Sharp error bounds for edge-element discretisations of the high-frequency Maxwell equations

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August 8, 2024

Abstract

We prove sharp wavenumber-explicit error bounds for first- or second-type-Nédélec-element (a.k.a. edge-element) conforming discretisations, of arbitrary (fixed) order, of the variable-coefficient time-harmonic Maxwell equations posed in a bounded domain with perfect electric conductor (PEC) boundary conditions. The PDE coefficients are allowed to be piecewise regular and complex-valued; this set-up therefore includes scattering from a PEC obstacle and/or variable real-valued coefficients, with the radiation condition approximated by a perfectly matched layer (PML).

In the analysis of the h-version of the finite-element method, with fixed polynomial degree p, applied to the time-harmonic Maxwell equations, the *asymptotic regime* is when the meshwidth, h, is small enough (in a wavenumber-dependent way) that the Galerkin solution is quasioptimal independently of the wavenumber, while the *preasymptotic regime* is the complement of the asymptotic regime.

The results of this paper are the first preasymptotic error bounds for the time-harmonic Maxwell equations using first-type Nédélec elements or higher-than-lowest-order second-type Nédélec elements. Furthermore, they are the first wavenumber-explicit results, even in the asymptotic regime, for Maxwell scattering problems with a non-empty scatterer.

1 Introduction

1.1 Statement of the main result

We consider the time-harmonic Maxwell equations

$$k^{-2}\mathrm{curl}\,(\mu^{-1}\mathrm{curl}\,E) - \epsilon E = f, \tag{1.1}$$

with wavenumber k, posed in a bounded domain $\Omega \subset \mathbb{R}^3$ with outward-pointing unit normal vector n and diameter L, where $E \in H_0(\mathrm{curl}\,,\Omega)$ (i.e., $E \in H(\mathrm{curl}\,,\Omega)$ with $E \times n = 0$ on $\partial\Omega$), the data $f \in (H(\mathrm{curl}\,,\Omega))^*$, and the coefficients μ and ϵ (the relative permeability and relative permittivity, respectively) satisfy $\mathrm{Re}\,\mu, \mathrm{Re}\,\epsilon \geq c > 0$ (in the sense of quadratic forms) in Ω. We are interested in this problem when $kL \gg 1$, i.e., the high-frequency regime.

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This setting includes the radial-perfectly-matched-layer approximation to the scattering problem where the scattering is caused by variable μ and ϵ and/or a perfect-electric-conductor obstacle; see §A.

We study approximations to the solution of (1.1) using the h-version of the finite-element method (h-FEM), where accuracy is increased by decreasing the meshwidth h while keeping the polynomial degree p constant, and the (conforming) approximation space consists of the first type of Nédélec finite elements [52], whose definition is recapped in §10.2 below [1]; note that we choose the convention that $p = 1$ corresponds to lowest-order Nédélec elements. Since the second type of Nédélec finite elements [53], [51, §8.2], [20, §15.5.1] contains the first type, and our results depend only on best-approximation properties of the space, our results also hold for second-type Nédélec finite elements.

We work in norms where each derivative is scaled by k^{-1}; in particular,

$$\|E\|^2_{H_k(\mathrm{curl}\,,\Omega)} := k^{-2}\|\mathrm{curl}\, E\|^2_{L^2(\Omega)} + \|E\|^2_{L^2(\Omega)}. \tag{1.2}$$

Definition 1.1 (C^ℓ with respect to a partition). *For $\ell \in \mathbb{N}$, Ω is C^ℓ with respect to the partition $\{\Omega_j\}_{j=1}^n$ if*

(i) $\overline{\Omega} = \cup_{j=1}^n \overline{\Omega_j}$, where $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$,

(ii) $\Gamma_{i,j}$ is C^ℓ for all (i,j), where $\partial\Omega_j = \sqcup_{i=1}^{L_j}\Gamma_{i,j}$ is the decomposition of $\partial\Omega_j$ into its connected components, and

(iii) for all i, i', j, j', if $\Gamma_{i,j} \cap \Gamma_{i',j'} \neq \emptyset$, then $\Gamma_{i,j} = \Gamma_{i',j'}$.

This definition implies that if Ω is C^ℓ with respect to a partition, then $\partial\Omega$ is C^ℓ (since $\partial\Omega = \Gamma_{i,j}$ for some i, j).

Assumption 1.2 (Regularity assumptions on Ω, ϵ, and μ). *For some $m \in \mathbb{N}$, Ω is C^{m+1} with respect to the partition $\{\Omega_j\}_{j=1}^n$ and $\epsilon \in C^{m,1}(\overline{\Omega_j})$ and $\mu \in C^m(\overline{\Omega_j})$ for all $j = 1, \ldots, n$.*

Theorem 1.3 (The main result). *Suppose that Assumption 1.2 holds for an integer $m \geq 1$. Let $K \subset (0, \infty)$ be the set of k such that, given $f \in (H_0(\mathrm{curl}\,,\Omega))^*$, there exists a unique solution E to (1.1); let C_{sol} then be the $L^2(\Omega) \to L^2(\Omega)$ norm of the solution operator $f \mapsto E$ (recall that $C_{\mathrm{sol}} \geq CkL$).*

Let $\mathcal{H}_h \subset H_0(\mathrm{curl}\,,\Omega)$ be the space of first- or second-type Nédélec finite-elements of degree $p \geq 0$ on a (curved) mesh satisfying Assumption 10.1 below, with maximal element size h.

If $p \leq m$ then, given $k_0 > 0$, there exist $C_1, C_2 > 0$ such that, for all $k \in K$ with $k \geq k_0$, if

$$(kh)^{2p} C_{\mathrm{sol}} \leq C_1 \tag{1.3}$$

then the Galerkin solution E_h exists, is unique, and satisfies

$$\|E - E_h\|_{H_k(\mathrm{curl}\,,\Omega)} \leq C_2\Big(1 + (kh)^p C_{\mathrm{sol}}\Big) \min_{v_h \in \mathcal{H}_h} \|E - E_h\|_{H_k(\mathrm{curl}\,,\Omega)}. \tag{1.4}$$

Furthermore, given $C_{\mathrm{osc}} > 0$ there exists $C_3 > 0$ such that if the data f is k-oscillatory with constant C_{osc} and regularity index m (in the sense of Definition 2.7 below), then

$$\frac{\|E - E_h\|_{H_k(\mathrm{curl}\,,\Omega)}}{\|E\|_{H_k(\mathrm{curl}\,,\Omega)}} \leq C_3\Big(1 + (kh)^p C_{\mathrm{sol}}\Big)(kh)^p; \tag{1.5}$$

[1] Recall that Nédélec elements are often called edge elements because at the lowest order basis functions and degrees of freedom are associated with the edges of the mesh; at higher order the geometrical identification of basis functions and degrees of freedom is more complicated.

i.e., the relative $H_k(\mathrm{curl}\,,\Omega)$ *error can be made controllably small by making* $(kh)^{2p}C_{\rm sol}$ *sufficiently small.*

We make two remarks:

- The abstract version of Theorem 1.3 – Theorem 2.9 below – is proved assuming only a Gårding inequality and elliptic-regularity-type assumptions (see Assumptions 2.1 and 2.3 below). Theorem 1.3 is then proved by showing that these regularity assumptions are satisfied using the classic regularity results of Weber [63] (see Lemma 11.2 below).

- Theorem 2.9 can also be applied to differential r-forms in any dimension. In this case, the operator $\epsilon^{-1}\mathrm{curl}\,(\mu^{-1}\mathrm{curl}\,)$ is replaced by $\mathcal{D} = *d*d$, where $*$ denotes the Hodge $*$ operator (with respect to the relevant metric); the kernel of $\mathcal{D}$ then consists of closed r-forms. (Recall that finite-element spaces in this setting are discussed in [15].)

1.2 The context and novelty of the main result

The asymptotic and preasymptotic regimes. We first discuss the analysis of the h-FEM applied to the Helmholtz equation $(k^{-2}\Delta+1)u = f$. The concepts of the asymptotic and preasymptotic regimes were first introduced by Ihlenburg and Babuška in [36, 37]. In the *asymptotic regime*, which is now known to be when $h = h(k)$ satisfies $(kh)^pC_{\rm sol} \ll 1$, the sequence of Galerkin solutions are quasioptimal, with quasioptimality constant independent of k. The *preasymptotic regime* is then when $(kh)^pC_{\rm sol} \gg 1$. In this regime, one expects that if $(kh)^{2p}C_{\rm sol}$ is sufficiently small then, for data oscillating at frequency $\lesssim k$, the relative error of the Galerkin solution is controllably small. Note that, since $C_{\rm sol}$ grows with kL, $hk = o(1)$ in the asymptotic regime, and this is the well-known *pollution effect* [3].

State of the art in the asymptotic regime for the Helmholtz h-FEM. The natural error bounds in the asymptotic regime were proved for Helmholtz problems satisfying only a Gårding inequality and an elliptic-regularity shift in [12] following earlier work by [45, 46, 21] for constant-coefficient Helmholtz problems. In fact, this earlier work showed that the *hp*-FEM does not suffer from the pollution effect when $hk/p \le C_1$ for C_1 sufficiently small, $p \ge C_2 \log(kL)$ for C_2 sufficiently large, and $C_{\rm sol} \le C_3(kL)^N$ for some $C_3, N > 0$; this result is now known for variable-coefficient Helmholtz problems by [38, 27, 28, 5].

The error bounds in the asymptotic regime rely on the fact that, since the Helmholtz adjoint solution operator is compact as a map from L^2 to H^1, the L^2 norm of the error is smaller than the H^1 norm by the Aubin–Nitsche lemma (see, e.g., [16, Theorem 19.1]). Indeed, Galerkin orthogonality $\langle P(u-u_h), v_h\rangle = 0$ for all finite-element functions v_h implies that, with Π_h the orthogonal projection onto the finite-element space,

$$\begin{aligned}\|u-u_h\|_{L^2}^2 &= \big\langle P^{-1}P(u-u_h), u-u_h\big\rangle,\\ &= \big\langle P(u-u_h), (P^*)^{-1}(u-u_h)\big\rangle,\\ &= \big\langle P(u-u_h), (I-\Pi_h)(P^*)^{-1}(u-u_h)\big\rangle,\\ &\le C\,\|u-u_h\|_{H^1}\,\big\|(I-\Pi_h)(P^*)^{-1}\big\|_{L^2\to H^1}\,\|u-u_h\|_{L^2}\,.\end{aligned}$$

Schatz [60] used this duality argument in conjunction with a Gårding inequality to bound the Helmholtz FEM error; see also [59] for a more modern perspective. (The Maxwell analogue of this result is Lemma 6.1 below.) The results [45, 46, 21, 12, 38, 27, 28, 5] discussed above then obtained quasi-optimality (with constant independent of k) when $(kh)^p C_{\rm sol}$ is sufficiently small by bounding $\|(I-\Pi_h)(P^*)^{-1}\|_{L^2\to H^1}$.

State of the art in the preasymptotic regime for the Helmholtz h-FEM. The natural bounds in the preasymptotic regime (i.e., the Helmholtz analogues of (1.4) and (1.5) above) were proved in [29] for Helmholtz problems satisfying only a Gårding inequality and an elliptic-regularity shift, following earlier work by [65, 66, 19, 4, 56, 10]. Central to this earlier work was the *elliptic projection argument* [23, 24], which used that the Helmholtz operator is coercive if a sufficiently large multiple of the identity is added. The key insight in [29] is that, in fact, this coercivity can be achieved by adding a smoothing operator, defined in terms of eigenfunctions of the real part of the Helmholtz operator (a Maxwell analogue of this is Lemma 7.10 below).

We highlight that the arguments of [29] immediately obtain a splitting analogous to that used to bound $\|(I-\Pi_h)(P^*)^{-1}\|_{L^2\to H^1}$ in [45, 46, 21, 12, 38, 27, 28, 5]. Indeed, since

$$(P^*+S)(P^*)^{-1} = I + S(P^*)^{-1},$$

then

$$(P^*)^{-1} = (P^*+S)^{-1} + (P^*+S)^{-1}S(P^*)^{-1}. \tag{1.6}$$

If S is smoothing operator such that $P+S$ is coercive (with coercivity constant independent of k) and P satisfies the natural assumptions for elliptic regularity, then $(P^*+S)^{-1}$ has the regularity shift associated with $(P^*)^{-1}$, but its norm is bounded independent of k. Furthermore, $(P^*+S)^{-1}S(P^*)^{-1}$ is smoothing, with norm bounded by the norm of $(P^*)^{-1}$.

Duality-argument analysis of the Maxwell h-FEM using Nédélec finite elements. Compared to the analysis of the Helmholtz h-FEM, the analysis of the Maxwell h-FEM is complicated by the large kernel of the curl operator. The kernel of curl does not consist of smooth functions; thus neither the solution operator nor its adjoint are compact. The duality arguments described above for Helmholtz therefore cannot immediately be applied.

If $\operatorname{div}(\zeta E)=0$ for some ζ with $\operatorname{Re}\zeta \geq c > 0$ (in the sense of quadratic forms), then E lies in a subspace transverse to the kernel of curl and the solution operator increases regularity by the regularity results of Weber [62]; see Theorem 9.1 and Lemma 11.2 below. This is related to the fact that, whereas the embedding $H_0(\operatorname{curl},\Omega) \hookrightarrow L^2(\Omega)$ is not compact, the embedding $H_0(\operatorname{curl},\Omega)\cap H(\operatorname{div},\zeta,\Omega) \hookrightarrow L^2(\Omega)$ is compact [64, 62, 57] [40, §8.4], where $H(\operatorname{div},\zeta,\Omega) := \{v\in L^2(\Omega) : \nabla\cdot(\zeta v)\in L^2(\Omega)\}$.

One strategy for proving bounds on the Galerkin error for Maxwell – first introduced by Monk [49] – is to

(i) bound the ϵ-divergence free part of the error using the duality arguments from [60] (discussed above), and

(ii) bound the part of the error that is not ϵ-divergence free using arguments originating from [30] (discussed below).

This argument is essentially equivalent to Lemma 6.1 below. Notable uses of this type of argument include in [31], in the analysis of Maxwell domain decomposition methods, and in [6], in the analysis of the h-FEM with Nédélec elements applied to the Maxwell PML problem.

Regarding Point (ii) above: by Galerkin orthogonality, the error is *discretely ϵ-divergence free*, in the sense that $(\epsilon(E - E_h), v_h)_{L^2(\Omega)} = 0$ for all $v_h \in \operatorname{Ker}\operatorname{curl} \cap \mathcal{H}_h$ (see (2.21) below). Therefore, the part of the error that is not ϵ-divergence free can be controlled by understanding how much a function that is discretely ϵ-divergence free is not pointwise ϵ-divergence free. These arguments crucially rely on the existence of an interpolation operator that leaves the finite-element space invariant and maps functions in $\operatorname{Ker}\operatorname{curl}$ to functions in $\operatorname{Ker}\operatorname{curl}$ (see §10.3 and Lemma 11.5 below). The initial versions of this argument in [30, 49] used standard interpolation operators, at the cost of demanding extra regularity of the Maxwell solution (see [30, Remark 3.1]). Later refinements of this argument [31, 50] then used quasi-interpolation operators with lower – and, ultimately, minimal – regularity assumptions; see [1, §4.1], [2, 61, 15, 14], [20, Chapter 23].

Current state of the art for wavenumber-explicit bounds on the Maxwell h-FEM using Nédélec finite elements.

- For real μ and ϵ, the natural asymptotic error bounds are proved by the combination of [9, Theorem 4.6, Lemma 5.2] and [13, Theorem 2].
- The papers [47, 48] show that the hp-FEM applied to (1.1) with constant μ and ϵ and analytic boundary does not suffer from the pollution effect if $p \geq C_1 \log(kL)$ for any $C_1 > 0$, $hk/p \leq C_2$ for sufficiently small $C_2 > 0$, and $C_{\rm sol} \leq C_3(kL)^N$ for some $C_3, N > 0$. The results in [48] for impedance boundary conditions contain the result that, when p is constant, the Galerkin solution is quasioptimal (with constant independent of k) when $(kh)^p C_{\rm sol}$ is sufficiently small; see [48, Proof of Lemma 9.5]; the results in [47] for when the radiation condition is realised exactly on $\partial\Omega$ are more restrictive; see [47, Remark 4.19]. These arguments essentially use a result equivalent to Lemma 6.1 below, and then prove approximation results about the adjoint solution operator to bound the second quantity in (6.1) (following the ideas introduced in the Helmholtz context in [45, 46, 21, 5]).

  Analogous results for a regularised formulation of (1.1) – where the space is embedded in H^1 if the boundary is smooth enough – were obtained in [55] (with the h-version of this method studied in a k-explicit way in [54]).
- Very recently, the natural preasymptotic error bounds were proved in [43, Theorem 4.2] when $p = 1$ for (1.1) with constant μ and ϵ and an impedance boundary condition on $\partial\Omega$, and when the h-FEM is implemented using Nédélec elements of the second type. Recall that the second-type elements have better approximation properties in the L^2 norm than the first type (see, e.g., [51, §8.2]), with this fact crucially used in [43, Equation A.2]. The analogous error bounds for continuous interior-penalty methods were proved in [43, Theorem 5.2]. These results do not use the duality arguments described above; instead the crucial ingredient is a bound on the norm of the Galerkin solution in terms of the data; see [43, Theorem 4.1] and the discussion in [43, Remark 4.2].

  The results of [43] built on earlier work studying the same set up and proving the analogous result for other h-version FEMs, including interior-penalty discontinuous Galerkin methods [25, Theorem 6.1], a different continuous interior penalty method using second-type Nédélec elements [42, Theorem 4.6], and hybridizable discontinuous Galerkin methods [22, Theorem 4.7], [41, Remark 5.1].

Summary of the ideas behind the proof of Theorem 1.3. Theorem 1.3 is proved by

(i) bounding the ϵ-divergence free part of the error using the ideas from the Helmholtz preasymptotic error analysis in [29], and

(ii) bounding the part of the error that is not ϵ-divergence free using the arguments originating from [30].

That is, compared to the classic duality argument introduced in [49, 50] (and discussed above) we replace the Schatz argument by the arguments in [29] and do everything in a k-explicit way.

Regarding Point (i): we highlight that even applying the basic elliptic-projection argument (which [29] generalises) to Nédélec-element discretisations of the time-harmonic Maxwell equations has proven difficult up to now, as described in [43, Remark 4.2(d)]. We use a projection Π_0 that maps into $\operatorname{Ker}\operatorname{curl}$, with then $\Pi_1 := I - \Pi_0$. A priori, there are many different choices for Π_0. However, the requirement that $\epsilon\Pi_1$ is L^2 orthogonal to $\operatorname{Ker}\operatorname{curl}$ (i.e., is ϵ divergence free) uniquely specifies Π_0; see Lemma 3.1 (d). This lemma also shows that Π_0 is uniquely determined by its other key properties (see Lemma 3.1 (b) and (c)).

Regarding Point (ii): these arguments are performed in a k-explicit way for (1.1) with μ and ϵ real-valued in [9, §3.3]; one slight difference between the arguments in [9] and those in the present paper is that [9] works in the L^2 inner product weighted with ϵ, but this is not possible here since ϵ is complex.

Finally, we highlight that the duality arguments in the present paper have the splitting (1.6) built in, so that only the adjoint solution operator applied to functions with high regularity appears; see (8.3) and (8.4) (and recall that the operator S is smoothing).

1.3 Outline

§2 states the main result (i.e., Theorem 1.3) in abstract form (see Theorem 2.9 below). Using the material in §3-§7, §8 proves Theorem 2.9. §9 recalls the regularity results of Weber [63]. §10 recalls the definition and properties of Nédélec finite elements. §11 proves Theorem 1.3. §A shows that the Maxwell PML problem falls into the class of Maxwell problems described in §1.1. §B recaps scaling arguments used to prove interpolation results for Nédélec elements on curved meshes.

2 The main result in abstract form

We now work in an abstract framework to highlight the underlying structure of the problem (independent of any Maxwell-specific notation).

2.1 Abstract framework and assumptions

Assumption 2.1. *$\mathcal{H}$ and $\mathcal{V}$ are Hilbert spaces with $\mathcal{H} \subset \mathcal{V}$, $\mathcal{H}$ dense in $\mathcal{V}$, and norms $|||\cdot|||_{\mathcal{H}}$ and $|||\cdot|||_{\mathcal{V}}$. $P : \mathcal{H} \to \mathcal{H}^*$ is bounded and such that $P = \mathcal{D} - \mathcal{E}$, where $\mathcal{E} : \mathcal{V} \to \mathcal{V}^*$ is bounded and there exists $C'_{\mathcal{E}} > 0$ such that*

$$\operatorname{Re} \langle \mathcal{E}v, v \rangle_{\mathcal{V}^* \times \mathcal{V}} \geq C'_{\mathcal{E}} |||v|||^2_{\mathcal{V}} \quad \textit{for all } v \in \mathcal{V}.$$

Finally $\operatorname{Ker}\mathcal{D}^* = \operatorname{Ker}\mathcal{D}$*, and there exist $C_1, C_2 > 0$ such that*

$$\operatorname{Re} \langle \mathcal{D}v, v \rangle_{\mathcal{H}^* \times \mathcal{H}} \geq C_1 |||v|||^2_{\mathcal{H}} - C_2 |||v|||^2_{\mathcal{V}} \quad \textit{for all } v \in \mathcal{H}.$$

We use later that if P satisfies Assumption 2.1, then so does P^*.

Let $\|\cdot\|_{\mathcal{V}} := \sqrt{C_2}|||\cdot|||_{\mathcal{V}}$ and

$$\|v\|_{\mathcal{H}}^2 := \operatorname{Re}\left\langle \mathcal{D}v, v\right\rangle_{\mathcal{H}^*\times\mathcal{H}} + C_2|||v|||_{\mathcal{V}}^2;$$

to see that this is indeed the square of a norm, note that the right-hand side can be written as $\langle \operatorname{Re}\mathcal{A}v, v\rangle_{\mathcal{H}^*\times\mathcal{H}} := \frac{1}{2}\langle(\mathcal{A}+\mathcal{A}^*)v, v\rangle_{\mathcal{H}^*\times\mathcal{H}}$ for $\mathcal{A}$ equal to $\mathcal{D}$ plus C_2 multiplied by the appropriate Riesz map $\mathcal{V}\to\mathcal{V}^*$ in the inner product corresponding to $|||\cdot|||_{\mathcal{V}}$.

