(p)| \approx k^{1/2}$$
and thus attains the maximal sup growth at p (similarily, at the south pole). At the classical level, the zonals $u_h = Y_0^k$ concentrate microlocally on the Lagrangian tori $\Lambda_0 = \mathcal{P}^{-1}(1,0)$. From the formula (4.1) it is clear that away from the poles (where (r,θ) are honest coordinates),
$$\Lambda_0 \setminus \{\pm p\} = \{(r,\theta,\xi_r = \pm 1, \xi_\theta = 0),\ r \in (0,\pi)\} \cong S^2 \setminus \{\pm p\}. \tag{4.3}$$
The choice of $\xi_r = \pm 1$ determines the Lagrangian torus (there are two of them) and also, either torus clearly covers the entire sphere. At the poles themselves, the projection $\pi_{\Lambda_0} : \Lambda_0 \to S^2$ has a blowdown singularity with
$$\pi_{\Lambda_0}^{-1}(\pm p) = S^*_{\pm}(S^2) \cong S^1. \tag{4.4}$$
To see this, consider the behaviour at p (with a similar computation at $-p$). Rewriting the integral in involution in Euclidean coordinates $(x,y,z) \in \mathbb{R}^3$ one has $H = (x\xi_y - y\xi_x)^2 + (x\xi_z - z\xi_x)^2 + (y\xi_z - z\xi_y)^2$ and $\xi_\theta = x\xi_y - y\xi_x$. Setting $H = 1$, $x\xi_y - y\xi_x = 0$ and $(x,y,z) = (0,0,1)$ gives
$$\pi_{\Lambda_0}^{-1}(p) \cong \{(\xi_x,\xi_y) \in \mathbb{R}^2; \xi_x^2 + \xi_y^2 = 1\}.$$
It is then clear from (4.3) and (4.4) that $\pi_{\Lambda_0} : \Lambda_0 \to S^2$ is surjective and a diffeomorphism away from the poles (modulo choice of Lagrangian cover) and the fibres above the poles are $S^*_{\pm}(S^2) \cong S^1$.

The defect measure μ associated with the zonals is
$$d\mu = |d\theta_1 d\theta_2|,$$
where $(\theta_1,\theta_2; I_1, I_2) \in \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^2$ are symplectic action-angle variables defined in a neighbourhood of the Lagrangian torus Λ_0 [TZ03]. One can choose one of the angle variables $\theta_1 \in S_p^*(S^2)$ to parametrize the circle fibre above p (a homology generator of the torus). Then, by the Liouville-Arnold Theorem, the geodesic flow on the torus $\Lambda_0 = \{I_1 = c_1, I_2 = c_2\}$ is affine with
$$\theta_j(t) = \theta_j(0) + \alpha_j t, \quad \alpha_j = \frac{\partial H}{\partial I_j} \neq 0.$$
It is then clear that
$$\mu(\Lambda_{p,\delta}) = \int_0^{2\pi} d\theta_1 \cdot \int_{|t|<\delta} \alpha_2 dt \approx \delta \neq 0$$
and $\operatorname{supp} \mu|_{\Lambda_p} = \Lambda_p$. Therefore, this case violates the assumption in Theorem 3 and that is of course consistent with the maximal L^∞ growth of zonal harmonics.

The analysis above extends in a straightforward fashion to the case of a more general sphere of rotation [TZ03].

5. Eigenfunctions of Schrödinger operators

Consider a Schrödinger operator $P(h) = -h^2\Delta_g + V$ with $V \in C^\infty(M;\mathbb{R})$ on a compact, closed Riemannian manifold (M,g) and let u_h be L^2-normalized eigenfunction with
$$P(h)u_h = E(h)u_h, \quad E(h) = E + o(1),\ E > \min V, \qquad \|u_h\|_{L^2} = 1. \tag{5.1}$$
Any sequence u_h of solutions to (5.1) has a subsequence u_{h_k} with a defect measure μ in the sense that for $a \in C_0^\infty(T^*M)$
$$\langle a(x,hD)u_h, u_h\rangle \to \int_{T^*M} a d\mu.$$
Such a measure μ is supported on $\{p = 0\}$ and is invariant under the bicharacteristic flow $G_t := \exp(tH_p)$.

In analogy with the homogeneous case, we define for $x \in M$ respectively the *flow out* and *time T flow out* by

$$\Lambda_{x,V} := \bigcup_{T=0}^{\infty} \Lambda_{x,T}, \qquad \Lambda_{x,T,V} := \bigcup_{t=-T}^{T} G_t(\Sigma_x)$$

where

$$\Sigma_x = \{\xi \in T^*_x M \mid |\xi|_g^2 + V(x) = E\}.$$

Definition **5.1.** *Let $\mathcal{H}^n$ be n-dimensional Hausdorff measure on $\{|\xi|_g^2 + V(x) = E\}$ induced by the Sasaki metric on T^*M. We say that the sequence u_h of solutions to* (5.1) *is* admissible at x *if*

$$\mathcal{H}^n(\operatorname{supp}\ \mu|_{\Lambda_{x,V}}) = 0. \tag{5.2}$$

With these definitions we have the analog of Theorem 3

Theorem **5.** *Let $B \subset V^{-1}(E)$ be a closed ball in the classically allowable region and μ be a defect measure associated with the eigenfunction sequence u_h. Then, if the eigenfunction sequence is admissible for all $x \in B$ in the sense of (5.2),*

$$\sup_{x \in B} |u_h(x)| = o(h^{\frac{1-n}{2}}).$$

Proof. In analogy with the homogeneous case [CHT15, Lemma 5.1], we have

$$\rho(h^{-1}[P(h) - E]))(x,y) = h^{\frac{1-n}{2}} a(x,y,h) e^{-iA(x,y)//h} + R(x,y,h)$$

where $A(x,y) \in [(2C_0)^{-1}\varepsilon, 2C_0\varepsilon]$ for some $C_0 > 1$ and is the action function defined to be the integral of the Lagrangian $L(x,\xi) = |\xi|_g^2 - V(x)$ along the bicharacteristic in $\{p = E\}$ starting at (y,η) and ending at (x,ξ). For (x,y) in a small neighborhood of the diagonal, there is a unique such η satisfying this condition. The remainder $R(x,y,h) = O(h^\infty)$ pointwise and with all derivatives. The proof then follows using the same argument as in the homogeneous case. □

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