These definitions imply that

$$\operatorname{Re}\left\langle \mathcal{D}v, v\right\rangle_{\mathcal{H}^*\times\mathcal{H}} = \|v\|_{\mathcal{H}}^2 - \|v\|_{\mathcal{V}}^2 \quad \text{for all } v\in\mathcal{H} \tag{2.1}$$

(so that $(u,v)_{\mathcal{H}} = \langle(\operatorname{Re}\mathcal{D})u, v\rangle_{\mathcal{H}^*\times\mathcal{H}} + (u,v)_{\mathcal{V}}$ by the polarization identity) and

$$\operatorname{Re}\left\langle \mathcal{E}v, v\right\rangle_{\mathcal{V}^*\times\mathcal{V}} \geq C_{\mathcal{E}}\|v\|_{\mathcal{V}}^2 \quad \text{for all } v\in\mathcal{V} \tag{2.2}$$

with $C_{\mathcal{E}} := C'_{\mathcal{E}}(C_2)^{-1}$. Furthermore, by (2.1),

$$\operatorname{Re}\left\langle Pv, v\right\rangle_{\mathcal{H}^*\times\mathcal{H}} \geq \|v\|_{\mathcal{H}}^2 - (1+\|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*})\|v\|_{\mathcal{V}}^2 \quad \text{for all } v\in\mathcal{H}. \tag{2.3}$$

Lemma 2.2. $\operatorname{Ker}\mathcal{D}$ *is closed in* $\mathcal{V}$.

Proof. Let $\{u_n\}\in\operatorname{Ker}\mathcal{D}$ with $u_n\to u$ in $\mathcal{V}$. We need to show that $u\in\operatorname{Ker}\mathcal{D}$. By (2.1), $\|u_n\|_{\mathcal{H}} = \|u_n\|_{\mathcal{V}}$. Since u_n is bounded in $\mathcal{V}$, u_n is bounded in $\mathcal{H}$. Since $\mathcal{H}$ is a Hilbert space, there exists $w\in\mathcal{H}$ such that $u_n\rightharpoonup w$ as $n\to\infty$. We now show that $w\in\operatorname{Ker}\mathcal{D}$. Let $\mathcal{R}:\mathcal{H}\to\mathcal{H}^*$ be the Riesz map such that $\langle a,b\rangle_{\mathcal{H}\times\mathcal{H}^*} = (a,\mathcal{R}b)_{\mathcal{H}}$ for all $a,b\in\mathcal{H}$. Since $u_n\in\operatorname{Ker}\mathcal{D}$ for all n and $u_n\rightharpoonup w$ as $n\to\infty$, for all $v\in\mathcal{H}$,

$$\begin{aligned} 0 = \langle\mathcal{D}u_n, v\rangle_{\mathcal{H}^*\times\mathcal{H}} &= \langle u_n, \mathcal{D}^*v\rangle_{\mathcal{H}\times\mathcal{H}^*} \\ &= (u_n, \mathcal{R}\mathcal{D}^*v)_{\mathcal{H}} \to (w, \mathcal{R}\mathcal{D}^*v)_{\mathcal{H}} = \langle w, \mathcal{D}^*v\rangle_{\mathcal{H}\times\mathcal{H}^*} = \langle\mathcal{D}w, v\rangle_{\mathcal{H}^*\times\mathcal{H}}. \end{aligned}$$

Therefore $\mathcal{D}w = 0$, i.e., $w\in\operatorname{Ker}\mathcal{D}$.

Since $\operatorname{Ker}\mathcal{D}^* = \operatorname{Ker}\mathcal{D}$, $(\operatorname{Re}\mathcal{D})u_n = (\operatorname{Re}\mathcal{D})w = 0$ and thus, since $(u,v)_{\mathcal{H}} = \langle(\operatorname{Re}\mathcal{D})u, v\rangle_{\mathcal{H}^*\times\mathcal{H}} + (u,v)_{\mathcal{V}}$,

$$(u_n, v)_{\mathcal{H}} = (u_n, v)_{\mathcal{V}} \quad \text{and} \quad (w,v)_{\mathcal{H}} = (w,v)_{\mathcal{V}} \quad \text{for all } v\in\mathcal{H}.$$

Therefore, on the one hand, since $u_n\rightharpoonup w$ as $n\to\infty$,

$$(u_n, v)_{\mathcal{V}} = (u_n, v)_{\mathcal{H}} \to (w,v)_{\mathcal{H}} = (w,v)_{\mathcal{V}} \quad \text{as } n\to\infty.$$

On the other hand $(u_n, v)_{\mathcal{V}}\to(u,v)_{\mathcal{V}}$ since $u_n\to u$ in $\mathcal{V}$. Therefore $(w,v)_{\mathcal{V}} = (u,v)_{\mathcal{V}}$ for all $v\in\mathcal{H}$; thus $u = w\in\operatorname{Ker}\mathcal{D}$. □

By Lemma 2.2, the $\mathcal{V}$-orthogonal projection onto $\operatorname{Ker}\mathcal{D}$ is well-defined; denote this $\Pi_0^{\mathcal{V}}$ and let $\Pi_1^{\mathcal{V}} := I - \Pi_0^{\mathcal{V}}$.

Let $\iota:\mathcal{V}\to\mathcal{V}^*$ be the Riesz map such that

$$\langle\iota u, v\rangle_{\mathcal{V}^*\times\mathcal{V}} := (u,v)_{\mathcal{V}} \quad \text{for all } u,v\in\mathcal{V}. \tag{2.4}$$

We highlight that we write the identification of $\mathcal{V}$ and $\mathcal{V}^*$ explicitly using ι because later we consider subspaces of $\mathcal{V}$ and $\mathcal{V}^*$ and need to write the identification of these in terms of the identification ι; see §7.1 and Part (ii) of Lemma 7.1 below.

We now define two non-orthogonal projections $\Pi_0, \Pi_1 : \mathcal{V} \to \mathcal{V}$. The action of $\iota^{-1}\mathcal{E} : \mathcal{V} \to \mathcal{V}$ with $\mathcal{V} = \operatorname{Ker} \mathcal{D} \oplus (\operatorname{Ker} \mathcal{D})^{\perp}$ can be written as

$$\begin{pmatrix} \Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0^{\mathcal{V}} & \Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_1^{\mathcal{V}} \\ \Pi_1^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0^{\mathcal{V}} & \Pi_1^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_1^{\mathcal{V}} \end{pmatrix} =: \begin{pmatrix} \mathcal{E}^{00} & \mathcal{E}^{01} \\ \mathcal{E}^{10} & \mathcal{E}^{11} \end{pmatrix}. \tag{2.5}$$

The inequality (2.2) implies, in particular, that $\mathcal{E}^{00}$ is invertible as a map from $\operatorname{Ker} \mathcal{D}$ to $\operatorname{Ker} \mathcal{D}$.

Let $\Pi_0, \Pi_1 : \mathcal{V} \to \mathcal{V}$ be defined by

$$\Pi_0 := (\mathcal{E}^{00})^{-1}\Pi_0^{\mathcal{V}}\iota^{-1}\mathcal{E} \quad \text{and} \quad \Pi_1 := I - \Pi_0. \tag{2.6}$$

By the matrix form of $\mathcal{E}$ above,

$$\Pi_0 = \begin{pmatrix} I & (\mathcal{E}^{00})^{-1}\mathcal{E}^{01} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi_1 = \begin{pmatrix} 0 & -(\mathcal{E}^{00})^{-1}\mathcal{E}^{01} \\ 0 & I \end{pmatrix}. \tag{2.7}$$

We emphasise that these projections depend on P, although we do not indicate this in the notation for brevity. Our arguments below use *both* Π_0 and Π_1 *and* the analogous projections with P replaced by P^* (note that, since $\operatorname{Ker} \mathcal{D} = \operatorname{Ker} \mathcal{D}^*$, replacing P by P^* amounts to replacing $\mathcal{E}$ by $\mathcal{E}^*$).

By (2.1),

$$\|\Pi_0 v\|_{\mathcal{H}} = \|\Pi_0 v\|_{\mathcal{V}} \quad \text{for all } v \in \mathcal{H}, \tag{2.8}$$

so that, in particular, $\Pi_0 : \mathcal{H} \to \mathcal{H}$ and $\Pi_1 := I - \Pi_0 : \mathcal{H} \to \mathcal{H}$ are both bounded.

Assumption 2.3 (Abstract regularity assumptions)**.** *Let $\mathcal{Z}^0 = \mathcal{V}$, $\mathcal{Z}^1 = \mathcal{H}$, and $\mathcal{Z}^j \subset \mathcal{Z}^{j-1}$ for $j = 1, \ldots, m+1$, with $\mathcal{V}$ dense in $(\mathcal{Z}^j)^*$ for $j \geq 1$. Given P satisfying Assumption 2.1, there exists $C > 0$ such that the following is true.*

(i) For $j = 1, \ldots, m+1$,

$$\left\|\Pi_0^{\mathcal{V}}\right\|_{\mathcal{Z}^j \to \mathcal{Z}^j} \leq C. \tag{2.9}$$

(ii) With D equal $\mathcal{D}$ or $\mathcal{D}^*$ or $\operatorname{Re} \mathcal{D}$, for $j = 2, \ldots, m+1$,

$$\|\Pi_1 u\|_{\mathcal{Z}^j} \leq C\Big(\|\Pi_1 u\|_{\mathcal{V}} + \sup_{v \in \mathcal{H}, \|\iota v\|_{(\mathcal{Z}^{j-2})^*} = 1} \left|\left\langle \mathsf{D}\Pi_1 u, \Pi_1 v\right\rangle_{\mathcal{H}^* \times \mathcal{H}}\right|\Big) \quad \textit{for all } u \in \mathcal{H}, \tag{2.10}$$

with $\Pi_1 u \in \mathcal{Z}^j$ if the right-hand side is finite.

(iii) With E equal $\iota^{-1}\mathcal{E}$ or $\iota^{-1}\mathcal{E}^*$ or $\iota^{-1} \operatorname{Re} \mathcal{E}$, for $j = 1, \ldots, m+1$,

$$\|\mathsf{E}\|_{\mathcal{Z}^j \to \mathcal{Z}^j} \leq C. \tag{2.11}$$

(iv) With E equal $\iota^{-1}\mathcal{E}$ or $\iota^{-1}\mathcal{E}^*$, for $j = 1, \ldots, m+1$,

$$\left\|\Pi_0^{\mathcal{V}} u\right\|_{\mathcal{Z}^j} \leq C\Big(\left\|\Pi_0^{\mathcal{V}}\mathsf{E}\Pi_0^{\mathcal{V}} u\right\|_{\mathcal{Z}^j} + \left\|\Pi_0^{\mathcal{V}} u\right\|_{\mathcal{V}} \Big) \quad \textit{for all } u \in \mathcal{V}. \tag{2.12}$$

Given the partition $\{\Omega_j\}_{j=1}^n$ from Assumption 1.2, let

$$H^j_{\rm pw}(\Omega) := \Big\{ v \in L^2(\Omega) \ : \ \text{for all multi-indices } \alpha \text{ with } |\alpha| \leq j,\ \partial^\alpha (v|_{\Omega_i}) \in L^2(\Omega_i) \Big\}, \tag{2.13}$$

and equip $H^j_{\rm pw}(\Omega)$ with the norm

$$\|v\|^2_{H^j_{{\rm pw},k}(\Omega)} := \sum_{|\alpha| \leq j} \sum_{i=1}^n \int_{\Omega_i} \left| (k^{-1}\partial)^\alpha (v|_{\Omega_i}) \right|^2. \tag{2.14}$$

Lemma 2.4 (Application to Maxwell). *Let $\mathcal{V} = L^2(\Omega)$ and let $\mathcal{H} = H_0(\operatorname{curl}, \Omega)$ (i.e., functions in $H(\operatorname{curl}, \Omega)$ with zero tangential trace). Given matrix-valued functions μ and ϵ with*

$$\operatorname{Re}\mu^{-1} \geq c > 0 \qquad \text{and} \qquad \operatorname{Re}\epsilon \geq c > 0 \tag{2.15}$$

in Ω (in the sense of quadratic forms), let

$$\mathcal{D} := k^{-2}\operatorname{curl}\mu^{-1}\operatorname{curl} \qquad \text{and} \qquad \mathcal{E} := \epsilon. \tag{2.16}$$

Let

$$\|v\|_{\mathcal{V}}^2 = \|v\|_{L^2(\Omega)}^2 \qquad \text{and} \qquad \|v\|_{\mathcal{H}}^2 = k^{-2}\big\|(\operatorname{Re}\mu)^{-1/2}\operatorname{curl} v\big\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 .$$

(a) Assumption 2.1 holds with $\operatorname{Ker}\mathcal{D} = \operatorname{Ker}\operatorname{curl}$, $\|P\|_{\mathcal{H}\to\mathcal{H}^*}$ independent of k, and (2.2) satisfied with $C_{\mathcal{E}} = c$.

(b) Assumption 2.3 holds, for both P and P^*, if Ω, ϵ, and μ satisfy Assumption 1.2 and, with $\{\Omega_i\}_{i=1}^n$ as in Assumption 1.2, $\mathcal{Z}^j = Z^j$ defined by

$$Z^j := H_0(\operatorname{curl}, \Omega) \cap \Big\{ v \in L^2(\Omega) \,:\, v \in H^{j-1}_{\rm pw}(\Omega) \text{ and } \operatorname{curl} v \in H^{j-1}_{\rm pw}(\Omega) \Big\} \tag{2.17}$$

(observe that $Z^1 = H_0(\operatorname{curl}, \Omega)$) and equipped with the norm

$$\|v\|_{Z^j_k}^2 := \|v\|_{H_k(\operatorname{curl}, \Omega)}^2 + \|v\|^2_{H^{j-1}_{{\rm pw},k}(\Omega)} + \big\|k^{-1}\operatorname{curl} v\big\|^2_{H^{j-1}_{{\rm pw},k}(\Omega)} . \tag{2.18}$$

Lemma 2.4 is proved in §11.1 below.

Remark 2.5 (The regularity assumptions on ϵ, μ, and $\partial\Omega$). *In §11.1.2 below, we see that*

- *(2.9) (2.10), and (2.12) hold when μ and ϵ are piecewise C^m and the connected components of $\partial\Omega_j$, $j = 1, \ldots, n$, are all C^{m+1}, and*
- *(2.11) holds when ϵ is piecewise $C^{m,1}$.*

The combination of these requirements is then Assumption 1.2.

Remark 2.6 (Π_1 projects to functions that are ϵ-divergence free). *By (3.4) below, $\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_1 = 0$, so that, in the Maxwell setting of Lemma 2.4, $\epsilon\Pi_1$ is L^2 orthogonal to $\operatorname{Ker}\mathcal{D} = \operatorname{Ker}\operatorname{curl}$. Since $\nabla H_0^1(\Omega) \subset \operatorname{Ker}\operatorname{curl}$, Π_1 projects, in particular, to functions that are ϵ-divergence free.*

Having defined the spaces Z^j, we now define the notion of k-oscillatory data used in Theorem 1.3.

Definition 2.7. *f is k-oscillatory with constant $C_{\rm osc} > 0$ and regularity index m if one of the two following conditions holds.*

(i) $f \in Z^{m+1}$ and

$$\|f\|_{Z_k^{m+1}} \leq C_{\rm osc}\, \|f\|_{(H_k(\operatorname{curl}, \Omega)^*)} . \tag{2.19}$$

(ii) $f \in Z^{m-1}$ with $\operatorname{div} f = 0$ and (2.19) holds with $m+1$ replaced by $m-1$.

2.2 The Galerkin method

Let $\mathcal{H}_h \subset \mathcal{H}$ be closed, and let $\Pi_h : \mathcal{H} \to \mathcal{H}_h$ be the orthogonal projection. Given $u \in \mathcal{H}$, we seek an approximation of u, u_h, satisfying

$$\big\langle P(u - u_h), v_h \big\rangle_{\mathcal{H}^*\times\mathcal{H}} = 0 \quad \text{for all } v_h \in \mathcal{H}_h. \tag{2.20}$$

Observe that, since $P = \mathcal{D} - \mathcal{E}$, the Galerkin orthogonality (2.20) implies that

$$\big\langle \mathcal{E}(u - u_h), v_h \big\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0 \quad \text{for all } v_h \in \mathcal{H}_h \cap \operatorname{Ker}\mathcal{D}. \tag{2.21}$$

2.3 The quantity $\gamma_{\rm dv}(P)$

Let

$$\gamma_{\rm dv}(P) := \sup\left\{ \frac{\|\Pi_0 w_h\|_{\mathcal{V}}}{\|w_h\|_{\mathcal{H}}} \;:\; w_h \in \mathcal{H}_h \text{ satisfies } \big\langle \mathcal{E} w_h, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0 \;\text{ for all } v_h \in \mathcal{H}_h \cap \operatorname{Ker}\mathcal{D}\right\}. \tag{2.22}$$

We write $\gamma_{\rm dv}(P)$, since we consider below both $\gamma_{\rm dv}(P)$ and $\gamma_{\rm dv}(P^*)$ (and we highlight again that Π_0 depends on P).

Note that if $v_h \in \mathcal{H}_h \cap \operatorname{Ker}\mathcal{D}$ in the definition of $\gamma_{\rm dv}(P)$ (2.22) is changed to $v \in \mathcal{H}\cap \operatorname{Ker}\mathcal{D}$, then $\gamma_{\rm dv}(P) = 0$. Indeed, if w satisfies $(\iota^{-1}\mathcal{E}w, v)_{\mathcal{V}} = 0$ for all $v \in \operatorname{Ker}\mathcal{D}$, then $\iota^{-1}\mathcal{E}w = \Pi_1^{\mathcal{V}} z$ for some $z \in \mathcal{H}$ (since $\mathcal{H} = (\operatorname{Ker}\mathcal{D}) \oplus (\operatorname{Ker}\mathcal{D})^\perp$); i.e. $w = (\iota^{-1}\mathcal{E})^{-1}\Pi_1^{\mathcal{V}} z$ for some $z \in \mathcal{H}$. Then, by (3.3) below, $\Pi_0 w = 0$.

Comparing (2.21) and (2.22), we see that, since the Galerkin error $u - u_h \notin \mathcal{H}_h$, $u - u_h$ is not contained in the set of w_h considered in (2.22). Nevertheless, controlling $\gamma_{\rm dv}(P)$ gives us a way to control $\Pi_0(u-u_h)$, with Lemma 5.4 below showing that, for a certain projection Π_h^+, $\Pi_0\Pi_h^+(u-u_h)$ is controlled by $\gamma_{\rm dv}(P)$, and then Lemma 5.1 controlling $\Pi_0(u-u_h)$.

Remark 2.8 ($\gamma_{\rm dv}(P)$ is the divergence conformity factor)**.** *In the Maxwell setting of Lemma 2.4, the w_h considered in (2.22) are discretely ϵ-divergence free. By Remark 2.6, if $\Pi_1 w_h = w_h$ (i.e., $\Pi_0 w_h = 0$) then w_h is ϵ-divergence free. The quantity $\gamma_{\rm dv}(P)$ is therefore the familiar* divergence conformity factor*, measuring how much a finite-element function that is discretely ϵ-divergence-free is not pointwise ϵ-divergence-free, with this mismatch central to the analysis of the Maxwell FEM using Nédélec elements, as discussed in §1.2 (see also [2, Lemma 5.2], [34, Lemma 4.5], [51, Lemma 7.6] [9, Lemma 5.2]).*

2.4 The main abstract theorem

Theorem 2.9 (The main result in abstract form)**.** *Suppose that Assumption 2.1 holds for P and that, for some $m \in \mathbb{Z}^+$ and spaces $\mathcal{Z}^j, j = 1,\ldots,m+1$, Assumption 2.3 holds for both P and P^*. Suppose that, given $f \in \mathcal{H}^*$, the solution to the equation $Pu = f$ exists and is unique.*

Then there exist C_1, C_2, and $C_3 > 0$ such that if

$$\max\big\{\gamma_{\rm dv}(P), \gamma_{\rm dv}(P^*)\big\} \le C_1 \quad \text{and} \quad \Big(\|I-\Pi_h\|_{\mathcal{Z}^{m+1}\to\mathcal{H}}\Big)^2\big(1+\big\|(P^*)^{-1}\Pi_1^*\big\|_{\mathcal{V}^*\to\mathcal{V}}\big) \le C_2, \tag{2.23}$$

then u_h defined by (2.20) exists, is unique, and satisfies

$$\|u-u_h\|_{\mathcal{H}} \le C_3\Big(1+\|I-\Pi_h\|_{\mathcal{Z}^{m+1}\to\mathcal{H}}\big(1+\big\|(P^*)^{-1}\Pi_1^*\big\|_{\mathcal{V}^*\to\mathcal{V}}\big)\Big)\,\|(I-\Pi_h)u\|_{\mathcal{H}}. \tag{2.24}$$

In addition, given $C_{\rm osc} > 0$, there exists $C_4 > 0$ such that if $\|\iota^{-1}f\|_{\mathcal{Z}^{m+1}} \le C_{\rm osc}\|f\|_{\mathcal{H}^*}$ and (2.23) holds, then

$$\frac{\|u-u_h\|_{\mathcal{H}}}{\|u\|_{\mathcal{H}}} \le C_4\Big(1+\|I-\Pi_h\|_{\mathcal{Z}^{m+1}\to\mathcal{H}}\big(1+\big\|(P^*)^{-1}\Pi_1^*\big\|_{\mathcal{V}^*\to\mathcal{V}}\big)\Big)\,\|I-\Pi_h\|_{\mathcal{Z}^{m+1}\to\mathcal{H}}. \tag{2.25}$$

Furthermore, if $\Pi_0^* f = \widetilde{\Pi}_0^* f$ for some $\widetilde{\Pi}_0 : \mathcal{V}\to\mathcal{V}$ satisfying $\Pi_0\widetilde{\Pi}_0 = \widetilde{\Pi}_0$ (i.e., $\widetilde{\Pi}_0$ maps into $\operatorname{Ker}\mathcal{D}$) and

$$\|\iota^{-1}\widetilde{\Pi}_0^*\iota\|_{\mathcal{Z}^{m-1}\to\mathcal{Z}^{m+1}} \le C, \tag{2.26}$$

then the assumption $\|\iota^{-1}f\|_{\mathcal{Z}^{m+1}} \le C_{\rm osc}\|f\|_{\mathcal{H}^*}$ can be relaxed to $\|\iota^{-1}f\|_{\mathcal{Z}^{m-1}} \le C_{\rm osc}\|f\|_{\mathcal{H}^*}$.

Theorem 2.9 is proved in §8 below. We make three remarks:

(i) By the order of quantifiers in Theorem 2.9, the constants C_1, C_2, C_3, and C_4 in the theorem depend on the quantities $\|P\|_{\mathcal{H}\to\mathcal{H}^*}$, $\|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*}$, $C'_{\mathcal{E}}$, C_1, and C_2 in Assumption 2.1 and the quantity C in Assumption 2.3.

(ii) The additional projection $\widetilde{\Pi}_0^*$ in the last part of the theorem caters for the fact that, in the Maxwell setting, the kernel of the curl does not only consist of gradients when Ω is not simply connected, but the condition that $\operatorname{div} f = \operatorname{div}(\epsilon E) = 0$ (i.e., f is orthogonal to gradients) is nevertheless enough for E to gain regularity with respect to f (see Theorem 9.1 below).

(iii) The relative-error bound (2.25) follows from the preasymptotic error bound (2.24) and the following regularity result (proved in §4 below).

Lemma 2.10 (k-oscillatory data implies k-oscillatory solution)**.** *Suppose that Assumption 2.3 holds for some $m \in \mathbb{Z}^+$ and spaces $\mathcal{Z}^j, j = 1, \ldots, m+1$. Given $C_{\rm osc} > 0$ there exists $C' > 0$ such that the following is true. If $Pu = f$ with $f \in \mathcal{V}^*$ satisfying*

$$\left\|\iota^{-1} f\right\|_{\mathcal{Z}^{m+1}} \le C_{\rm osc} \left\|f\right\|_{\mathcal{H}^*}, \qquad \text{then} \qquad \left\|u\right\|_{\mathcal{Z}^{m+1}} \le C' \left\|u\right\|_{\mathcal{H}}. \tag{2.27}$$

Furthermore, if $\Pi_0^* f = \widetilde{\Pi}_0^* f$ for some $\widetilde{\Pi}_0 : \mathcal{V} \to \mathcal{V}$ satisfying $\Pi_0 \widetilde{\Pi}_0 = \widetilde{\Pi}_0$ and (2.26), then the assumption $\|\iota^{-1} f\|_{\mathcal{Z}^{m+1}} \le C_{\rm osc} \|f\|_{\mathcal{H}^*}$ can be relaxed to $\|\iota^{-1} f\|_{\mathcal{Z}^{m-1}} \le C \|f\|_{\mathcal{H}^*}$.

3 Properties of Π_0 and Π_1

By its definition (2.6), $\Pi_0 : \mathcal{V} \to \operatorname{Ker} \mathcal{D}$. Since $\operatorname{Ker} \mathcal{D} = \operatorname{Ker} \mathcal{D}^*$,

$$\mathcal{D} = \mathcal{D}\Pi_1 = \Pi_1^* \mathcal{D} = \Pi_1^* \mathcal{D} \Pi_1. \tag{3.1}$$

Lemma 3.1 (Properties and equivalent definitions of Π_0)**.** *The following are equivalent*

(a) $\Pi_0 := (\mathcal{E}^{00})^{-1} \Pi_0^{\mathcal{V}} (\iota^{-1}\mathcal{E})$; i.e., Π_0 is given by (2.7).

(b) $\Pi_0 : \mathcal{V} \to \operatorname{Ker} \mathcal{D}$ is a projection satisfying

$$\Pi_0^* \mathcal{E} \Pi_1 = 0. \tag{3.2}$$

(c) $\Pi_0 : \mathcal{V} \to \operatorname{Ker} \mathcal{D}$ is a projection satisfying

$$\Pi_0 (\iota^{-1}\mathcal{E})^{-1} \Pi_1^{\mathcal{V}} = 0. \tag{3.3}$$

(d) $\Pi_0 : \mathcal{V} \to \operatorname{Ker} \mathcal{D}$ is a projection satisfying

$$\Pi_0^{\mathcal{V}} (\iota^{-1}\mathcal{E}) \Pi_1 = 0. \tag{3.4}$$

We highlight that the property (3.2) is essential in the duality arguments below (since it means that the matrix representation of P as a map $(\Pi_0\mathcal{H}, \Pi_1\mathcal{H}) \to (\Pi_0^*\mathcal{H}^*, \Pi_1^*\mathcal{H}^*)$ is lower triangular; see (4.2) below). The property (3.3) is essential for $\gamma_{\rm dv}(P)$ to $\to 0$ as $\mathcal{H}_h \to \mathcal{H}$ (as explained in the text after (2.22)). Finally, recall from Remark 2.6 that (3.4) implies, in the Maxwell setting, that $\epsilon \Pi_1$ is L^2 orthogonal to $\operatorname{Ker} \mathcal{D} = \operatorname{Ker} \operatorname{curl}$ and thus Π_1 projects, in particular, to functions that are ϵ-divergence free.

Proof of Lemma 3.1. (a) immediately implies (c) since $\Pi_0^{\mathcal{V}} \Pi_1^{\mathcal{V}} = 0$. To see that (a) implies (d), observe that

$$\iota^{-1}\mathcal{E}\Pi_1 = \begin{pmatrix} \mathcal{E}^{00} & \mathcal{E}^{01} \\ \mathcal{E}^{10} & \mathcal{E}^{11} \end{pmatrix} \begin{pmatrix} 0 & -(\mathcal{E}^{00})^{-1}\mathcal{E}^{01} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\mathcal{E}^{10}(\mathcal{E}^{00})^{-1}\mathcal{E}^{01} + \mathcal{E}^{11} \end{pmatrix}, \tag{3.5}$$

so that (d) holds. To see that (a) implies (b), we first claim that

$$\iota = \begin{pmatrix} \iota & 0 \\ 0 & \iota \end{pmatrix} \tag{3.6}$$

as a map from $\mathcal{V} = \big(\Pi_0^{\mathcal{V}}\mathcal{V}, \Pi_1^{\mathcal{V}}\mathcal{V}\big)$ to $\mathcal{V}^* = \big((\Pi_0^{\mathcal{V}})^*\mathcal{V}^*, (\Pi_1^{\mathcal{V}})^*\mathcal{V}^*\big)$. Indeed, since $\Pi_0^{\mathcal{V}}$ and $\Pi_1^{\mathcal{V}}$ are are $\mathcal{V}$-orthogonal projections, they are self-adjoint in $(\cdot,\cdot)_{\mathcal{V}}$. Therefore, since also $\Pi_0^{\mathcal{V}}\Pi_1^{\mathcal{V}} = 0$,

$$\begin{aligned} \langle \iota u, v\rangle_{\mathcal{V}^*\times\mathcal{V}} = (u,v)_{\mathcal{V}} &= (\Pi_0^{\mathcal{V}}u, \Pi_0^{\mathcal{V}}v)_{\mathcal{V}} + (\Pi_1^{\mathcal{V}}u, \Pi_1^{\mathcal{V}}v)_{\mathcal{V}} \\ &= \langle \iota\Pi_0^{\mathcal{V}}u, \Pi_0^{\mathcal{V}}v\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \iota\Pi_1^{\mathcal{V}}u, \Pi_1^{\mathcal{V}}v\rangle_{\mathcal{V}^*\times\mathcal{V}}, \end{aligned}$$

which implies (3.6). The combination of (3.5) and (3.6) implies that

$$\begin{aligned} \Pi_1^*\mathcal{E}\Pi_1 &= \begin{pmatrix} 0 & 0 \\ -(\mathcal{E}^{01})^*((\mathcal{E}^{00})^{-1})^* & I \end{pmatrix} \begin{pmatrix} \iota & 0 \\ 0 & \iota \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\mathcal{E}^{10}(\mathcal{E}^{00})^{-1}\mathcal{E}^{01} + \mathcal{E}^{11} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \iota(-\mathcal{E}^{10}(\mathcal{E}^{00})^{-1}\mathcal{E}^{01} + \mathcal{E}^{11}) \end{pmatrix} = \mathcal{E}\Pi_1, \end{aligned}$$

i.e., (b) holds.

For (b) implies (a): since $\Pi_0^2 = \Pi_0$ and $\Pi_0 : \mathcal{V} \to \operatorname{Ker}\mathcal{D}$, the matrix representation of Π_0 as a map from $\mathcal{V} = \big(\Pi_0^{\mathcal{V}}\mathcal{V}, \Pi_1^{\mathcal{V}}\mathcal{V}\big)$ to itself is

$$\Pi_0 = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix} \tag{3.7}$$

for some A. Then, by a similar calculation to that in (3.5),

$$\Pi_0^*\mathcal{E}\Pi_1 = \begin{pmatrix} 0 & \iota(-\mathcal{E}^{00}A + \mathcal{E}^{01}) \\ 0 & A^*\iota(-\mathcal{E}^{00}A + \mathcal{E}^{01}) \end{pmatrix},$$

so that $A = (\mathcal{E}^{00})^{-1}\mathcal{E}^{01}$.

Similarly, for (d) implies (a): if Π_0 is given by (3.7), then

$$\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_1 = \begin{pmatrix} 0 & -\mathcal{E}^{00}A + \mathcal{E}^{01} \\ 0 & 0 \end{pmatrix},$$

so that, again, $A = (\mathcal{E}^{00})^{-1}\mathcal{E}^{01}$.

For (c) implies (a): $\Pi_0(\iota^{-1}\mathcal{E})^{-1}\Pi_1^{\mathcal{V}} = 0$ implies that $\Pi_0(\iota^{-1}\mathcal{E})^{-1} = \Pi_0(\iota^{-1}\mathcal{E})^{-1}\Pi_0^{\mathcal{V}}$, so that $\Pi_0 = \Pi_0(\iota^{-1}\mathcal{E})^{-1}\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})$; i.e.,

$$\Pi_0 = B\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E}) \quad \text{ for some } B : \mathcal{V} \to \mathcal{V}. \tag{3.8}$$

We'll show that $B = (\mathcal{E}^{00})^{-1}$ to complete the proof. Since $\Pi_0, \Pi_0^{\mathcal{V}} : \mathcal{V} \to \operatorname{Ker}\mathcal{D}$, (3.8) implies that $B : \operatorname{Ker}\mathcal{D} \to \operatorname{Ker}\mathcal{D}$. Furthermore, by (2.7),

$$\Pi_0^{\mathcal{V}} = \Pi_0\Pi_0^{\mathcal{V}}. \tag{3.9}$$

The combination of (3.8) and (3.9) implies that

$$\Pi_0^{\mathcal{V}} = B\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0^{\mathcal{V}} = B\mathcal{E}^{00}$$

(by the definition of $\mathcal{E}^{00}$ in (2.5)); i.e., on $\operatorname{Ker}\mathcal{D}$, $B\mathcal{E}^{00}$ is the identity. Therefore, $B = (\mathcal{E}^{00})^{-1}$ as an operator $\operatorname{Ker}\mathcal{D} \to \operatorname{Ker}\mathcal{D}$ and the proof is complete. □

Lemma 3.2 (Π_0 and Π_1 preserve regularity)**.** *If Assumption 2.3 holds then there exists $C > 0$ such that for $j = 0, \dots, m+1$,*

$$\|\Pi_0\|_{\mathcal{Z}^j \to \mathcal{Z}^j} \le C \tag{3.10}$$

and

$$\left\|\iota^{-1}\Pi_0^*\iota\right\|_{\mathcal{Z}^j \to \mathcal{Z}^j} = \left\|\iota\Pi_0\iota^{-1}\right\|_{(\mathcal{Z}^j)^* \to (\mathcal{Z}^j)^*} \le C, \tag{3.11}$$

with analogous bounds holding for Π_1 since $\Pi_0 = I - \Pi_1$.

Proof. By the definitions of Π_0 (2.6), $\mathcal{E}^{00}$ (2.5), and $\mathcal{E}^{01}$, to prove (3.10) it is sufficient to prove that

$$\big(\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0^{\mathcal{V}}\big)^{-1}\big(\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_1^{\mathcal{V}}\big) : \Pi_1^{\mathcal{V}}\mathcal{Z}^j \to \Pi_0^{\mathcal{V}}\mathcal{Z}^j. \tag{3.12}$$

By (2.9) and (2.11) (with $\mathsf{E} = \iota^{-1}\mathcal{E}$),

$$\big\|\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_1^{\mathcal{V}}\big\|_{\mathcal{Z}^j \to \mathcal{Z}^j} \le C.$$

Therefore, to prove (3.10) it is sufficient to prove that

$$\big\|\big(\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0^{\mathcal{V}}\big)^{-1}\big\|_{\Pi_0^{\mathcal{V}}\mathcal{Z}^j \to \Pi_0^{\mathcal{V}}\mathcal{Z}^j} \le C. \tag{3.13}$$

However, (3.13) follows from (2.12) with $\mathsf{E} = \iota^{-1}\mathcal{E}$ and the fact that $(\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0^{\mathcal{V}})^{-1} : \Pi_0^{\mathcal{V}}\mathcal{V} \to \Pi_0^{\mathcal{V}}\mathcal{V}$ is bounded by (2.2).

For (3.11), by (3.6) and the definition of Π_0,

$$\iota^{-1}\Pi_0^*\iota = \begin{pmatrix} I & 0 \\ \iota^{-1}(\mathcal{E}^{01})^*\big((\mathcal{E}^{00})^*\big)^{-1}\iota & 0 \end{pmatrix}.$$

Now, by (2.4), $\iota^* = \iota$ and $(\Pi_0^{\mathcal{V}})^* = \iota\Pi_0^{\mathcal{V}}\iota^{-1}$. Therefore,

$$\iota^{-1}(\mathcal{E}^{01})^*\big((\mathcal{E}^{00})^*\big)^{-1}\iota = \Pi_1^{\mathcal{V}}(\iota^{-1}\mathcal{E}^*)\Pi_0^{\mathcal{V}}\big(\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E}^*)\Pi_0^{\mathcal{V}}\big)^{-1}$$

(compare to (3.12)). Thus (3.11) follows in a similar way to (3.10), now using (2.12) with $\mathsf{E} = \iota^{-1}\mathcal{E}^*$. □

4 Matrix representation of P, regularity shift of $(P^*)^{-1}\Pi_1^*$, and proof of Lemma 2.10

Since $\Pi_0\Pi_1 = 0$, for all $f \in \mathcal{H}^*$ and $v \in \mathcal{H}$,

$$\langle f, v\rangle_{\mathcal{H}^*\times\mathcal{H}} = \langle \Pi_0^* f, \Pi_0 v\rangle_{\mathcal{H}^*\times\mathcal{H}} + \langle \Pi_1^* f, \Pi_1 v\rangle_{\mathcal{H}^*\times\mathcal{H}}.$$

Thus, given $A : \mathcal{H} \to \mathcal{H}^*$, for all $u, v \in \mathcal{H}$,

$$\langle Au, v\rangle_{\mathcal{H}^*\times\mathcal{H}} = \langle \Pi_0^* A(\Pi_0 u + \Pi_1 u), \Pi_0 v\rangle_{\mathcal{H}^*\times\mathcal{H}} + \langle \Pi_1^* A(\Pi_0 u + \Pi_1 u), \Pi_1 v\rangle_{\mathcal{H}^*\times\mathcal{H}}.$$

Therefore, given $A : \mathcal{H} \to \mathcal{H}^*$, its matrix representation as a map $(\Pi_0\mathcal{H}, \Pi_1\mathcal{H}) \to (\Pi_0^*\mathcal{H}^*, \Pi_1^*\mathcal{H}^*)$ is

$$\begin{pmatrix} \Pi_0^* A\Pi_0 & \Pi_0^* A\Pi_1 \\ \Pi_1^* A\Pi_0 & \Pi_1^* A\Pi_1 \end{pmatrix} =: \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \tag{4.1}$$

With this notation, by (3.1) and (3.2),

$$\mathcal{D} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{D}_{11} \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} \mathcal{E}_{00} & 0 \\ \mathcal{E}_{10} & \mathcal{E}_{11} \end{pmatrix} \quad \text{and thus} \quad P = \begin{pmatrix} -\mathcal{E}_{00} & 0 \\ -\mathcal{E}_{10} & \mathcal{D}_{11} - \mathcal{E}_{11} \end{pmatrix}. \tag{4.2}$$

We now show that Part (iv) of Assumption 2.3 (i.e., (2.12)) implies the following result.

Lemma 4.1. *If Assumption 2.3 holds then there exists $C > 0$ such that, for $j = 0, \ldots, m+1$,*

$$\|\Pi_0 u\|_{\mathcal{Z}^j} \le C \min\left\{ \left\|\iota^{-1}\mathcal{E}_{00}\Pi_0 u\right\|_{\mathcal{Z}^j}, \left\|\iota^{-1}\mathcal{E}_{00}^*\Pi_0 u\right\|_{\mathcal{Z}^j} \right\} \quad \textit{for all } u \in \mathcal{V}. \tag{4.3}$$

Proof. We first prove the bound in (4.3) involving $\mathcal{E}_{00}$. By (2.7), $\Pi_0^{\mathcal{V}}\Pi_0 = \Pi_0$. Therefore, by (2.12) with $\mathsf{E} = \iota^{-1}\mathcal{E}$,

$$\begin{aligned}\|\Pi_0 u\|_{\mathcal{Z}^j} &= \left\|\Pi_0^{\mathcal{V}}\Pi_0 u\right\|_{\mathcal{Z}^j} \\ &\le C\Big(\left\|\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0^{\mathcal{V}}\Pi_0 u\right\|_{\mathcal{Z}^j} + \left\|\Pi_0^{\mathcal{V}}\Pi_0 u\right\|_{\mathcal{V}}\Big) \\ &= C\Big(\left\|\Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0 u\right\|_{\mathcal{Z}^j} + \left\|\Pi_0 u\right\|_{\mathcal{V}}\Big). \end{aligned} \tag{4.4}$$

Now, by (2.7), $\Pi_0^{\mathcal{V}}\iota^{-1}\Pi_0^* = \Pi_0^{\mathcal{V}}\iota^{-1}$. By this and the definition $\mathcal{E}_{00} := \Pi_0^*\mathcal{E}\Pi_0$ (4.1),

$$\Pi_0^{\mathcal{V}}\iota^{-1}\mathcal{E}_{00}\Pi_0 u = \Pi_0^{\mathcal{V}}\iota^{-1}\Pi_0^*\mathcal{E}\Pi_0 u = \Pi_0^{\mathcal{V}}(\iota^{-1}\mathcal{E})\Pi_0 u,$$

By the last displayed equation, (4.4), and (2.9),

$$\|\Pi_0 u\|_{\mathcal{Z}^j} \le C\Big(\left\|\iota^{-1}\mathcal{E}_{00}\Pi_0 u\right\|_{\mathcal{Z}^j} + \left\|\Pi_0 u\right\|_{\mathcal{V}}\Big). \tag{4.5}$$

To remove the second term on the right-hand side of (4.5) and obtain the bound in (4.3) involving $\mathcal{E}_{00}$, we use (2.2) to obtain that

$$C_{\mathcal{E}}\|\Pi_0 u\|_{\mathcal{V}}^2 \le \left|\langle\mathcal{E}_{00}\Pi_0 u, \Pi_0 u\rangle_{\mathcal{V}^*\times\mathcal{V}}\right| \le \left\|\iota^{-1}\mathcal{E}_{00}\Pi_0 u\right\|_{\mathcal{V}}\|\Pi_0 u\|_{\mathcal{V}}; \tag{4.6}$$

the result then follows by combining (4.5) and (4.6). The bound in (4.3) involving $\mathcal{E}_{00}^*$ follows in an analogous way, now using (2.12) with $\mathsf{E} = \iota^{-1}\mathcal{E}^*$. □

Lemma 4.2 (Regularity of $(P^*)^{-1}\Pi_1^*\iota$)**.** *If $(P^*)^{-1}$ exists and Assumption 2.3 holds, then there exists $C > 0$ such that*

$$\left\|(P^*)^{-1}\Pi_1^*\iota\right\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^j} \le C\big(1 + \left\|(P^*)^{-1}\Pi_1^*\right\|_{\mathcal{V}^*\to\mathcal{V}}\big) \quad \textit{for } j = 2, \ldots, m+1.$$

Proof. Let $g \in \mathcal{Z}^{j-2} \subset \mathcal{V}$ and let $u = (P^*)^{-1}\Pi_1^*\iota g \in \mathcal{H}$ so that $P^*u = \Pi_1^*\iota g \in \mathcal{H}^*$. Now, by (3.1) and (2.4), for $v \in \mathcal{H} \subset \mathcal{V}$,

$$\begin{aligned}\left|\langle\mathcal{D}^*\Pi_1 u, \Pi_1 v\rangle_{\mathcal{H}^*\times\mathcal{H}}\right| &= \left|\langle\mathcal{D}^* u, v\rangle_{\mathcal{H}^*\times\mathcal{H}}\right| = \left|\langle\Pi_1^*\iota g + \mathcal{E}^* u, v\rangle_{\mathcal{V}^*\times\mathcal{V}}\right| \\ &\le \left|\langle g, \iota\Pi_1 v\rangle_{\mathcal{V}\times\mathcal{V}^*}\right| + \left|\langle u, \mathcal{E}v\rangle_{\mathcal{V}\times\mathcal{V}^*}\right| \\ &\le \Big(\|g\|_{\mathcal{Z}^{j-2}}\left\|\iota\Pi_1\iota^{-1}\right\|_{(\mathcal{Z}^{j-2})^*\to(\mathcal{Z}^{j-2})^*} + \left\|\mathcal{E}\iota^{-1}\right\|_{(\mathcal{Z}^{j-2})^*\to(\mathcal{Z}^{j-2})^*}\|u\|_{\mathcal{Z}^{j-2}}\Big)\|\iota v\|_{(\mathcal{Z}^{j-2})^*}.\end{aligned}$$

Therefore, by (3.11) and (2.11),

$$\left|\langle\mathcal{D}^*\Pi_1 u, \Pi_1 v\rangle_{\mathcal{H}^*\times\mathcal{H}}\right| \le C\Big(\|g\|_{\mathcal{Z}^{j-2}} + \|u\|_{\mathcal{Z}^{j-2}}\Big)\|\iota v\|_{(\mathcal{Z}^{j-2})^*},$$

so that, by (2.10) with $\mathsf{D} = \mathcal{D}^*$,

$$\|\Pi_1 u\|_{\mathcal{Z}^j} \le C\Big(\|\Pi_1 u\|_{\mathcal{V}} + \|g\|_{\mathcal{Z}^{j-2}} + \|u\|_{\mathcal{Z}^{j-2}}\Big) \le C'\Big(\|g\|_{\mathcal{Z}^{j-2}} + \|u\|_{\mathcal{Z}^{j-2}}\Big). \tag{4.7}$$

Now, from the matrix form of P (4.2),

$$\mathcal{E}_{00}^*\Pi_0 u + \mathcal{E}_{10}^*\Pi_1 u = 0 \quad \text{in } \Pi_0^*\mathcal{H}^*.$$

By (4.3), the definition $\mathcal{E}_{10}^* := (\Pi_1^*\mathcal{E}\Pi_0)^* = \Pi_0^*\mathcal{E}^*\Pi_1$ (4.1), (3.11), and (2.11) with $\mathsf{E} = \iota^{-1}\mathcal{E}^*$, for $j = 2, \ldots, m+1$,

$$\big\|\Pi_0 u\big\|_{\mathcal{Z}^j} \le C\big\|\iota^{-1}\Pi_0^*\mathcal{E}^*\Pi_1 u\big\|_{\mathcal{Z}^j} = C\big\|\iota^{-1}\Pi_0^*\iota(\iota^{-1}\mathcal{E}^*)\Pi_1 u\big\|_{\mathcal{Z}^j} \le C'\big\|\Pi_1 u\big\|_{\mathcal{Z}^j}.$$

Combining this with (4.7) we obtain that

$$\|u\|_{\mathcal{Z}^j} \le C\Big(\|g\|_{\mathcal{Z}^{j-2}} + \|u\|_{\mathcal{Z}^{j-2}} \Big). \tag{4.8}$$

When $j = 2$, (4.8) implies that

$$\|u\|_{\mathcal{Z}^2} \le C\Big(1 + \big\|(P^*)^{-1}\Pi_1^*\big\|_{\mathcal{V}^*\to\mathcal{V}}\Big) \|g\|_{\mathcal{V}};$$

the result then follows by combining this with (4.8). □

We now also prove Lemma 2.10, since its proof is similar to that of Lemma 4.2.

Proof of Lemma 2.10. From the matrix form of P (4.2), $-\mathcal{E}_{00}\Pi_0 u = \Pi_0^* f$. By (4.3), (3.11), and the bound on f in (2.27),

$$\begin{aligned}\|\Pi_0 u\|_{\mathcal{Z}^{m+1}} \le C\big\|\iota^{-1}\Pi_0^* f\big\|_{\mathcal{Z}^{m+1}} \le C\big\|\iota^{-1} f\big\|_{\mathcal{Z}^{m+1}} &\le CC_{\rm osc}\,\|f\|_{\mathcal{H}^*}\\ &= CC_{\rm osc}\sup_{v\in\mathcal{H}}\frac{\big|\langle Pu, v\rangle_{\mathcal{H}^*\times\mathcal{H}}\big|}{\|v\|_{\mathcal{H}}}\\ &\le CC_{\rm osc}C'\,\|u\|_{\mathcal{H}}.\end{aligned} \tag{4.9}$$

We now argue as in the proof of Lemma 4.2 – but now with P^* replaced by P and (2.10) applied with $\mathsf{D} = \mathcal{D}$ – to obtain that (4.7) holds for $j = 2, \ldots, m+1$ and $g = \iota^{-1} f$ (recall that in Lemma 4.2 we started with $g \in \mathcal{Z}^{j-2} \subset \mathcal{V}$, and here we started with $f \in \mathcal{H}^*$). When $j = 2$, this bound implies that

$$\|\Pi_1 u\|_{\mathcal{Z}^2} \le C\Big(\big\|\iota^{-1} f\big\|_{\mathcal{V}} + \|u\|_{\mathcal{V}}\Big) \le C\Big(\big\|\iota^{-1} f\big\|_{\mathcal{Z}^{m+1}} + \|u\|_{\mathcal{H}}\Big) \le C'\,\|u\|_{\mathcal{H}}, \tag{4.10}$$

where in the last inequality we have argued as in (4.9). The combination of (4.9) and (4.10) implies that $\|u\|_{\mathcal{Z}^2} \le C\|u\|_{\mathcal{H}}$. The result then follows from iterating the argument involving (4.7) for increasing j, up to $m+1$.

For the second assertion, now $-\mathcal{E}_{00}\Pi_0 u = \widetilde{\Pi}_0^* f$. By (4.3) and the assumption (2.26),

$$\|\Pi_0 u\|_{\mathcal{Z}^{m+1}} \le C\big\|\iota^{-1}\widetilde{\Pi}_0^* f\big\|_{\mathcal{Z}^{m+1}} \le C\,\big\|\iota^{-1} f\big\|_{\mathcal{Z}^{m-1}}. \tag{4.11}$$

Therefore, since $\Pi_1 u$ gains two derivatives over f via (4.7),

$$\|u\|_{\mathcal{Z}^{m+1}} \le \|\Pi_0 u\|_{\mathcal{Z}^{m+1}} + \|\Pi_1 u\|_{\mathcal{Z}^{m+1}} \le C\Big(\big\|\iota^{-1} f\big\|_{\mathcal{Z}^{m-1}} + \|u\|_{\mathcal{Z}^{m-1}}\Big).$$

Repeatedly applying (4.7) and using (4.11) (similar to in the first part of the proof) then gives that

$$\|u\|_{\mathcal{Z}^{m+1}} \le C\Big(\big\|\iota^{-1} f\big\|_{\mathcal{Z}^{m-1}} + \|u\|_{\mathcal{H}}\Big);$$

the result then follows in an analogous way to (4.9). □

5 Bounding $\|\Pi_0(u-u_h)\|_{\mathcal{H}}$ using $\gamma_{\rm dv}(P)$

To prove the bound (2.24) in Theorem 2.9, it is sufficient to prove this bound under the assumption of existence. Indeed, by the assumed uniqueness of the solution to $Pu=f$, the bound (2.24) under the assumption of existence implies uniqueness of u_h, and uniqueness implies existence for the finite-dimensional Galerkin linear system. From now on, therefore, we assume that u_h exists.

The main result of this section is the following.

Lemma 5.1. *Given P satisfying Assumption 2.1, define Π_0 by (2.6), $\gamma_{\rm dv}(P)$ by (2.22). Given $u\in\mathcal{H}$, assume that the solution $u_h\in\mathcal{H}_h$ of (2.20) exists. Then*

$$\|\Pi_0(u-u_h)\|_{\mathcal{H}} \leq C\Big(\|(I-\Pi_h)u\|_{\mathcal{H}} + \gamma_{\rm dv}(P)\,\|u-u_h\|_{\mathcal{H}}\Big). \tag{5.1}$$

To prove Lemma 5.1, we introduce the sesquilinear form

$$b^+(u,v) := \langle \mathcal{D}u, v\rangle_{\mathcal{H}^*\times\mathcal{H}} + (C_{\mathcal{E}})^{-1}\langle \mathcal{E}u, v\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{5.2}$$

Lemma 5.2. *b^+ is continuous and coercive on $\mathcal{H}$.*

Proof. Continuity is immediate. For coercivity, by (2.2) and (2.1),

$$\operatorname{Re} b^+(v,v) = \operatorname{Re}\langle \mathcal{D}v,v\rangle + (C_{\mathcal{E}})^{-1}\operatorname{Re}\langle \mathcal{E}u, v\rangle_{\mathcal{V}^*\times\mathcal{V}}. = \|v\|^2_{\mathcal{H}} - \|v\|^2_{\mathcal{V}} + \|v\|^2_{\mathcal{V}} = \|v\|^2_{\mathcal{H}}\,.$$

□

Corollary 5.3 (Definition and boundedness of Π_h^+)**.** *Given $u\in\mathcal{H}$, define $\Pi_h^+u\in\mathcal{H}_h$ as the solution of*

$$b^+\big(\Pi_h^+u, v_h\big) = b^+\big(u,v_h\big) \quad \textit{for all } v_h\in\mathcal{H}_h;$$

i.e.,

$$b^+\big((I-\Pi_h^+)u, v_h\big) = 0 \quad \textit{for all } v_h\in\mathcal{H}_h \tag{5.3}$$

Then $\Pi_h^+:\mathcal{H}\to\mathcal{H}_h$ is well-defined, bounded, satisfies $\Pi_h^+w_h=w_h$ for all $w_h\in\mathcal{H}_h$, and satisfies

$$\big\|I-\Pi_h^+\big\|_{\mathcal{H}\to\mathcal{H}} \leq C\,\|I-\Pi_h\|_{\mathcal{H}\to\mathcal{H}}\,. \tag{5.4}$$

Proof. The fact that Π_h^+ is well defined and bounded follows from Lemma 5.2 combined with the Lax–Milgram lemma [39], [44, Lemma 2.32], and the bound (5.4) then follows from Céa's lemma [8], [16, Theorem 13.1]. The fact that $\Pi_h^+w_h=w_h$ for all $w_h\in\mathcal{H}_h$ follows from the facts that (i) Π_h^+ is well-defined, and (ii) if $w_h\in\mathcal{H}_h$, then $\Pi_h^+w_h=w_h$ is a solution of (5.3). □

By the definitions of b^+ (5.2) and Π_h^+ (5.3),

$$\big\langle \mathcal{E}(I-\Pi_h^+)u, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0 \quad \text{for all } v_h\in\mathcal{H}_h\cap\operatorname{Ker}\mathcal{D}. \tag{5.5}$$

In the terminology of Remark 2.8, a consequence of (5.5) is that if u is discretely ϵ-divergence free, then so is Π_h^+u.

Lemma 5.4. *Given P satisfying Assumption 2.1, define Π_0 by (2.6), $\gamma_{\rm dv}(P)$ by (2.22), and Π_h^+ by (5.3). If w satisfies*

$$\big\langle \mathcal{E}w, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0 \quad \textit{for all } v_h\in\mathcal{H}_h\cap\operatorname{Ker}\mathcal{D},$$

then

$$\big\|\Pi_0\Pi_h^+w\big\|_{\mathcal{V}} \leq C\gamma_{\rm dv}(P)\,\|w\|_{\mathcal{H}}\,.$$

Proof. By (5.5),

$$\left\langle \mathcal{E}\Pi_h^+ w, v_h \right\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0 \quad \text{ for all } v_h \in \mathcal{H}_h \cap \operatorname{Ker}\mathcal{D}.$$

Therefore, by the definition of $\gamma_{\rm dv}$ (2.22) and Corollary 5.3,

$$\left\|\Pi_0\Pi_h^+ w\right\|_{\mathcal{V}} \le \gamma_{\rm dv}(P) \left\|\Pi_h^+ w\right\|_{\mathcal{H}} \le C\gamma_{\rm dv}(P) \left\|w\right\|_{\mathcal{H}}.$$

□

Proof of Lemma 5.1. Since $\Pi_h^+ u_h = u_h$ (by Corollary 5.3), $\Pi_0 : \mathcal{V} \to \mathcal{V}$ is bounded, and $I - \Pi_h^+$ satisfies (5.4),

$$\begin{aligned}
\|\Pi_0(u-u_h)\|_{\mathcal{V}}^2 &= \Big(\Pi_0(u-u_h), \Pi_0\Big((I-\Pi_h^+)u + \Pi_h^+(u-u_h)\Big)\Big)_{\mathcal{V}} \\
&\le \|\Pi_0(u-u_h)\|_{\mathcal{V}} \Big(C\left\|(I-\Pi_h)u\right\|_{\mathcal{H}} + \left\|\Pi_0\Pi_h^+(u-u_h)\right\|_{\mathcal{V}}\Big).
\end{aligned}$$

By (2.21) and Lemma 5.4,

$$\left\|\Pi_0\Pi_h^+(u-u_h)\right\|_{\mathcal{V}} \le C\gamma_{\rm dv}(P)\left\|u-u_h\right\|_{\mathcal{H}},$$

and the result then follows since $\|\Pi_0(u-u_h)\|_{\mathcal{H}} = \|\Pi_0(u-u_h)\|_{\mathcal{V}}$ by (2.8). □

6 Asymptotic quasi-optimality

As mentioned in §1.2, the following result, Lemma 6.1, is morally equivalent to the classic duality argument introduced in [49, 50]. The main abstract result of this paper (in the form of Lemma 8.3 below) provides a stronger result than Lemma 6.1, but the proof of Lemma 8.3 uses Lemma 6.1 applied to an auxiliary operator, $P^\#$; see Lemma 7.18 below.

Lemma 6.1 (Asymptotic quasi-optimality)**.** *If P satisfies Assumption 2.1 and $(P^*)^{-1}$ exists, then there exist $C_1, C_2, C_3 > 0$ such that if*

$$\gamma_{\rm dv}(P) \le C_1 \quad \text{and} \quad \left\|(I-\Pi_h)(P^*)^{-1}\Pi_1^*\right\|_{\mathcal{V}^*\to\mathcal{H}} \le C_2, \tag{6.1}$$

then u_h exists, is unique, and satisfies

$$\|u-u_h\|_{\mathcal{H}} \le C_3\left\|(I-\Pi_h)u\right\|_{\mathcal{H}}.$$

Lemma 6.1 combined with Lemma 4.2 with $m = 1$ gives the following corollary.

Corollary 6.2 (Asymptotic quasi-optimality under low regularity)**.** *If P satisfies Assumptions 2.1 and 2.3, the latter with $m = 1$, and $(P^*)^{-1}$ exists, then there exist $C_1, C_2, C_3 > 0$ such that if*

$$\gamma_{\rm dv}(P) \le C_1 \quad \text{and} \quad \|I-\Pi_h\|_{\mathcal{Z}^2\to\mathcal{H}}\left(1 + \left\|(P^*)^{-1}\Pi_1^*\right\|_{\mathcal{V}^*\to\mathcal{V}}\right) \le C_2,$$

then u_h exists, is unique, and satisfies

$$\|u-u_h\|_{\mathcal{H}} \le C_3\left\|(I-\Pi_h)u\right\|_{\mathcal{H}}.$$

Lemma 6.1 is an immediate consequence of the following two results.

Lemma 6.3 (Quasi-optimality of the Galerkin solution, modulo $\|\Pi_1(u-u_h)\|_{\mathcal{H}}$)**.** *Suppose that P satisfies Assumption 2.1. Given $u \in \mathcal{H}$, assume that the solution $u_h \in \mathcal{H}_h$ of (2.20) exists. Then there exists $C_1, C_2 > 0$ such that*

$$\big(1 - C_1\gamma_{\rm dv}(P)\big)\left\|u-u_h\right\|_{\mathcal{H}} \le C_2\Big(\left\|(I-\Pi_h)u\right\|_{\mathcal{H}} + \left\|\Pi_1(u-u_h)\right\|_{\mathcal{V}}\Big) \quad \text{for all } v \in \mathcal{H}. \tag{6.2}$$

Lemma 6.4. *Suppose that P satisfies Assumption 2.1. Given $u \in \mathcal{H}$, assume that the solution $u_h \in \mathcal{H}_h$ of (2.20) exists. Then there exists $C > 0$ such that*

$$\|\Pi_1(u - u_h)\|_{\mathcal{V}} \leq \big\|(I - \Pi_h)(P^*)^{-1}\Pi_1^*\|_{\mathcal{V}^*\to\mathcal{H}} \|u - u_h\|_{\mathcal{H}} .$$

The proof of Lemma 6.4 is short, and so we give it first.

Proof of Lemma 6.4. By the definition of ι (2.4), the definition of $(P^*)^{-1} : \mathcal{H}^* \to \mathcal{H}$, Galerkin orthogonality (2.20), and boundedness of $P : \mathcal{H} \to \mathcal{H}^*$,

$$\begin{aligned}
\|\Pi_1(u - u_h)\|_{\mathcal{V}}^2 &= \big\langle \Pi_1(u - u_h), \iota\Pi_1(u - u_h)\big\rangle_{\mathcal{V}\times\mathcal{V}^*}, \\
&= \big\langle u - u_h, \Pi_1^*\iota\Pi_1(u - u_h)\big\rangle_{\mathcal{V}\times\mathcal{V}^*}, \\
&= \big\langle P(u - u_h), (P^*)^{-1}\Pi_1^*\iota\Pi_1(u - u_h)\big\rangle_{\mathcal{H}^*\times\mathcal{H}}, \\
&= \big\langle P(u - u_h), (I - \Pi_h)(P^*)^{-1}\Pi_1^*\iota\Pi_1(u - u_h)\big\rangle_{\mathcal{H}^*\times\mathcal{H}}, \\
&\leq C \|u - u_h\|_{\mathcal{H}} \big\|(I - \Pi_h)(P^*)^{-1}\Pi_1^*\|_{\mathcal{V}^*\to\mathcal{H}} \|\Pi_1(u - u_h)\|_{\mathcal{V}} ,
\end{aligned}$$

and the result follows. □

Proof of Lemma 6.3. By the triangle inequality, (2.8), and (5.1),

$$\begin{aligned}
\|u - u_h\|_{\mathcal{H}} &\leq \|\Pi_0(u - u_h)\|_{\mathcal{V}} + \|\Pi_1(u - u_h)\|_{\mathcal{H}} \\
&\leq C\Big(\|(I - \Pi_h)u\|_{\mathcal{H}} + \gamma_{\rm dv}(P) \|u - u_h\|_{\mathcal{H}} \Big) + \|\Pi_1(u - u_h)\|_{\mathcal{H}} ;
\end{aligned}$$

i.e.,

$$\big(1 - C\gamma_{\rm dv}(P)\big) \|u - u_h\|_{\mathcal{H}} \leq C' \|(I - \Pi_h)u\|_{\mathcal{H}} + \|\Pi_1(u - u_h)\|_{\mathcal{H}} . \tag{6.3}$$

We claim that it is now sufficient to prove that, for all $\varepsilon > 0$,

$$\|\Pi_1(u - u_h)\|_{\mathcal{H}} \leq \varepsilon \|u - u_h\|_{\mathcal{H}} + C\varepsilon^{-1}\Big(\|(I - \Pi_h)u\|_{\mathcal{H}} + \|\Pi_1(u - u_h)\|_{\mathcal{V}} + \|\Pi_0(u - u_h)\|_{\mathcal{H}} \Big). \tag{6.4}$$

Indeed, inputting (6.4) into (6.3) and using again (5.1), we find (6.2).

We now prove (6.4). By the Gårding inequality (2.3),

$$\|\Pi_1(u - u_h)\|_{\mathcal{H}}^2 \leq \mathrm{Re}\,\big\langle P\Pi_1(u - u_h), \Pi_1(u - u_h)\big\rangle_{\mathcal{H}^*\times\mathcal{H}} + (1 + \|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*}) \|\Pi_1(u - u_h)\|_{\mathcal{V}}^2 . \tag{6.5}$$

Now, since $\Pi_0 : \mathcal{H} \to \mathrm{Ker}\,\mathcal{D}$ and $\mathrm{Ker}\,\mathcal{D} = \mathrm{Ker}\,\mathcal{D}^*$, for all $v \in \mathcal{H}$,

$$\begin{aligned}
&\mathrm{Re}\,\big\langle P\Pi_1 v, \Pi_1 v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \\
&\quad = \langle Pv, v\rangle_{\mathcal{H}^*\times\mathcal{H}} - \langle P\Pi_0 v, v\rangle_{\mathcal{H}^*\times\mathcal{H}} - \langle Pv, \Pi_0 v\rangle_{\mathcal{H}^*\times\mathcal{H}} + \langle P\Pi_0 v, \Pi_0 v\rangle_{\mathcal{H}^*\times\mathcal{H}} \\
&\quad = \langle Pv, v\rangle_{\mathcal{H}^*\times\mathcal{H}} + \langle \mathcal{E}\Pi_0 v, v\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathcal{E}v, \Pi_0 v\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle \mathcal{E}\Pi_0 v, \Pi_0 v\rangle_{\mathcal{V}^*\times\mathcal{V}}.
\end{aligned}$$

Therefore, by the boundedness of $\mathcal{E} : \mathcal{V} \to \mathcal{V}^*$ and the inequality

$$2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2, \qquad \text{for } a, b, \varepsilon > 0, \tag{6.6}$$

$$\mathrm{Re}\,\big\langle P\Pi_1 v, \Pi_1 v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \leq \mathrm{Re}\,\big\langle Pv, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} + \varepsilon^{-1} \|\Pi_0 v\|_{\mathcal{V}}^2 + \varepsilon \|v\|_{\mathcal{V}}^2 .$$

Applying this last inequality with $v = u - u_h$, combining with (6.5), and then using Galerkin orthogonality (2.20), we find that

$$\|\Pi_1(u - u_h)\|_{\mathcal{H}}^2 \leq \mathrm{Re}\,\big\langle P(u - u_h), (I - \Pi_h)u\big\rangle$$

$$+ C\Big(\varepsilon^{-1} \|\Pi_0(u-u_h)\|_{\mathcal{V}}^2 + \varepsilon \|u-u_h\|_{\mathcal{V}}^2 + \|\Pi_1(u-u_h)\|_{\mathcal{V}}^2\Big).$$

Therefore, by (6.6) and the boundedness of $P : \mathcal{H} \to \mathcal{H}^*$,

$$\|\Pi_1(u-u_h)\|_{\mathcal{H}}^2 \le \varepsilon \|u-u_h\|_{\mathcal{H}}^2 + C\Big(\varepsilon^{-1} \|(I-\Pi_h)u\|_{\mathcal{H}}^2 + \varepsilon^{-1} \|\Pi_0(u-u_h)\|_{\mathcal{V}}^2 + \|\Pi_1(u-u_h)\|_{\mathcal{V}}^2\Big). \quad (6.7)$$

By (2.8), this last inequality implies (6.4) and the proof is complete. □

7 Definition of the operator $P^{\#}$ and associated results

7.1 Identification of $\Pi_1\mathcal{V}$ with $\Pi_1^*\mathcal{V}^*$ and $(\Pi_1\mathcal{V})^*$

Since $\Pi_1\mathcal{V}$ is the kernel of the bounded operator $\Pi_0 : \mathcal{V} \to \mathcal{V}$, $\Pi_1\mathcal{V}$ is closed in $\mathcal{V}$, and thus $\Pi_1\mathcal{V}$ is a Hilbert space. We define

$$(u, v)_{\Pi_1\mathcal{V}} := (\Pi_1 u, \Pi_1 v)_{\mathcal{V}} \quad \text{for } u, v \in \Pi_1\mathcal{V}. \quad (7.1)$$

We now define the maps identifying $\Pi_1\mathcal{V}$ with $\Pi_1^*\mathcal{V}^*$ and $(\Pi_1\mathcal{V})^*$ and then prove that these maps are bijective (see Corollary 7.4 below). In particular, the rest of §7 crucially uses the fact that the identification of $\Pi_1\mathcal{V}$ with $\Pi_1^*\mathcal{V}^*$, denoted by η, is invertible.

Identification of $\Pi_1\mathcal{V}$ with $\Pi_1^*\mathcal{V}^*$. Let $\eta : \Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^*$ be the identification of $\Pi_1\mathcal{V}$ with $\Pi_1^*\mathcal{V}^*$ defined by

$$\langle \eta u, v\rangle_{\Pi_1^*\mathcal{V}^* \times \Pi_1\mathcal{V}} = (u, v)_{\Pi_1\mathcal{V}} \quad \text{for } u, v \in \Pi_1\mathcal{V}. \quad (7.2)$$

Lemma 7.1 (Properties of η).
(i) $\eta : \Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^*$ *is injective.*
(ii) $\eta = \Pi_1^*\iota\Pi_1$*, and this formula extends* η *to a map* $\mathcal{V} \to \mathcal{V}^*$.

Proof. (i) Suppose $\eta u = 0$ for some $u \in \Pi_1\mathcal{V}$. Then, by (7.2), $(u, v)_{\Pi_1\mathcal{V}} = 0$ for all $v \in \Pi_1\mathcal{V}$, so that $u = 0$.

(ii) By (7.2), (7.1), and the definition of the Riesz map $\iota : \mathcal{V} \to \mathcal{V}^*$ (2.4), for $u, v \in \Pi_1\mathcal{V}$,

$$\langle \eta u, v\rangle_{\Pi_1^*\mathcal{V}^* \times \Pi_1\mathcal{V}} = (\Pi_1 u, \Pi_1 v)_{\mathcal{V}} = \big\langle \iota\Pi_1 u, \Pi_1 v\big\rangle_{\mathcal{V}^* \times \mathcal{V}} = \big\langle \Pi_1^*\iota\Pi_1 u, v\big\rangle_{\mathcal{V}^* \times \mathcal{V}} = \big\langle \Pi_1^*\iota\Pi_1 u, v\big\rangle_{\Pi_1^*\mathcal{V}^* \times \Pi_1\mathcal{V}} \quad (7.3)$$

(where in the last step we treat $\Pi_1\mathcal{V}$ and $\Pi_1^*\mathcal{V}^*$ as subsets of $\mathcal{V}$ and $\mathcal{V}^*$, respectively). Therefore (7.3) shows that $\eta = \Pi_1^*\iota\Pi_1$ as a map $\Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^*$. Since $\eta = \Pi_1^*\eta\Pi_1$, the formula $\eta = \Pi_1^*\iota\Pi_1$ extends η to a map $\mathcal{V} \to \mathcal{V}^*$. □

Identification of $\Pi_1\mathcal{V}$ with $(\Pi_1\mathcal{V})^*$. Let $\widetilde{\eta} : \Pi_1\mathcal{V} \to (\Pi_1\mathcal{V})^*$ be defined by

$$\langle \widetilde{\eta} u, v\rangle_{(\Pi_1\mathcal{V})^* \times \Pi_1\mathcal{V}} = (u, v)_{\Pi_1\mathcal{V}} \quad \text{for } u, v \in \Pi_1\mathcal{V} \quad (7.4)$$

(compare to (7.2)). By the Riesz representation theorem, $\widetilde{\eta}$ is bijjective $\Pi_1\mathcal{V} \to (\Pi_1\mathcal{V})^*$.

Identification of $\Pi_1^*\mathcal{V}^*$ with $(\Pi_1\mathcal{V})^*$. Let $\rho : \Pi_1^*\mathcal{V}^* \to (\Pi_1\mathcal{V})^*$ be defined by: given $\phi \in \Pi_1^*\mathcal{V}^*$ (so that $\phi = \Pi_1^*\phi$),

$$\langle \rho\phi, v\rangle_{(\Pi_1\mathcal{V})^*\times\Pi_1\mathcal{V}} = \langle \Pi_1^*\phi, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \quad \text{for all } v \in \Pi_1\mathcal{V}. \tag{7.5}$$

Lemma 7.2. *$\rho : \Pi_1^*\mathcal{V}^* \to (\Pi_1\mathcal{V})^*$ is injective.*

Proof. If $\rho\phi = 0$, where $\phi = \Pi_1^*\phi$, then, by definition, $\langle \Pi_1^*\phi, v\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0$ for all $v \in \Pi_1\mathcal{V}$. Since $\Pi_0\Pi_1 = 0$, this last equality holds in fact for all $v \in \mathcal{V}$, so that $\phi = \Pi_1^*\phi = 0$ as an element of $\mathcal{V}^*$, and hence also as an element of $\Pi_1^*\mathcal{V}^*$. □

Lemma 7.3. *$\rho\eta = \widetilde{\eta}$ as maps $\Pi_1\mathcal{V} \to (\Pi_1\mathcal{V})^*$.*

Corollary 7.4. *$\eta : \Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^*$, $\widetilde{\eta} : \Pi_1\mathcal{V} \to (\Pi_1\mathcal{V})^*$, and $\rho : \Pi_1^*\mathcal{V}^* \to (\Pi_1\mathcal{V})^*$ are all bijective.*

Proof of Corollary 7.4. The bijectivity of $\widetilde{\eta} : \Pi_1\mathcal{V} \to (\Pi_1\mathcal{V})^*$ is a consequence of the Riesz representation theorem (as noted above). Since ρ and η are both injective and $\widetilde{\eta}$ is bijective, Lemma 7.3 implies that ρ and η are bijective. □

Proof of Lemma 7.3. By the definition of ρ (7.5), Part (ii) of Lemma 7.1, the definition of ι (2.4), (7.1), and (7.4), for all $u, v \in \Pi_1\mathcal{V}$,

$$\begin{aligned}\langle \rho\eta u, v\rangle_{(\Pi_1\mathcal{V})^*\times\Pi_1\mathcal{V}} = \langle \Pi_1^*\Pi_1^*\iota\Pi_1 u, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \langle \iota\Pi_1 u, \Pi_1 v\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &= (\Pi_1 u, \Pi_1 v)_{\mathcal{V}} \\ &= (u, v)_{\Pi_1\mathcal{V}} = \langle \widetilde{\eta} u, v\rangle_{(\Pi_1\mathcal{V})^*\times\Pi_1\mathcal{V}}.\end{aligned}$$

□

Having proved that η^{-1} exists, we now prove that η^{-1} and η are both self adjoint.

Lemma 7.5 (η and η^{-1} are both self-adjoint)**.** *With $\eta : \Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^*$ defined by (7.2),*

$$\langle \phi, v\rangle_{\Pi_1^*\mathcal{V}^*\times\Pi_1\mathcal{V}} = \langle \eta^{-1}\phi, \eta v\rangle_{\Pi_1\mathcal{V}\times\Pi_1^*\mathcal{V}^*} \quad \textit{for all } \phi \in \Pi_1^*\mathcal{V}^* \textit{ and } v \in \Pi_1\mathcal{V}.$$

Proof. Let $u := \eta^{-1}\phi \in \Pi_1\mathcal{V}$ (which exists by Corollary 7.4). Then, by two applications of (7.2),

$$\langle \phi, v\rangle_{\Pi_1^*\mathcal{V}^*\times\Pi_1\mathcal{V}} = \langle \eta u, v\rangle_{\Pi_1^*\mathcal{V}^*\times\Pi_1\mathcal{V}} = (u, v)_{\Pi_1\mathcal{V}} = \langle u, \eta v\rangle_{\Pi_1\mathcal{V}\times\Pi_1^*\mathcal{V}^*},$$

and the result follows. □

7.2 Definition of $P^\#$ and $(P^\#)^{-1}$

Recalling the matrix form of P (4.2), we define $\mathcal{P} : \Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$ by

$$\mathcal{P} := \mathrm{Re}\,\big(\mathcal{D}_{11} - \mathcal{E}_{11}\big). \tag{7.6}$$

By definition, if $v \in \Pi_1\mathcal{H}$ then $\langle \mathcal{P}v, v\rangle_{\mathcal{H}^*\times\mathcal{H}} = \mathrm{Re}\langle Pv, v\rangle_{\mathcal{H}^*\times\mathcal{H}}$. Therefore, by (2.3),

$$\big\langle \mathcal{P}v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \geq \|v\|_{\mathcal{H}}^2 - (1 + \|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*})\,\|v\|_{\mathcal{V}}^2 \quad \text{for all } v \in \Pi_1\mathcal{H}. \tag{7.7}$$

Theorem 7.6 (Friedrichs extension theorem)**.** *Suppose that V is a Hilbert space and H is dense in V. Suppose that $Q : H \times H \to \mathbb{C}$ is a sesquilinear form such that (i) $Q(u,v) = \overline{Q(v,u)}$ for all $u, v \in H$, (ii) there exists $C > 0$ such that*

$$Q(v,v) \geq -C\,\|v\|_V^2 \qquad \text{for all } v \in H,$$

and (iii) H is complete under the norm

$$|\!|\!|v|\!|\!| := \sqrt{Q(v,v) + (1+C)\,\|v\|_V^2}.$$

Then there exists a densely-defined, self-adjoint operator $\mathcal{Q} : V \to V^*$ such that

$$Q(u,v) = \langle \mathcal{Q}u, v\rangle_{V^*\times V} \qquad \text{for all } u \in \mathrm{Dom}(\mathcal{Q}) \text{ and } v \in V,$$

where the domain of $\mathcal{Q}$, $\mathrm{Dom}(\mathcal{Q})$, is defined by

$$\mathrm{Dom}(\mathcal{Q}) := \Big\{ u \in H \,:\, |Q(u,v)| \leq C_u \,\|v\|_V \qquad \text{for all } v \in V \Big\}.$$

References for the proof. See, e.g., [58, Theorem VIII.15, Page 278], [33, Theorem 12.24, Page 360] (with [26] the original paper). □

Corollary 7.7. *$\mathcal{P}$ defined by (7.6) extends to a densely-defined, self-adjoint operator $\Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^*$ with*

$$\big\langle \mathcal{P}u, v\big\rangle_{\Pi_1^*\mathcal{H}^*\times\Pi_1\mathcal{H}} = \big\langle \mathcal{P}u, v\big\rangle_{\Pi_1^*\mathcal{V}^*\times\Pi_1^*\mathcal{V}} \qquad \text{for all } u \in \mathrm{Dom}(\mathcal{P}) \subset \Pi_1\mathcal{H} \text{ and } v \in \Pi_1\mathcal{V}. \quad (7.8)$$

Furthermore $\eta^{-1}\mathcal{P}$ is a densely-defined, self-adjoint operator $\Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$, with its spectrum bounded below by $-\|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*}$.

Proof. We apply Theorem 7.6 with $H = \Pi_1\mathcal{H}$, $V = \Pi_1\mathcal{V}$, and $Q(u,v) = \langle \mathcal{P}u, v\rangle_{\mathcal{H}^*\times\mathcal{H}}$. We also identify $(\Pi_1\mathcal{H})^*$ and $\Pi_1^*\mathcal{H}^*$ (so that $\mathcal{P} : \Pi_1\mathcal{H} \to (\Pi_1\mathcal{H})^*$, i.e., $\mathcal{P} : V \to V^*$); this identification is analogous to the identification between $(\Pi_1\mathcal{V})^*$ and $\Pi_1^*\mathcal{V}^*$ described in §7.1. However, we do not introduce any notation for it since this identification is only used inside this proof and inside the proof of Lemma 7.10 (when using the the Lax–Milgram lemma). The proof of Lemma 7.11 uses the Lax–Milgram lemma with the analogous identification of $\Pi_0^*\mathcal{H}^*$ and $(\Pi_0\mathcal{H})^*$.

We now check the assumptions of Theorem 7.6. Since $\mathcal{H}$ is dense in $\mathcal{V}$, and $\Pi_1 : \mathcal{V} \to \mathcal{V}$ is bounded, $H = \Pi_1\mathcal{H}$ is dense in $V = \Pi_1\mathcal{V}$. By its definition (7.6), $\mathcal{P} : \Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$ is self adjoint; i.e., Assumption (i) of Theorem 7.6 is satisifed. The Gårding inequality (7.7) then implies that Assumptions (ii) and (iii) of Theorem 7.6 are satisfied with $C = C_{\mathcal{E}}$.

We denote the extension $\mathcal{Q}$ given by Theorem 7.6 also by $\mathcal{P}$, so that we extend $\mathcal{P}$ to a densely-defined, self-adjoint operator $\Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^*$.

By (7.2), the self-adjointness of $\mathcal{P} : \Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^*$, and (7.2) again, for all $u \in \mathrm{Dom}(\eta^{-1}\mathcal{P})$,

$$\big(\eta^{-1}\mathcal{P}u, v\big)_{\Pi_1\mathcal{V}} = \big\langle \mathcal{P}u, v\big\rangle_{\Pi_1^*\mathcal{V}^*\times\Pi_1\mathcal{V}} = \big\langle u, \mathcal{P}v\big\rangle_{\Pi_1\mathcal{V}\times\Pi_1^*\mathcal{V}^*} = \big(u, \eta^{-1}\mathcal{P}v\big)_{\Pi_1\mathcal{V}};$$

thus $\eta^{-1}\mathcal{P} : \Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$ is a densely-defined self-adjoint operator. Finally, by (7.7), for all $v \in \mathrm{Dom}(\eta^{-1}\mathcal{P}) \subset \Pi_1\mathcal{V}$,

$$\big(\eta^{-1}\mathcal{P}v, v\big)_{\Pi_1\mathcal{V}} \geq \|v\|_{\mathcal{H}}^2 - (1 + \|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*})\,\|v\|_{\mathcal{V}}^2 \geq -\|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*}\,\|v\|_{\mathcal{V}}^2\,.$$

For all $\varepsilon > 0$, $\eta^{-1}\mathcal{P} + \|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*} + \varepsilon : \mathrm{Dom}(\eta^{-1}\mathcal{P}) \to \Pi_1\mathcal{V}$ is then invertible by a variant of the Lax–Milgram lemma for densely-defined operators; see, e.g., [33, Theorem 12.18] or the proof of [58, Theorem VIII.15]. Thus the spectrum of $\eta^{-1}\mathcal{P}$ (i.e., the set of λ such that $\eta^{-1}\mathcal{P} - \lambda : \mathrm{Dom}(\eta^{-1}\mathcal{P}) \to \Pi_1\mathcal{V}$ is not invertible) is bounded below by $-\|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*}$. □

We now use the functional calculus for $\eta^{-1}\mathcal{P} : \Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$ to define

$$S := \psi\big(\eta^{-1}\mathcal{P}\big), \tag{7.9}$$

where $\psi \in C^\infty_{\rm comp}(\mathbb{R};[0,\infty))$ is such that

$$x + \psi^2(x) \geq 1 \quad \text{ for } \quad x \geq -\|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*}. \tag{7.10}$$

We recap the following results about the functional calculus.

Theorem 7.8 (Functional-calculus results)**.** *Let $\mathcal{L}$ be a densely-defined, self-adjoint operator on a Hilbert space V, and let $\sigma(\mathcal{L})$ denote its spectrum.*

(i) If $\psi \in L^\infty(\mathbb{R};\mathbb{R})$ then $\psi(\mathcal{L}) : V \to V$ is self-adjoint, in the sense that $(\psi(\mathcal{L})u, v)_V = (u, \psi(L)v)_V$ for all $u, v \in V$.

(ii) If $\psi \in L^\infty(\mathbb{R};\mathbb{C})$ then $\|\psi(\mathcal{L})\|_{V\to V} \leq \sup_{\lambda\in\sigma(\mathcal{L})} |\psi(\lambda)|$.

(iii) If $\psi \in L^\infty(\mathbb{R};\mathbb{R})$ is such that $\psi \geq c > 0$ on $\sigma(\mathcal{L})$, then

$$\big(\psi(\mathcal{L})v, v\big)_V \geq c\,\|v\|^2_V \quad \textit{for all } v \in V.$$

References for the proof. See, e.g., [58, §VIIII.3, Page 259]. □

Lemma 7.9 (Properties of S inherited from the functional calculus)**.** *If $S := \psi(\eta^{-1}\mathcal{P})$ then*

(a) $S : \Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$.

(b) $\eta S = S^*\eta$, where $S^* : \Pi_1^*\mathcal{V}^* \to \Pi_1\mathcal{V}$.

(c) Given $m \in \mathbb{Z}^+$, there exists $C > 0$ such that

$$\big\|\Pi_1(\eta^{-1}\mathcal{P})^m\psi(\eta^{-1}\mathcal{P})\Pi_1\big\|_{\mathcal{V}\to\mathcal{V}} \leq C.$$

Proof. Since ψ is real valued, by Part (i) of Theorem 7.8, $S : \Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$ satisfies

$$(Su, v)_{\Pi_1\mathcal{V}} = (u, Sv)_{\Pi_1\mathcal{V}} \quad \text{ for all } u, v \in \Pi_1\mathcal{V}.$$

Therefore, by the definition of η (7.2),

$$\langle \eta Su, v\rangle_{\Pi_1^*\mathcal{V}^*\times\Pi_1\mathcal{V}} = \langle \eta u, Sv\rangle_{\Pi_1^*\mathcal{V}^*\times\Pi_1\mathcal{V}},$$

so that Part (b) follows.

Finally, since ψ has compact support, the function $t \mapsto t^m\psi(t)$ is bounded for all $m \geq 0$; Part (c) then follows by Part (ii) of Theorem 7.8. □

Lemma 7.10. *With S defined by (7.9), $\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2 : \Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$ is continuous,*

$$\operatorname{Re}\big\langle\big(\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2\big)v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \geq C\,\|v\|^2_{\mathcal{H}} \quad \textit{for all } v \in \Pi_1\mathcal{H}, \tag{7.11}$$

and thus $\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2 : \Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$ is invertible.

Proof. Since $S^2 : \Pi_1\mathcal{H} \subset \Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$ and $\eta : \Pi_1\mathcal{V} \to \Pi_1^*\mathcal{V}^* \subset \Pi_1^*\mathcal{H}^*$ are continuous, $\eta S^2 : \Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$ is continuous. Since $\mathcal{D}_{11}$ and $\mathcal{E}_{11}$ are continuous $\Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$ by assumption, the continuity result follows.

For the coercivity, by the definition of S (7.9),

$$\operatorname{Re}\big\langle\big(\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2\big)v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} = \big\langle\big(\mathcal{P} + \eta\psi^2(\eta^{-1}\mathcal{P})\big)v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}}.$$

By (7.8) and the fact that η^{-1} is the identification map $\Pi_1^*\mathcal{V}^* \to \Pi_1\mathcal{V}$,

$$\langle \mathcal{P}v, v\rangle_{\mathcal{H}^*\times\mathcal{H}} = \langle \mathcal{P}v, v\rangle_{\mathcal{V}^*\times\mathcal{V}} = \big(\eta^{-1}\mathcal{P}v, v\big)_{\Pi_1\mathcal{V}} \quad \text{for all } v \in \mathrm{Dom}(\mathcal{P}) \subset \Pi_1\mathcal{V}.$$

Therefore, by the inequality (7.10) and Part (iii) of Theorem 7.8, for all $v \in \mathrm{Dom}(\mathcal{P})$,

$$\mathrm{Re}\,\big\langle \big(\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2\big)v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} = \big(\big(\eta^{-1}\mathcal{P} + \psi^2(\eta^{-1}\mathcal{P})\big)v, v\big)_{\Pi_1\mathcal{V}} \geq \|v\|_{\mathcal{V}}^2. \tag{7.12}$$

Since $\mathrm{Dom}(\mathcal{P})$ is dense in $\Pi_1\mathcal{V}$ (since $\mathcal{P}$ is densely-defined by Corollary 7.7), (7.12) holds for all $v \in \Pi_1\mathcal{H}$.

We now use the Gårding inequality (7.7) to replace the $\mathcal{V}$ norm on the right-hand side of (7.12) by a $\mathcal{H}$ norm and obtain (7.11). Since $\psi^2 \geq 0$, $S^2 \geq 0$. Using this, along with (7.7) and (7.12), we find that, for all $\varepsilon > 0$ and $v \in \Pi_1\mathcal{H}$,

$$\begin{aligned}
&\mathrm{Re}\,\big\langle \big(\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2\big)v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \\
&\quad\geq \varepsilon\,\mathrm{Re}\,\big\langle \big(\mathcal{D}_{11} - \mathcal{E}_{11}\big)v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} + (1-\varepsilon)\,\mathrm{Re}\,\big\langle \big(\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2\big)v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \\
&\quad\geq \varepsilon\Big(\|v\|_{\mathcal{H}}^2 - (1 + \|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*})\,\|v\|_{\mathcal{V}}^2 \Big) + (1-\varepsilon)\,\|v\|_{\mathcal{V}}^2
\end{aligned}$$

so that, choosing $0 < \varepsilon \leq (2 + \|\mathcal{E}\|_{\mathcal{V}\to\mathcal{V}^*})^{-1}$, we see that $\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2$ is coercive $\Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$.

Invertibility of $\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2 : \Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$ then follows from the Lax–Milgram lemma (where, as in the application of Theorem 7.6 we identify $(\Pi_1\mathcal{H})^*$ and $\Pi_1^*\mathcal{H}^*$). □

Let

$$P^\# = \begin{pmatrix} -\mathcal{E}_{00} & 0 \\ -\mathcal{E}_{10} & \mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2 \end{pmatrix} = P + \Pi_1^*\eta S^2\Pi_1. \tag{7.13}$$

With this definition, we record for later that (7.11) is equivalent to

$$\mathrm{Re}\,\big\langle P^\#\Pi_1 v, \Pi_1 v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \geq C\,\|\Pi_1 v\|_{\mathcal{H}}^2 \quad \text{for all } v \in \mathcal{H}. \tag{7.14}$$

Lemma 7.11. *$(P^\#)^{-1} : \mathcal{H}^* \to \mathcal{H}$ is well defined with $\|(P^\#)^{-1}\|_{\mathcal{H}^*\to\mathcal{H}} \leq C$.*

Proof. By the matrix form of $P^\#$ (7.13) and the fact that $\mathcal{D}_{11} - \mathcal{E}_{11} + \eta S^2 : \Pi_1\mathcal{H} \to \Pi_1^*\mathcal{H}^*$ is invertible, the result follows if $\mathcal{E}_{00} : \Pi_0\mathcal{H} \to \Pi_0^*\mathcal{H}^*$ is invertible. We claim that $\mathcal{E}_{00} : \Pi_0\mathcal{H} \to \Pi_0^*\mathcal{H}^*$ satisfies

$$\mathrm{Re}\,\big\langle \mathcal{E}_{00}v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \geq C_{\mathcal{E}}\,\|v\|_{\mathcal{H}}^2 \quad \text{for all } v \in \Pi_0\mathcal{H}, \tag{7.15}$$

from which the result follows by the Lax–Milgram lemma (where we identify $\Pi_0^*\mathcal{H}^*$ and $(\Pi_0\mathcal{H})^*$).

By (in this order) the definition $\mathcal{E}_{00} := \Pi_0^*\mathcal{E}\Pi_0$ (4.1), the inclusion $\Pi_0\mathcal{H} \subset \mathcal{V}$ (by (2.8)), the boundedness of $\mathcal{E} : \mathcal{V} \to \mathcal{V}^*$, the coercivity of $\mathcal{E}$ (2.2), and (2.8), for all $v \in \mathcal{H}$,

$$\begin{aligned}
\mathrm{Re}\,\big\langle \mathcal{E}_{00}\Pi_0 v, \Pi_0 v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} = \mathrm{Re}\,\big\langle \mathcal{E}\Pi_0 v, \Pi_0 v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} &= \mathrm{Re}\,\big\langle \mathcal{E}\Pi_0 v, \Pi_0 v\big\rangle_{\mathcal{V}^*\times\mathcal{V}} \\
&\geq C_{\mathcal{E}}\,\|\Pi_0 v\|_{\mathcal{V}}^2 = C_{\mathcal{E}}\,\|\Pi_0 v\|_{\mathcal{H}}^2;
\end{aligned}$$

i.e., (7.15) holds and the proof is complete. □

Lemma 7.12. *$P^\# : \mathcal{H} \to \mathcal{H}^*$ satisfies a Gårding inequality; i.e., there exists $C_1, C_2 > 0$ such that*

$$\mathrm{Re}\,\big\langle P^\# v, v\big\rangle_{\mathcal{H}^*\times\mathcal{H}} \geq C_1\,\|v\|_{\mathcal{H}}^2 - C_2\,\|v\|_{\mathcal{V}}^2 \quad \textit{for all } v \in \mathcal{H}. \tag{7.16}$$

Proof. We first claim that it is sufficient to prove that there exist $C_1', C_2' > 0$ such that, for all $v \in \mathcal{H}$,

$$\operatorname{Re}\left\langle P^{\#} v, v\right\rangle_{\mathcal{H}^* \times \mathcal{H}} \geq C_1' \left\|\Pi_1 v\right\|_{\mathcal{H}}^2 - C_2' \left\|\Pi_0 v\right\|_{\mathcal{V}}^2 \quad \text{for all } v \in \mathcal{H}. \tag{7.17}$$

Indeed, by (2.8) and (6.6),

$$\|v\|_{\mathcal{H}}^2 \leq \left(\|\Pi_1 u\|_{\mathcal{H}} + \|\Pi_0 u\|_{\mathcal{V}}\right)^2 \leq (1+\varepsilon)\|\Pi_1 u\|_{\mathcal{H}}^2 + (1+\varepsilon^{-1})\|\Pi_0 u\|_{\mathcal{V}}^2,$$

for all $\varepsilon > 0$, so that, if (7.17) holds, then

$$\operatorname{Re}\left\langle P^{\#} v, v\right\rangle_{\mathcal{H}^* \times \mathcal{H}} \geq C_1'(1+\varepsilon)^{-1}\|v\|_{\mathcal{H}}^2 - \left[C_1'(1+\varepsilon^{-1})(1+\varepsilon)^{-1} + C_2'\right]\|\Pi_0 v\|_{\mathcal{V}}^2,$$

and (7.16) follows since $\Pi_0 : \mathcal{V} \to \mathcal{V}$ is bounded.

We therefore now prove (7.17). By the coercivity of $P^{\#}$ on $\Pi_1\mathcal{H}$ (7.14), the boundedness of $P^{\#} : \mathcal{H} \to \mathcal{H}^*$, and (2.8),

$$\begin{aligned}
\operatorname{Re}\langle P^{\#} v, v\rangle_{\mathcal{H}^* \times \mathcal{H}} &= \operatorname{Re}\left\langle P^{\#}\Pi_1 v, \Pi_1 v\right\rangle_{\mathcal{H}^* \times \mathcal{H}} + \operatorname{Re}\left\langle P^{\#}\Pi_0 v, \Pi_1 v\right\rangle_{\mathcal{H}^* \times \mathcal{H}} \\
&\qquad + \operatorname{Re}\left\langle P^{\#}\Pi_1 v, \Pi_0 v\right\rangle_{\mathcal{H}^* \times \mathcal{H}} + \operatorname{Re}\left\langle P^{\#}\Pi_0 v, \Pi_0 v\right\rangle_{\mathcal{H}^* \times \mathcal{H}}, \\
&\geq C_3 \|\Pi_1 v\|_{\mathcal{H}}^2 - C_4 \|\Pi_0 v\|_{\mathcal{H}} \|\Pi_1 v\|_{\mathcal{H}} - C_5 \|\Pi_0 v\|_{\mathcal{H}}^2, \\
&= C_3 \|\Pi_1 v\|_{\mathcal{H}}^2 - C_4 \|\Pi_0 v\|_{\mathcal{V}} \|\Pi_1 v\|_{\mathcal{H}} - C_5 \|\Pi_0 v\|_{\mathcal{V}}^2,
\end{aligned}$$

and (7.17) follows from the inequality (6.6). □

7.3 $S = \psi(\eta^{-1}\mathcal{P})$ increases regularity

The main result of this subsection is the following.

Lemma 7.13 (S increases regularity)**.** *Suppose that Assumption 2.3 holds for some $m \in \mathbb{Z}^+$ and spaces $\mathcal{Z}^j, j = 1, \ldots, m+1$. Then there exists $C > 0$ such that*

$$\|S\|_{\Pi_1\mathcal{V} \to \Pi_1\mathcal{Z}^j} \leq C \quad \text{for } j = 0, \ldots, m-1.$$

To prove Lemma 7.13, we first combine the regularity assumptions (2.10) and (2.11).

Lemma 7.14. *Suppose that Assumption 2.3 holds for some $m \in \mathbb{Z}^+$ and spaces $\mathcal{Z}^j, j = 1, \ldots, m+1$. Then there exists $C > 0$ such that for $j = 2, \ldots, m+1$,*

$$\|\Pi_1 u\|_{\mathcal{Z}^j} \leq C\Big(\|\Pi_1 u\|_{\mathcal{Z}^{j-2}} + \left\|\Pi_1 \eta^{-1}\mathcal{P}\Pi_1 u\right\|_{\mathcal{Z}^{j-2}}\Big) \quad \text{for all } u \in \mathcal{H}.$$

To prove Lemma 7.14 we use the following result.

Lemma 7.15. *Suppose that Assumptions 2.1 and 2.3 hold, the latter for some $m \in \mathbb{Z}^+$ and spaces $\mathcal{Z}^j, j = 1, \ldots, m+1$. Suppose that $\phi \in \Pi_1^*\mathcal{H}^*$ and, given $j \geq 2$, there exists $C_\phi < \infty$ such that*

$$\sup_{v \in \Pi_1\mathcal{H}, \|\iota v\|_{(\mathcal{Z}^{j-2})^*} = 1} \left|\langle \phi, v\rangle_{\mathcal{H}^* \times \mathcal{H}}\right| = C_\phi.$$

Then $\eta^{-1}\phi \in \Pi_1\mathcal{V} \cap \mathcal{Z}^{j-2}$ and $\|\eta^{-1}\phi\|_{\mathcal{Z}^{j-2}} = C_\phi$.

Proof. Since $\|v\|_{\mathcal{V}} = \|\iota v\|_{\mathcal{V}^*}$ (by (2.4)) and $\|\iota v\|_{\mathcal{V}^*} \geq \|\iota v\|_{(\mathcal{Z}^{j-2})^*}$ (since $\mathcal{Z}^{j-2} \subset \mathcal{V}$ and thus $\mathcal{V}^* \subset (\mathcal{Z}^{j-2})^*$).

$$\sup_{v\in\mathcal{H},\|v\|_{\mathcal{V}}=1} \left|\langle \phi, v\rangle_{\mathcal{H}^*\times\mathcal{H}}\right| \leq C_\phi.$$

Since $\mathcal{H}$ is dense in $\mathcal{V}$, the last inequality implies that $\phi \in \mathcal{V}^*$, and thus

$$\langle \phi, v\rangle_{\mathcal{H}^*\times\mathcal{H}} = \langle \phi, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \quad \text{for all } v \in \mathcal{H} \subset \mathcal{V}. \tag{7.18}$$

Since $\phi = \Pi_1^*\phi$, $\phi \in \Pi_1^*\mathcal{V}^*$, and thus $\eta^{-1}\phi \in \Pi_1\mathcal{V}$ (since $\eta^{-1} : \Pi_1^*\mathcal{V}^* \to \Pi_1\mathcal{V}$ by Corollary 7.4). Furthermore, by (7.18),

$$\sup_{v\in\Pi_1\mathcal{H},\|\iota v\|_{(\mathcal{Z}^{j-2})^*}=1} \left|\langle \phi, v\rangle_{\mathcal{V}^*\times\mathcal{V}}\right| = \sup_{v\in\Pi_1\mathcal{H},\|\iota v\|_{(\mathcal{Z}^{j-2})^*}=1} \left|\langle \phi, v\rangle_{\mathcal{H}^*\times\mathcal{H}}\right| = C_\phi. \tag{7.19}$$

By (in this order) the facts that $\phi \in \Pi_1^*\mathcal{V}^*$ and $v \in \Pi_1\mathcal{V}$, Lemma 7.5, the fact that $\eta = \Pi_1^*\iota\Pi_1$, and the fact that $\Pi_1\eta^{-1} = \eta^{-1}$ (since $\eta^{-1} : \Pi_1^*\mathcal{V}^* \to \Pi_1\mathcal{V}$),

$$\begin{aligned}\left|\langle \phi, v\rangle_{\mathcal{V}^*\times\mathcal{V}}\right| &= \left|\langle \phi, v\rangle_{\Pi_1^*\mathcal{V}^*\times\Pi_1\mathcal{V}}\right| = \left|\langle \eta^{-1}\phi, \eta v\rangle_{\Pi_1\mathcal{V}\times\Pi_1^*\mathcal{V}^*}\right| \\ &= \left|\langle \eta^{-1}\phi, \iota v\rangle_{\Pi_1\mathcal{V}\times\Pi_1^*\mathcal{V}^*}\right| \leq \left\|\eta^{-1}\phi\right\|_{\mathcal{Z}^{j-2}} \|\iota v\|_{(\mathcal{Z}^{j-2})^*}.\end{aligned} \tag{7.20}$$

Since $\mathcal{H}$ is dense in $\mathcal{V}$ (by Assumption 2.1) and $\mathcal{V}$ is dense in $(\mathcal{Z}^{j-2})^*$ (by Assumption 2.3), $\mathcal{H}$ is dense in $(\mathcal{Z}^{j-2})^*$. The result then follows from combining this density with (7.19) and (7.20). □

Proof of Lemma 7.14. In preparation for applying (2.10) with $\mathsf{D} = \operatorname{Re}\mathcal{D}$, observe that, by (4.2),

$$\left\langle \operatorname{Re}\mathcal{D}\Pi_1 u, v\right\rangle_{\mathcal{H}^*\times\mathcal{H}} = \left\langle \Pi_1^*(\operatorname{Re}\mathcal{D}_{11})\Pi_1 u, v\right\rangle_{\mathcal{H}^*\times\mathcal{H}} = \left\langle \Pi_1^*(\mathcal{P} + \operatorname{Re}\mathcal{E}_{11})\Pi_1 u, v\right\rangle_{\mathcal{H}^*\times\mathcal{H}} \tag{7.21}$$

Therefore, by (7.21) and Lemma 7.15,

$$\begin{aligned}&\sup_{v\in\Pi_1\mathcal{H},\|\iota v\|_{(\mathcal{Z}^{j-2})^*}=1} \left|\left\langle \operatorname{Re}\mathcal{D}\Pi_1 u, v\right\rangle_{\mathcal{H}^*\times\mathcal{H}}\right| \\ &\qquad= \sup_{v\in\Pi_1\mathcal{H},\|\iota v\|_{(\mathcal{Z}^{j-2})^*}=1} \left|\left\langle \Pi_1^*(\mathcal{P} + \operatorname{Re}\mathcal{E}_{11})\Pi_1 u, v\right\rangle_{\mathcal{H}^*\times\mathcal{H}}\right| \\ &\qquad= \left\|\eta^{-1}\Pi_1^*(\mathcal{P} + \operatorname{Re}\mathcal{E}_{11})\Pi_1 u\right\|_{\mathcal{Z}^{j-2}}.\end{aligned}$$

Therefore, by (2.10) with $\mathsf{D} = \operatorname{Re}\mathcal{D}$,

$$\|\Pi_1 u\|_{\mathcal{Z}^j} \leq C\Big(\|\Pi_1 u\|_{\mathcal{Z}^{j-2}} + \left\|\eta^{-1}\Pi_1^*(\mathcal{P} + \operatorname{Re}\mathcal{E}_{11})\Pi_1 u\right\|_{\mathcal{Z}^{j-2}}\Big). \tag{7.22}$$

Now

$$\begin{aligned}\left\|\eta^{-1}\Pi_1^*\operatorname{Re}\mathcal{E}_{11}\Pi_1 u\right\|_{\mathcal{Z}^{j-2}} &\leq \left\|\eta^{-1}\Pi_1^*\iota\right\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^{j-2}} \left\|\iota^{-1}\operatorname{Re}\mathcal{E}_{11}\right\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^{j-2}} \|\Pi_1 u\|_{\mathcal{Z}^{j-2}} \\ &\leq C\left\|\eta^{-1}\Pi_1^*\iota\right\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^{j-2}} \|\Pi_1 u\|_{\mathcal{Z}^{j-2}}\end{aligned} \tag{7.23}$$

by (2.11) with $\mathsf{E} = \operatorname{Re}\mathcal{E}$. By Part (ii) of Lemma 7.1, $\eta^{-1} : \Pi_1^*\mathcal{V}^* \to \Pi_1\mathcal{V}$ is given by $\eta^{-1} = \Pi_1\iota^{-1}\Pi_1^*$ (since the inverse of the inclusion map is the projection map and vice versa). Therefore, by (3.10) and (3.11),

$$\left\|\eta^{-1}\Pi_1^*\iota\right\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^{j-2}} = \left\|\Pi_1\iota^{-1}\Pi_1^*\iota\right\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^{j-2}} \leq C,$$

and combining this with (7.23) we obtain that

$$\left\|\eta^{-1}\Pi_1^* \operatorname{Re}\mathcal{E}_{11}\Pi_1 u\right\|_{\mathcal{Z}^{j-2}} \leq C \left\|\Pi_1 u\right\|_{\mathcal{Z}^{j-2}}. \tag{7.24}$$

Now $\eta^{-1} = \Pi_1\eta^{-1} = \Pi_1\eta^{-1}\Pi_1^*$ (either by the formula for η^{-1} above, or just the fact that $\eta^{-1} : \Pi_1^*\mathcal{V}^* \to \Pi_1\mathcal{V}$), so that

$$\eta^{-1}\Pi_1^*\mathcal{P}\Pi_1 = \Pi_1\eta^{-1}\Pi_1^*\mathcal{P}\Pi_1 = \Pi_1\eta^{-1}\mathcal{P}\Pi_1; \tag{7.25}$$

the result then follows from combining (7.25) with (7.22) and (7.24). □

The final result we need to prove Lemma 7.13 is the following.

Lemma 7.16. *$S : \mathcal{V} \to \mathcal{H}$.*

Proof. By its definition (7.9), $S := \psi(\eta^{-1}\mathcal{P}) : \Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$. Therefore, by the Gårding inequality (7.7), it is sufficient to prove that

$$\left|\left\langle \mathcal{P}\psi(\eta^{-1}\mathcal{P})v, \psi(\eta^{-1}\mathcal{P})v\right\rangle_{\mathcal{H}^*\times\mathcal{H}}\right| \leq C \left\|v\right\|_{\mathcal{V}}^2 \quad \text{for all } v \in \Pi_1\mathcal{V}. \tag{7.26}$$

By (7.8) and the fact that η^{-1} is the identification $\Pi_1^*\mathcal{V}^* \to \Pi_1\mathcal{V}$,

$$\begin{aligned}\left\langle \mathcal{P}\psi(\eta^{-1}\mathcal{P})v, \psi(\eta^{-1}\mathcal{P})v\right\rangle_{\mathcal{H}^*\times\mathcal{H}} &= \left\langle \mathcal{P}\psi(\eta^{-1}\mathcal{P})v, \psi(\eta^{-1}\mathcal{P})v\right\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &= \left(\eta^{-1}\mathcal{P}\psi(\eta^{-1}\mathcal{P})v, \psi(\eta^{-1}\mathcal{P})v\right)_{\mathcal{V}}.\end{aligned}$$

The bound (7.26) then follows from Lemma 7.9. □

Proof of Lemma 7.13. We apply Lemma 7.14 with $u = \psi(\eta^{-1}\mathcal{P})\Pi_1 v$ for arbitrary $v \in \mathcal{V}$; observe that this is allowed since $u \in \mathcal{H}$ by Lemma 7.16. Since $\Pi_1\psi(\eta^{-1}\mathcal{P}) = \psi(\eta^{-1}\mathcal{P})$ (since $\psi(\eta^{-1}\mathcal{P})$ is defined using the functional calculus on $\Pi_1\mathcal{V}$), this application of Lemma 7.14 implies that

$$\left\|\Pi_1\psi(\eta^{-1}\mathcal{P})\Pi_1\right\|_{\mathcal{V}\to\mathcal{Z}^j} \leq C\Big(\left\|\Pi_1\psi(\eta^{-1}\mathcal{P})\Pi_1\right\|_{\mathcal{V}\to\mathcal{Z}^{j-2}} + \left\|\Pi_1\eta^{-1}\mathcal{P}\psi(\eta^{-1}\mathcal{P})\Pi_1\right\|_{\mathcal{V}\to\mathcal{Z}^{j-2}}\Big). \tag{7.27}$$

We now apply Lemma 7.14 with $u = (\eta^{-1}\mathcal{P})^m\psi(\eta^{-1}\mathcal{P})\Pi_1 v$ for arbitrary $v \in \mathcal{V}$. The proof that this $u \in \mathcal{H}$ is very similar to the proof of Lemma 7.16, using Lemma 7.9 – the key points are that (i) any compactly supported function of $\eta^{-1}\mathcal{P}$ is bounded $\Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$, and (ii) the $\mathcal{H}$ norm essentially just adds another power of $\eta^{-1}\mathcal{P}$.

Lemma 7.14 and the fact that $\Pi_1\eta^{-1}\mathcal{P} = \eta^{-1}\mathcal{P}$ (since $\eta^{-1}\mathcal{P} : \Pi_1\mathcal{V} \to \Pi_1\mathcal{V}$) therefore imply that

$$\begin{aligned}&\left\|\Pi_1(\eta^{-1}\mathcal{P})^m\psi(\eta^{-1}\mathcal{P})\Pi_1\right\|_{\mathcal{V}\to\mathcal{Z}^j} \\ &\quad\leq C\Big(\left\|\Pi_1(\eta^{-1}\mathcal{P})^m\psi(\eta^{-1}\mathcal{P})\Pi_1\right\|_{\mathcal{V}\to\mathcal{Z}^{j-2}} + \left\|\Pi_1(\eta^{-1}\mathcal{P})^{m+1}\psi(\eta^{-1}\mathcal{P})\Pi_1\right\|_{\mathcal{V}\to\mathcal{Z}^{j-2}}\Big).\end{aligned} \tag{7.28}$$

The combination of (7.27) and (7.28) implies that

$$\left\|\Pi_1\psi(\eta^{-1}\mathcal{P})\Pi_1\right\|_{\mathcal{V}\to\mathcal{Z}^{m-1}} \leq C_m \sum_{j=0}^{\lceil (m-1)/2\rceil} \left\|\Pi_1(\eta^{-1}\mathcal{P})^j\psi(\mathcal{P})\Pi_1\right\|_{1\mathcal{V}\to\mathcal{V}},$$

and the result then follows from Lemma 7.9. □

7.4 Regularity of $(P^{\#})^{-1}\Pi_1^*$

Lemma 7.17. *Suppose that Assumptions 2.1 and 2.3 hold, the latter for some $m \in \mathbb{Z}^+$ and spaces $\mathcal{Z}^j, j = 1, \ldots, m+1$. Then*

$$\big\|(P^{\#})^{-1}\Pi_1^*\iota\big\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^j} \le C \qquad \textit{for } j = 2, \ldots, m+1.$$

Proof. The proof is similar to the proof of Lemma 4.2, but it is simpler since it turns out that now $\Pi_0 u = 0$. Given $f \in \mathcal{Z}^{j-2}$, let $u = (P^{\#})^{-1}\Pi_1^*\iota f$ so that $P^{\#}u = \Pi_1^*\iota f$. By the definition of $P^{\#}$ (7.13), $\mathcal{E}_{00}\Pi_0 u = 0$; i.e., $\Pi_0^*\mathcal{E}\Pi_0 u = 0$ by (4.1), and thus $\Pi_0 u = 0$ by (2.2). Therefore, for $f \in \mathcal{Z}^{j-2}$, by (3.1), the definition of $P^{\#}$ (7.13), and the fact that $\eta = \Pi_1^*\iota\Pi_1$ (by Part (ii) of Lemma 7.1),

$$\begin{aligned}
\big|\big\langle \mathcal{D}\Pi_1 u, \Pi_1 v\big\rangle_{\mathcal{H}\times\mathcal{H}^*}\big| &= \big|\big\langle \Pi_1^*\iota f + \mathcal{E}u - \Pi_1^*\eta S^2\Pi_1 u, \Pi_1 v\big\rangle_{\mathcal{V}\times\mathcal{V}^*}\big| \\
&= \big|\big\langle \iota^{-1}\big(\Pi_1^*\iota f + \mathcal{E}u - \Pi_1^*\eta S^2\Pi_1 u\big), \iota\Pi_1 v\big\rangle_{\mathcal{V}\times\mathcal{V}^*}\big| \\
&\le C\Big[\big\|\iota^{-1}\Pi_1^*\iota\big\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^{j-2}}\|f\|_{\mathcal{Z}^{j-2}} + \big\|\iota^{-1}\mathcal{E}\big\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^{j-2}}\|u\|_{\mathcal{Z}^{j-2}} \\
&\qquad + \big\|\iota^{-1}\Pi_1^*\iota\big\|_{\mathcal{Z}^{j-2}\to\mathcal{Z}^{j-2}}\big\|\Pi_1 S^2\Pi_1\big\|_{\mathcal{V}\to\mathcal{Z}^{j-2}}\|\Pi_1 u\|_{\mathcal{V}}\Big] \\
&\qquad\qquad \big\|\iota\Pi_1\iota^{-1}\big\|_{(\mathcal{Z}^{j-2})^*\to(\mathcal{Z}^{j-2})^*}\|\iota v\|_{(\mathcal{Z}^{j-2})^*}.
\end{aligned}$$

Thus, by (3.11), (2.11), and Lemma 7.13,

$$\big|\big\langle \mathcal{D}\Pi_1 u, \Pi_1 v\big\rangle_{\mathcal{H}\times\mathcal{H}^*}\big| \le C\Big(\|f\|_{\mathcal{Z}^{j-2}} + \|u\|_{\mathcal{Z}^{j-2}} + \|\Pi_1 u\|_{\mathcal{V}}\Big)\|\iota v\|_{(\mathcal{Z}^{j-2})^*}$$

for $j = 2, \ldots, m+1$. Inputting this last inequality into (2.10) with $\mathsf{D} = \mathcal{D}$ and recalling that $u = \Pi_1 u$, we see that

$$\|u\|_{\mathcal{Z}^j} = \|\Pi_1 u\|_{\mathcal{Z}^j} \le C\Big(\|\Pi_1 u\|_{\mathcal{V}} + \|f\|_{\mathcal{Z}^{j-2}} + \|u\|_{\mathcal{Z}^{j-2}}\Big) \le C\Big(\|f\|_{\mathcal{Z}^{j-2}} + \|u\|_{\mathcal{Z}^{j-2}}\Big) \tag{7.29}$$

for $j = 2, \ldots, m+1$. Now $\|(P^{\#})^{-1}\|_{\mathcal{V}^*\to\mathcal{V}} \le C$ by Lemma 7.11 and the fact that $\mathcal{H} \subset \mathcal{V}$ and $\mathcal{V}^* \subset \mathcal{H}^*$. Therefore, by (7.29) with $j = 2$, $\|u\|_{\mathcal{Z}^2} \le C\|f\|_{\mathcal{V}^*}$; the result then follows by combining this with (7.29). □

7.5 Quasi-optimality of $\Pi_h^{\#}$

Our final task in §7 is to prove quasi-optimality of the projection $\Pi_h^{\#} : \mathcal{H} \to \mathcal{H}_h$ defined by

$$\big\langle P^{\#}v_h, (I - \Pi_h^{\#})w\big\rangle_{\mathcal{H}^*\times\mathcal{H}} = 0 \qquad \text{for all } v_h \in \mathcal{H}_h; \tag{7.30}$$

i.e.,

$$\big\langle (P^{\#})^*(I - \Pi_h^{\#})w, v_h\big\rangle_{\mathcal{H}^*\times\mathcal{H}} = 0 \qquad \text{for all } v_h \in \mathcal{H}_h. \tag{7.31}$$

Lemma 7.18 (Quasi-optimality of $\Pi_h^{\#}$)**.** *If P satisfies Assumptions 2.1 and 2.3, the latter with $m = 1$, then there exist $C_1, C_2 > 0$ such that if*

$$\gamma_{\rm dv}(P^*) \le C_1 \quad \textit{then} \quad \big\|(I - \Pi_h^{\#})v\big\|_{\mathcal{H}} \le C_2\big\|(I - \Pi_h)v\big\|_{\mathcal{H}} \quad \textit{for all } v \in \mathcal{H}. \tag{7.32}$$

Proof. The idea is to apply Corollary 6.2 with P replaced by $(P^{\#})^*$ (so that P^* is replaced by $P^{\#}$). We now need to check that the assumptions of Corollary 6.2 are satisfied with this replacement.

We first claim that

$$(P^{\#})^* = \mathcal{D}^* - \mathcal{E}^* + \Pi_1^* \eta S^2 \Pi_1. \tag{7.33}$$

Indeed, by the definition of $P^{\#}$ (7.13), (7.33) holds if $\Pi_1^* \eta S^2 \Pi_1$ is self-adjoint, and this holds by Part (b) of Lemma 7.9 and the fact that η is self-adjoint (by Lemma 7.5).

Now, since $P^{\#}$ satisfies the Gårding inequality (7.16), $(P^{\#})^*$ satisfies Assumption 2.1 with $\mathcal{D}$ set to $\mathcal{D}^* + \Pi_1^* \eta S^2 \Pi_1$ (which has the same kernel as $\mathcal{D}$) and $\mathcal{E}$ set to $\mathcal{E}^*$. Because of the regularity property of S in Lemma 7.13, if P satisfies Assumption 2.3 with $m = 1$, then so does $(P^{\#})^*$; i.e., the assumptions of Corollary 6.2 are satisfied with P replaced by $(P^{\#})^*$.

The result then follows if we can show that (i) $\gamma_{\rm dv}((P^{\#})^*) = \gamma_{\rm dv}(P^*)$, and (ii) $\|(P^{\#})^{-1}\Pi_1^*\|_{\mathcal{V}\to\mathcal{V}} \leq C$. Point (ii) is satisfied by Lemma 7.11 since $\mathcal{H} \subset \mathcal{V} \subset \mathcal{H}^*$. To show Point (i), observe that the projections Π_0 and Π_1 are now defined with $\mathcal{E}$ replaced by $\mathcal{E}^*$, and the analogue of (5.5) is now

$$\left\langle \mathcal{E}^*(I - \Pi_h^{\#})w, v_h \right\rangle = 0 \quad \text{ for all } v_h \in \mathcal{H}_h \cap \operatorname{Ker} \mathcal{D}$$

(this follows from (7.31) since $(P^{\#})^* v_h = -\mathcal{E}^* v_h$ for $v_h \in \operatorname{Ker}\mathcal{D}$ by (7.33)). By (2.22), $\gamma_{\rm dv}((P^{\#})^*) = \gamma_{\rm dv}(P^*)$ and the proof is complete. □

8 Proof of Theorem 2.9 (the main abstract theorem)

As noted below the statement of Theorem 2.9 the relative-error bound (2.25) follows from the error bound (2.24) and the regularity result of Lemma 2.10.

We now use a duality argument involving $P^{\#}$ to prove the error bound (2.24).

8.1 Reducing bounding the Galerkin error to bounding $\|S\Pi_1(u - u_h)\|_{\mathcal{V}}$

The following lemma is an improved version of Lemma 6.3 (due to the presence of S on the right-hand side).

Lemma 8.1 (Galerkin quasi-optimality, modulo a norm of the error involving S)**.** *Suppose that P satisfies Assumption 2.1. Given $u \in \mathcal{H}$, assume that the solution $u_h \in \mathcal{H}_h$ of (2.20) exists. Then there exists $C_1, C_2 > 0$ such that*

$$\big(1 - C_1 \gamma_{\rm dv}(P)\big) \|u - u_h\|_{\mathcal{H}} \leq C_2 \Big(\|(I - \Pi_h)u\|_{\mathcal{H}} + \|S\Pi_1(u - u_h)\|_{\mathcal{V}} \Big) \quad \textit{for all } v \in \mathcal{H}.$$

Proof. The arguments at the start of Lemma 6.3 show that it is sufficient to prove the bound

$$\|\Pi_1(u - u_h)\|_{\mathcal{H}} \leq \varepsilon \|u - u_h\|_{\mathcal{H}} + C\varepsilon^{-1} \Big(\|(I - \Pi_h)u\|_{\mathcal{H}} + \|S\Pi_1(u - u_h)\|_{\mathcal{V}} + \|\Pi_0(u - u_h)\|_{\mathcal{H}} \Big) \tag{8.1}$$

(note that (8.1) is identical to (6.4) apart from the S multiplying $\Pi_1(u - u_h)$ on the right-hand side).

By coercivity of $P^{\#} = P + \Pi_1^* S^2 \Pi_1$ on $\Pi_1 \mathcal{H}$ (7.14),

$$\|\Pi_1(u - u_h)\|_{\mathcal{H}}^2 \leq \operatorname{Re}\left\langle P\Pi_1(u - u_h), \Pi_1(u - u_h)\right\rangle + \|S\Pi_1(u - u_h)\|_{\mathcal{V}}^2 \tag{8.2}$$

(compare to (6.5)). The arguments after (6.5) then show that

$$\|\Pi_1(u - u_h)\|_{\mathcal{H}}^2 \leq \varepsilon \|u - u_h\|_{\mathcal{H}}^2 + C\Big(\varepsilon^{-1} \|(I - \Pi_h)u\|_{\mathcal{H}}^2 + \varepsilon^{-1} \|\Pi_0(u - u_h)\|_{\mathcal{V}}^2 + \|S\Pi_1(u - u_h)\|_{\mathcal{V}}^2 \Big)$$

(compare to (6.7)); this implies (8.1) and the proof is complete. □

8.2 Duality argument using $P^\#$ to bound $\|S\Pi_1(u-u_h)\|_{\mathcal{V}}$

Lemma 8.2. *Suppose that P satisfies Assumption 2.1. Given $u \in \mathcal{H}$, assume that the solution $u_h \in \mathcal{H}_h$ of (2.20) exists. Suppose further that the projection $\Pi_h^\#$ (7.30) is well-defined. Then there exists $C_1, C_2 > 0$ such that*

$$\Big(1 - C_1\big\|(I-\Pi_h)(P^\#)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\big\|(I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\Big)\,\|S\Pi_1(u-u_h)\|_{\mathcal{V}}$$
$$\leq C_2\big\|(I-\Pi_h)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\,\|(I-\Pi_h)u\|_{\mathcal{H}}.$$

Combining Lemmas 8.1 and 8.2 immediately gives the following result.

Lemma 8.3 (The main abstract result without using regularity of $(P^*)^{-1}$ or $(P^\#)^{-1}$)**.** *Suppose that P satisfies Assumption 2.1. Given $u \in \mathcal{H}$, assume that the solution $u_h \in \mathcal{H}_h$ of (2.20) exists. Suppose further that the projection $\Pi_h^\#$ (7.30) is well-defined. Then there exists $C_1, C_2, C_3, C_4 > 0$ such that*

$$\big(1 - C_1\gamma_{\rm dv}(P)\big)\times$$
$$\Big(1 - C_2\big\|(I-\Pi_h)(P^\#)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\big\|(I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\Big)\,\|u-u_h\|_{\mathcal{H}}$$
$$\leq C_3\bigg(1 - C_2\Big\|(I-\Pi_h)(P^\#)^{-1}\Pi_1^*\eta S\Pi_1\Big\|_{\mathcal{V}\to\mathcal{H}}\big\|(I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}$$
$$+ C_4\big\|(I-\Pi_h)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\bigg)\,\|(I-\Pi_h)u\|_{\mathcal{H}}.$$

That is, if

$$\gamma_{\rm dv}(P) \quad \text{and} \quad \big\|(I-\Pi_h)(P^\#)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\big\|(I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}} \tag{8.3}$$

are both sufficiently small, then

$$\|u-u_h\|_{\mathcal{H}} \leq C\Big(1 + \big\|(I-\Pi_h)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\Big)\,\|(I-\Pi_h)u\|_{\mathcal{H}}. \tag{8.4}$$

Proof of Lemma 8.2. By the definition of η (7.2), the definition of $(P^*)^{-1}: \mathcal{H}^* \to \mathcal{H}$, Part (b) of Lemma 7.9, Galerkin orthogonality (2.20), the definition of $P^\#$ (7.13), and Galerkin orthogonality for $P^\#$ (7.30),

$$\begin{aligned}
\|S\Pi_1(u-u_h)\|_{\mathcal{V}}^2 &= \big\langle S\Pi_1(u-u_h), \eta S\Pi_1(u-u_h)\big\rangle_{\mathcal{V}\times\mathcal{V}^*},\\
&= \big\langle u-u_h, \Pi_1^* S^*\eta S\Pi_1(u-u_h)\big\rangle_{\mathcal{V}\times\mathcal{V}^*},\\
&= \big\langle P(u-u_h), (P^*)^{-1}\Pi_1^*\eta S^2\Pi_1(u-u_h)\big\rangle_{\mathcal{H}^*\times\mathcal{H}},\\
&= \big\langle P(u-u_h), (I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S^2\Pi_1(u-u_h)\big\rangle_{\mathcal{H}^*\times\mathcal{H}},\\
&= \big\langle P^\#(I-\Pi_h)u, (I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S^2\Pi_1(u-u_h)\big\rangle_{\mathcal{H}^*\times\mathcal{H}},\\
&\qquad - \big\langle \Pi_1^*\eta S^2\Pi_1(u-u_h), (I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S^2\Pi_1(u-u_h)\big\rangle_{\mathcal{H}^*\times\mathcal{H}},\\
&\leq C\,\|(I-\Pi_h)u\|_{\mathcal{H}}\,\big\|(I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\,\|S\Pi_1(u-u_h)\|_{\mathcal{V}}\\
&\qquad + \big|\big\langle \Pi_1^*\eta S^2\Pi_1(u-u_h), (I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S^2\Pi_1(u-u_h)\big\rangle\big|.
\end{aligned} \tag{8.5}$$

We now use a duality argument involving $P^\#$ to bound the final term. By (7.30), for $\phi \in \mathcal{V}^*$ and $w \in \mathcal{H}$,

$$\big\langle \Pi_1^*\phi, (I-\Pi_h^\#)w\big\rangle_{\mathcal{H}^*\times\mathcal{H}} = \big\langle P^\#(P^\#)^{-1}\Pi_1^*\phi, (I-\Pi_h^\#)w\big\rangle_{\mathcal{H}^*\times\mathcal{H}}$$

$$= \langle P^\#(I-\Pi_h)(P^\#)^{-1}\Pi_1^*\phi, (I-\Pi_h^\#)w\rangle_{\mathcal{H}^*\times\mathcal{H}},$$

so that

$$\big|\langle \Pi_1^*\phi, (I-\Pi_h^\#)w\rangle\big| \le C\big\|(I-\Pi_h)(P^\#)^{-1}\Pi_1^*\phi\big\|_{\mathcal{H}}\big\|(I-\Pi_h^\#)w\big\|_{\mathcal{H}}. \tag{8.6}$$

We apply (8.6) with $\phi = \eta S^2\Pi_1(u-u_h) \in \mathcal{V}^*$ and $w = (P^*)^{-1}\Pi_1^*\eta S^2\Pi_1(u-u_h) \in \mathcal{H}$ and combine it with (8.5) to obtain

$$\begin{aligned}
&\|S\Pi_1(u-u_h)\|_{\mathcal{V}}^2 \\
&\quad\le C\,\|(I-\Pi_h)u\|_{\mathcal{H}}\,\big\|(I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}^*\to\mathcal{H}}\,\|S\Pi_1(u-u_h)\|_{\mathcal{V}} \\
&\qquad + C\big\|(I-\Pi_h)(P^\#)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}}\,\|S\Pi_1(u-u_h)\|_{\mathcal{V}}^2\,\big\|(I-\Pi_h^\#)(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{H}},
\end{aligned}$$

and the result follows. □

8.3 Proof of the error bound (2.24)

We now use Lemma 8.3 to prove the error bound (2.24) under the condition that the quantities in (2.23) are sufficiently small.

By Lemma 7.18 the projection $\Pi_h^\#$ is well-defined and satisfies (7.32) if $\gamma_{\rm dv}(P^*)$ is sufficiently small. Therefore, the instances of $(I-\Pi_h^\#)$ in Lemma 8.3 can be replaced (up to constants) by $(I-\Pi_h)$.

The result (2.24) then follows if we can show that

$$\big\|(P^\#)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{Z}^{m+1}} \le C$$

and

$$\big\|(P^*)^{-1}\Pi_1^*\eta S\Pi_1\big\|_{\mathcal{V}\to\mathcal{Z}^{m+1}} \le C\big(1+\big\|(P^*)^{-1}\Pi_1^*\big\|_{\mathcal{V}^*\to\mathcal{V}}\big).$$

By the regularity property of S in Lemma 7.13, it is sufficient to prove that

$$\big\|(P^\#)^{-1}\Pi_1^*\eta\Pi_1\big\|_{\mathcal{Z}^{m-1}\to\mathcal{Z}^{m+1}} \le C \tag{8.7}$$

and

$$\big\|(P^*)^{-1}\Pi_1^*\eta\Pi_1\big\|_{\mathcal{Z}^{m-1}\to\mathcal{Z}^{m+1}} \le C\big(1+\big\|(P^*)^{-1}\Pi_1^*\big\|_{\mathcal{V}^*\to\mathcal{V}}\big). \tag{8.8}$$

By Part (b) of Lemma 7.1, $(P^\#)^{-1}\Pi_1^*\eta\Pi_1 = (P^\#)^{-1}\Pi_1^*\iota\Pi_1$ and $(P^*)^{-1}\Pi_1^*\eta\Pi_1 = (P^*)^{-1}\Pi_1^*\iota\Pi_1$. The bounds in (8.7) and (8.8) then follow from Lemmas 7.17 and 4.2, respectively, combined with (3.10) (with Π_0 replaced by Π_1).

9 Recap of the regularity result of Weber [63]

The following result is [63, Theorem 2.2], where we observe that this result – originally proved for real-valued coefficients – immediately generalises to complex-valued coefficients. Recall the definitions of the piecewise spaces $H^j_{\rm pw}(\Omega)$ (2.13) and the associated norm $\|\cdot\|_{H^j_{{\rm pw},k}(\Omega)}$ (2.14).

Theorem 9.1 (Regularity result for curl and div)**.** *Suppose that ζ is a complex matrix-valued function on Ω satisfying ${\rm Re}\,\zeta \ge c > 0$ (in the sense of quadratic forms). Suppose further that, for some integer $\kappa \ge 1$, Ω is $C^{\kappa+1}$ with respect to the partition $\{\Omega_i\}_{i=1}^n$ (in the sense of Definition 1.1) and $\zeta \in C^\kappa(\Omega_j)$ for all $j = 1,\ldots,n$.*

Then there exists $C > 0$ such that, for all $0 \le \ell \le \kappa-1$, if either $u\times n = 0$ or $(\zeta u)\cdot n = 0$ *on $\partial\Omega$ then*

$$\|u\|_{H^{\ell+1}_{{\rm pw},k}(\Omega)} \le C\Big(\|u\|_{L^2(\Omega)} + \big\|k^{-1}{\rm curl}\,u\big\|_{H^\ell_{{\rm pw},k}(\Omega)} + \big\|k^{-1}\,{\rm div}(\zeta u)\big\|_{H^\ell_{{\rm pw},k}(\Omega)}\Big). \tag{9.1}$$

Proof. The result for ζ real-valued and symmetric positive-definite and with norms not weighted with k is [63, Theorem 2.2]. Repeating the proof but now weighting each derivative by k^{-1} gives the bound (9.1).

We now outline why the result holds for complex-valued ζ with $\operatorname{Re}\zeta \geq c > 0$. The proof of [63, Theorem 2.2] begins by localising and mapping the boundary to a half-plane (using [63, Lemma 3.1]) – this is unaffected by the change in assumptions on ζ. The parts of the proof that depend on ζ then involve

1. difference-quotient arguments, and
2. the decomposition of an arbitrary L^2 vector field F into $F_1 + F_2$, where $F_1 = \nabla f$ and $\operatorname{div}(\zeta F_2) = 0$, and *either* $f \in H_0^1(\Omega)$ and $\zeta F_2 \in H(\operatorname{div};\Omega)$ *or* $f \in H^1(\Omega)$ and $\zeta F_2 \in H(\operatorname{div};\Omega)$ with $(\zeta F_2)\cdot n = 0$ on $\partial\Omega$ [63, Lemmas 3.4 and 3.5].

The arguments in Point 1 go through verbatim (noting that ζ is still invertible). The results in Point 2 are quoted from [62, Lemmas 3.8 and 3.9], where they are proved using projections in the $L^2(\Omega)$ inner product weighted with ζ. When ζ is complex valued, the results in Point 2 can be proved via the following. For the first result (when $f \in H_0^1(\Omega)$), given F, let $f \in H_0^1(\Omega)$ be the solution of the variational problem

$$\big(\zeta\nabla f, \nabla w\big)_{L^2(\Omega)} = \big(\zeta F, \nabla w\big)_{L^2(\Omega)} \quad \text{for all } w \in H_0^1(\Omega). \tag{9.2}$$

When $\operatorname{Re}\zeta \geq c > 0$, the solution of (9.2) is unique by the Lax–Milgram lemma. Let $F_2 := F - \nabla f$, so that (9.2) is the statement that $\operatorname{div}(\zeta F_2) = 0$. For the second result (when $f \in H^1(\Omega)$), given F, let $f \in H^1(\Omega)$ be the solution of the variational problem

$$\big(\zeta\nabla f, \nabla w\big)_{L^2(\Omega)} = \big(\zeta F, \nabla w\big)_{L^2(\Omega)} \quad \text{for all } w \in H^1(\Omega). \tag{9.3}$$

When $\operatorname{Re}\zeta \geq c > 0$, the solution of (9.3) (a Laplace Neumann problem) is unique up to constants, so that $F_2 := F - \nabla f$ is uniquely defined. Now (9.3) for $w \in H_0^1(\Omega)$ is the statement that $\operatorname{div}(\zeta F_2) = 0$ and (9.3) for $w \in H^1(\Omega)$ implies that $n\cdot(\zeta F) = 0$ by, e.g., [51, Equation 3.33]. □

10 Definition and properties of Nédélec finite elements

10.1 Curved tetrahedral mesh

We consider a partition of Ω into a conforming mesh $\mathcal{T}_h$ of (curved) tetrahedral elements K as in, e.g., [47, Assumption 3.1]. For $K \in \mathcal{T}_h$ we denote by $\mathscr{F}_K : \widehat{K} \to K$ the mapping between the reference tetrahedra $\widehat{K}$ and the element K. We further assume that the mesh $\mathcal{T}_h$ is conforming with the partition $\{\Omega_i\}_{i=1}^n$ of Ω from Assumption 1.2. This means that for each $K \in \mathcal{T}_h$, there is a unique $i \in \{1,\dots,n\}$ such that $K \subset \overline{\Omega_i}$.

10.2 Nédélec finite element space

Fix a polynomial degree $p \geq 1$. Then, following [52] (see also, e.g., [20, Chapter 15]), we introduce the Nédélec polynomial space

$$N_p(\widehat{K}) = P_{p-1}(\widehat{K}) + x \times P_{p-1}(\widehat{K}),$$

where $P_s(\widehat{K})$ consists of functions such that each component is a polynomial of degree $\leq s$ defined over $\widehat{K}$. (Note that in [20, Chapter 15] the lowest-order elements correspond to $p = 0$, whereas here they correspond to $p = 1$.) The associated approximation space is obtained by mapping the Nédélec polynomial space to the mesh cells through a Piola mapping (see (B.2) below), leading to

$$\mathcal{H}_h := \left\{ v_h \in H_0(\mathrm{curl}\,, \Omega) \,:\, \left(D(\mathscr{F}_K^{-1})\right)^T \left(v_h|_K \circ \mathscr{F}_K^{-1}\right) \in N_p(\widehat{K}) \quad \text{for all } K \in \mathcal{T}_h \right\},$$

where $D(\mathscr{F}_K^{-1})$ is the Jacobian matrix of $\mathscr{F}_K^{-1}$.

Assumption 10.1 (Curved finite-element mesh). *The maps $\mathscr{F}_K$ satisfy*

$$\|\partial^\alpha \mathscr{F}_K\|_{L^\infty(\widehat{K})} \leq CL \left(\frac{h_K}{L}\right)^{|\alpha|} \qquad \text{and} \qquad \|(D\mathscr{F}_K)^{-1}\|_{L^\infty(K)} \leq C h_K^{-1}, \tag{10.1}$$

for $1 \leq |\alpha| \leq p+1$, where h_K is the diameter of K.

Note that the bound (10.1) with $|\alpha| = 1$ corresponds to the mesh elements $K \in \mathcal{T}_h$ being shape regular (as in, e.g., [47, Equation 3.3]).

10.3 High-order interpolation

Theorem 10.2 (Interpolation results in $\mathcal{H}_h$). *Given Ω (with diameter L) and $\mathcal{H}_h$ there exists an interpolation operator $\mathcal{J}_h : Z^2 \to \mathcal{H}_h$ (with Z^j defined by (2.17)) and a constant C such that for all $\ell \in \{1, \dots, p\}$, for all $K \in \mathcal{T}_h$, and for all $v \in Z^{\ell+1}$,*

$$\|v - \mathcal{J}_h v\|_{L^2(K)} \leq C \left(\frac{h_K}{L}\right)^\ell \sum_{j=1}^{\ell} L^j \Big(|v|_{H^j(K)} + h_K |\mathrm{curl}\, v|_{H^j(K)} \Big) \tag{10.2}$$

and

$$\|\mathrm{curl}\,(v - \mathcal{J}_h v)\|_{L^2(K)} \leq C \left(\frac{h_K}{L}\right)^\ell \sum_{j=1}^{\ell} L^j |\mathrm{curl}\, v|_{H^j(K)}. \tag{10.3}$$

Theorem 10.2 is proved using Assumption 10.1 and standard scaling arguments for curved elements in Appendix B.

Observe that (10.3) implies that

$$\text{if } \ \mathrm{curl}\, v = 0 \ \text{ then } \ \mathrm{curl}\,(\mathcal{J}_h v) = 0. \tag{10.4}$$

Corollary 10.3 (Best approximation result in $\mathcal{H}_h$). *Given Ω, $p \in \mathbb{N}$, and $k_0 > 0$, there exists $C > 0$ such that the following is true. With Π_h the orthogonal projection $\mathcal{H} \to \mathcal{H}_h$, for all $\ell \in \{1, \dots, p\}$ and for all $v \in Z^{\ell+1}$,*

$$\|(I - \Pi_h) v\|_{H_k(\mathrm{curl}\,, \Omega)} \leq C (kh)^\ell \, \|v\|_{Z_k^{\ell+1}}. \tag{10.5}$$

Proof. By summing (10.2) and (10.3) over $K \in \mathcal{T}_h$, recalling the assumption that $\mathcal{T}_h$ is conforming with the partition $\{\Omega_i\}_{i=1}^n$ of Ω from Assumption 1.2, and using that $h_K \leq h$,

$$\|v - \mathcal{J}_h v\|_{L^2(\Omega)} \leq C \left(\frac{h}{L}\right)^r \sum_{i=1}^{n} \sum_{j=0}^{\ell} L^j \Big(|u|_{H^j(\Omega_i)} + (k h_K) |k^{-1} \mathrm{curl}\, u|_{H^j(\Omega_i)} \Big)$$

and

$$\|k^{-1} \mathrm{curl}\,(v - \mathcal{J}_h v)\|_{L^2(\Omega)} \leq C \left(\frac{h}{L}\right)^r \sum_{i=1}^{n} \sum_{j=1}^{\ell} L^j |\mathrm{curl}\, u|_{H^j(\Omega_i)}.$$

The result then follows by using the definitions of the norms $\|\cdot\|_{H_k(\mathrm{curl}\,, \Omega)}$ (1.2), $\|\cdot\|_{H^\ell_{\mathrm{pw},k}(\Omega)}$ (2.14), and $\|\cdot\|_{Z_k^{\ell+1}}$ (2.18), along with the fact that $kL \geq k_0 L$ (to absorb factors of $(kL)^{-1}$ into the constant C). □

11 Proof of Theorem 1.3

To show that Theorem 1.3 follows from the abstract result Theorem 2.9, we need to

1. prove Lemma 2.4 (i.e., show that Maxwell fits into the abstract framework),
2. show that the assumptions of the second part of Lemma 2.10 hold when $\operatorname{div} f = 0$,
3. bound $\gamma_{\rm dv}(P)$ and $\gamma_{\rm dv}(P^*)$ and show that the condition on these in (2.23) is weaker (when $kL \gg 1$) than the condition (2.23) involving $(P^*)^{-1}$ (since only the latter appears in (1.3)), and
4. show that, given $m \in \mathbb{N}$ and $p \leq m$, there exists $C > 0$ such that

$$\|I - \Pi_h\|_{Z_k^{m+1} \to H_k(\operatorname{curl},\Omega)} \leq C(kh)^p. \tag{11.1}$$

Indeed, Theorem 1.3 follows from Theorem 2.9 using these points, as well as the fact that the $L^2 \to L^2$ norm of the adjoint solution operator equals the $L^2 \to L^2$ norm of the solution operator (so that $\|(P^*)^{-1}\|_{\mathcal{V}^* \to \mathcal{V}} = C_{\rm sol}$).

Points 1, 2, and 3 are proved in §11.1, §11.2, and §11.3, respectively. Regarding Point 3: we show in Lemma 11.4 below that $\max\{\gamma_{\rm dv}(P), \gamma_{\rm dv}(P^*)\} \leq Ckh(1 + kh)$, i.e., for the first inequality in (2.23) to hold, kh must be sufficiently small. This condition is indeed weaker when $kL \gg 1$ than the condition "$(kh)^{2p} C_{\rm sol}$ is sufficiently small" arising from the second inequality in (2.23), since $C_{\rm sol} \geq CkL$ (recall that this can be proved when μ and ϵ are constant in part of the domain by considering data that is a cut-off function multiplied by a plane wave; see, e.g., [11, §1.4.1]).

Point 4 follows from (10.5); indeed, given $m \in \mathbb{N}$ and $p \leq m$, the bound (10.5) with $\ell = p$ implies that there exists $C > 0$ such that

$$\|(I - \Pi_h)v\|_{H_k(\operatorname{curl},\Omega)} \leq C(kh)^p \|v\|_{Z_k^{p+1}} \leq C(kh)^p \|v\|_{Z_k^{m+1}},$$

so that (11.1) follows.

11.1 Proof of Lemma 2.4

11.1.1 Proof of Part (a).

Since $H_0(\operatorname{curl};\Omega) \supset C_0^\infty(\Omega)$ (by, e.g., [51, Equation 3.42]) and $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, $\mathcal{H}$ is dense in $\mathcal{V}$.

Lemma 11.1. *If* $\mathcal{D} := k^{-2} \operatorname{curl} \mu^{-1} \operatorname{curl}$ *then, in* $H_0(\operatorname{curl},\Omega)$, $\operatorname{Ker} \mathcal{D} = \operatorname{Ker} \mathcal{D}^* = \operatorname{Ker}(\operatorname{curl})$.

Proof. With $\mathcal{H} = H_0(\operatorname{curl},\Omega)$, by, e.g., [51, Theorem 3.31],

$$\langle \mathcal{D}u, v\rangle_{\mathcal{H}^* \times \mathcal{H}} = \langle u, \mathcal{D}^* v\rangle_{\mathcal{H}^* \times \mathcal{H}} = k^{-2}\big(\mu^{-1} \operatorname{curl} u, \operatorname{curl} v\big)_{L^2(\Omega)}$$

for all $u, v \in H_0(\operatorname{curl},\Omega)$. Therefore, if $\mathcal{D}u = 0$, then

$$0 = k^{-2}\big(\mu^{-1} \operatorname{curl} u, \operatorname{curl} u\big)_{L^2(\Omega)},$$

and thus $\operatorname{curl} u = 0$ by (2.15). Identical arguments show that if $\mathcal{D}^* v = 0$, then $\operatorname{curl} v = 0$. Clearly if $\operatorname{curl} u = 0$ then $u \in \operatorname{Ker} \mathcal{D} \cap \operatorname{Ker} \mathcal{D}^*$, and the result follows. □

The rest of Part (i) of Lemma 2.4 follows immediately from the definitions.

11.1.2 Proof of Part (b).

Proof that Parts (ii) and (iii) of Assumption 2.3 hold. By, e.g., [32, Theorem 1.4.1.1, page 21], the bound (2.11) (i.e., Part (iii) of Assumption 2.3) holds if ϵ is piecewise $C^{m,1}$ with respect to the partition $\{\Omega_i\}_{i=1}^n$. We now use Theorem 9.1 to prove that the bound (2.10) (i.e., Part (ii) of Assumption 2.3) holds when Ω is C^{m+1} with respect to the partition $\{\Omega_j\}_{j=1}^n$ (in the sense of Definition 1.1) and $\mu, \epsilon \in C^m(\overline{\Omega_j})$ for all $j = 1, \ldots, n$ (recall from Remark 2.5 that the combination of these regularity requirements is then Assumption 1.2).

Lemma 11.2. *Suppose that Ω is C^{m+1} with respect to the partition $\{\Omega_i\}_{i=1}^n$ (in the sense of Definition 1.1). Suppose that ζ_1, ζ_2 are complex, matrix-valued functions on Ω satisfying $\mathrm{Re}\,\zeta_j \geq c > 0$, $j = 1, 2$, (in the sense of quadratic forms) and $\zeta_1, \zeta_2 \in C^m(\Omega_j)$ for all $j = 1, \ldots, n$.*

Then there exists $C > 0$ such that the following is true for $j = 2, \ldots, m+1$. Given $f \in Z^{j-2}$ (defined by (2.17)), if $v \in H_0(\mathrm{curl}\,, \Omega)$ is such that

$$k^{-2}\mathrm{curl}\,(\zeta_1 \mathrm{curl}\, v) = f \in Z^{j-2} \quad \textit{and} \quad \mathrm{div}(\zeta_2 v) = 0 \quad \textit{in } \Omega, \tag{11.2}$$

then

$$\|v\|_{Z^j} \leq C\Big(\|v\|_{L^2} + \|f\|_{Z^{j-2}} \Big). \tag{11.3}$$

Proof. First observe that it is sufficient to prove the bound

$$\|v\|_{Z^j} \leq C\Big(\|v\|_{L^2} + \big\|k^{-1}\mathrm{curl}\, v\big\|_{L^2} + \|f\|_{Z^{j-2}} \Big). \tag{11.4}$$

Indeed, the weak form of the PDE (11.2) and the fact that $\mathrm{Re}\,\zeta_1 \geq c > 0$ imply that

$$\big\|k^{-1}\mathrm{curl}\, v\big\|^2_{L^2(\Omega)} \leq c^{-1} \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} ;$$

the term involving $\mathrm{curl}\, v$ on the right-hand side of (11.4) can therefore be removed (since $j \geq 2$), with (11.3) the result.

Let $w := k^{-1}\zeta_1 \mathrm{curl}\, v$ so that $k^{-1}\mathrm{curl}\, w = f$. Observe that $\mathrm{div}(\zeta_1^{-1} w) = 0$ and $\mathrm{div}\, f = 0$. With div_T the surface divergence on $\partial\Omega$, by, e.g., [51, Equation 3.52], on $\partial\Omega$,

$$n \cdot \mathrm{curl}\, v = \mathrm{div}_T(v \times n).$$

Since $v \in H_0(\mathrm{curl}\,, \Omega)$, $v \times n = 0$ on $\partial\Omega$, and thus $n \cdot (\zeta_1^{-1} w) = 0$ on $\partial\Omega$.

The regularity assumptions on Ω, ζ_1, and ζ_2 imply that we can apply Theorem 9.1 with $\kappa = m$ and ζ equal one of ζ_1, ζ_2, or their inverses. Therefore, Theorem 9.1 applied with $u = w$, $\zeta = \zeta_1^{-1}$, $\kappa = m$, and $\ell = j-2$, $j = 2, \ldots, m+1$ (so that $\ell \leq \kappa - 1$ as required by Theorem 9.1), implies that

$$\|w\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \leq C\Big(\|w\|_{L^2} + \big\|k^{-1}\mathrm{curl}\, w\big\|_{H^{j-2}_{\mathrm{pw},k}(\Omega)} \Big) \leq C\Big(\big\|k^{-1}\mathrm{curl}\, v\big\|_{L^2} + \|f\|_{H^{j-2}_{\mathrm{pw}}(\Omega)} \Big). \tag{11.5}$$

Similarly, Theorem 9.1 applied with $u = f$, $\zeta = I$, and $\kappa = m$, and $\ell = j-3$ ($j = 3, \ldots, m+1$, so that $\ell \leq \kappa - 1$) implies that

$$\|f\|_{H^{j-2}_{\mathrm{pw},k}(\Omega)} \leq C\Big(\|f\|_{L^2(\Omega)} + \|\mathrm{curl}\, f\|_{H^{j-3}_{\mathrm{pw},k}(\Omega)} \Big) \quad \text{for } j = 3, \ldots, m+1. \tag{11.6}$$

Since $k^{-1}\mathrm{curl}\, v = \zeta_1^{-1} w$ and ζ_1^{-1} is piecewise C^m,

$$\big\|k^{-1}\mathrm{curl}\, v\big\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} = \big\|\zeta_1^{-1} w\big\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \leq C\big\|w\big\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \quad \text{for } j = 2, \ldots, m+1 \tag{11.7}$$

by, e.g., [32, Theorem 1.4.1.1, page 21].

The combination of (11.5), (11.6), and (11.7) imply that

$$\left\|k^{-1}\mathrm{curl}\, v\right\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \le C\Big(\left\|k^{-1}\mathrm{curl}\, v\right\|_{L^2}+\|f\|_{L^2(\Omega)}+\|\mathrm{curl}\, f\|_{H^{j-3}_{\mathrm{pw},k}(\Omega)}\Big) \text{ for } j=3,\ldots,m{+}1. \tag{11.8}$$

Theorem 9.1 applied with $u=v$, $\zeta=\zeta_2$, $\kappa=m$, and $\ell=j-2$, $j=2,\ldots,m+1$ (so that again $\ell\le\kappa-1$ as required by Theorem 9.1), implies that

$$\|v\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \le C\Big(\|v\|_{L^2} + \left\|k^{-1}\mathrm{curl}\, v\right\|_{H^{j-2}_{\mathrm{pw},k}(\Omega)}\Big). \tag{11.9}$$

The combination of (11.8) and (11.9) implies that the bound (11.4) holds for $j=3,\ldots,m+1$. The bound (11.4) when $j=2$ then follows from combining (11.5) and (11.8), both with $j=2$, and the proof is complete. □

To prove that Part (ii) of Assumption 2.3 holds, we seek to apply Lemma 11.2 with $v=\Pi_1 u$, $\zeta_1=\mu^{-1}$, and $\zeta_2=\mathcal{E}=\epsilon$. Since $C^\infty(\overline{D})$ is dense in $H^s(D)$ for all $s\in\mathbb{R}$ and $D\subset\mathbb{R}^3$ open (see, e.g., [44, Page 77]), $L^2(\Omega)$ is dense in $(H^j_{\mathrm{pw}}(\Omega))^*$ for $j\ge1$, so the assumption that $\mathcal{V}$ is dense in $(\mathcal{Z}^j)^*=(Z^j)^*$ for $j\ge1$ is satisfied. The bound (2.10) then follows from (11.3) if $\Pi_1 u$ is in $H_0(\mathrm{curl}\,,\Omega)$ and satisfies $\mathrm{div}(\epsilon\Pi_1 u)=0$. Recall from (2.8) that $\Pi_0:\mathcal{H}\to\mathcal{H}$, and thus also $\Pi_1:\mathcal{H}\to\mathcal{H}$. Thus, $u\in H_0(\mathrm{curl}\,,\Omega)$ implies that $\Pi_1 u\in H_0(\mathrm{curl}\,,\Omega)$. For the zero-divergence condition, observe that, since gradients are always inside the kernel of curl ,

$$\big\langle\, \mathrm{grad}\,\phi, \Pi_1^* v\big\rangle_{L^2(\Omega)} = \big\langle \Pi_1(\mathrm{grad}\,\phi), v\big\rangle_{L^2(\Omega)} = 0 \qquad \text{for all } \phi\in H^1_0(\Omega) \text{ and } v\in L^2(\Omega).$$

Thus, $\mathrm{div}\,\Pi_1^* v=0$ for all $v\in L^2(\Omega)$ (by, e.g., [51, Equation 3.33]). By (3.2), $\mathrm{div}(\epsilon\Pi_1 u)= -\mathrm{div}(\mathcal{E}\Pi_1 u)=-\mathrm{div}(\Pi_1^*\mathcal{E}\Pi_1 u)$ so that $\mathrm{div}(\epsilon\Pi_1 u)=0$ as required.

Proof that Part (i) of Assumption 2.3 holds. By, e.g., [51, Theorem 3.41], the kernel of the curl operator in $H_0(\mathrm{curl}\,,\Omega)$ equals $(\nabla H^1_0(\Omega))\oplus K_N(\Omega)$, where

$$K_N(\Omega) := \Big\{u\in H_0(\mathrm{curl}\,,\Omega)\;:\; \mathrm{curl}\, u=0 \text{ and } \mathrm{div}\, u=0 \text{ in } \Omega\Big\}.$$

Let $\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)}$ and $\Pi^{\mathcal{V}}_{K_N(\Omega)}$ be the $\mathcal{V}$-orthogonal projections onto $\nabla H^1_0(\Omega)$ and $K_N(\Omega)$, respectively, so that

$$\Pi^{\mathcal{V}}_0 = \Pi^{\mathcal{V}}_{K_N(\Omega)} + \Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)}. \tag{11.10}$$

The definition of $K_N(\Omega)$ implies that elements of $K_N(\Omega)$ are solutions of the equation

$$\mathrm{curl}\,\mathrm{curl}\, u - \mathrm{grad}(\mathrm{div}\, u) = 0 \qquad \text{in } \Omega.$$

By [18, §4.5], this PDE is strongly elliptic in the sense of [18, Definition 3.2.2], and the PDE plus the boundary condition $u\times n=0$ on $\partial\Omega$ are then elliptic in the sense of [18, Definition 2.2.31]; see [18, Theorem 3.2.6]. Since $\partial\Omega\in C^{m+1}$ (by Assumption 1.2), the elliptic-regularity result [18, Theorem 3.4.5] implies that $K_N(\Omega)\subset H^{m+1}(\Omega)$. Since every $u\in K_N(\Omega)$ has $\mathrm{curl}\, u=0$ by definition, $K_N(\Omega)$ is therefore contained in $\mathcal{Z}^{m+1}=Z^{m+1}$ defined by (2.17). Thus $\Pi^{\mathcal{V}}_{K_N(\Omega)}$ smooths to the maximal extent possible given the spaces $\{\mathcal{Z}^j\}_{j=0}^{m+1}$, and, in particular, preserves regularity, as required.

Given $f\in L^2(\Omega)$, $\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} f=\nabla\phi$, where $\phi\in H^1_0(\Omega)$ is the unique solution of the variational problem

$$(\nabla\phi,\nabla v)_{L^2(\Omega)} = (f,\nabla v)_{L^2(\Omega)} \qquad \text{for all } v\in H^1_0(\Omega). \tag{11.11}$$

Observe that this is the weak form of the PDE $\Delta\phi = \operatorname{div} f$.

We now apply Theorem 9.1 with $u = \nabla\phi$, $\zeta = I$, and $\kappa = m$ (note that this is allowed because Ω is C^{m+1} with respect to the partition) so that, for $\ell = 0, \ldots, m-1$,

$$\|\nabla\phi\|_{H^{\ell+1}_{\mathrm{pw},k}(\Omega)} \leq C\Big(\|\nabla\phi\|_{L^2(\Omega)} + \big\|k^{-1} \operatorname{div} f\big\|_{H^{\ell}_{\mathrm{pw},k}(\Omega)} \Big). \tag{11.12}$$

By (11.11) with $v = \phi$, $\|\nabla\phi\|_{L^2(\Omega)} \leq C\,\|f\|_{L^2(\Omega)}$,. Therefore, by (11.12) applied with $\ell+1 = j-1$ (so that $\ell = 0, \ldots, k-1$ corresponds to $j = 2, \ldots, m+1$) and the definition of $\|\cdot\|_{H^j_{\mathrm{pw},k}(\Omega)}$ (2.18),

$$\big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} f\big\|_{Z^j_k} = \|\nabla\phi\|_{Z^j_k} = \|\nabla\phi\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \leq C\|f\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \leq C\|f\|_{Z^j_k}$$

for $j = 2, \ldots, m+1$. The splitting (11.10) therefore implies that (2.9) holds for $j = 2, \ldots, m+1$. Since $\mathcal{Z}^1 = \mathcal{H}$, and $\Pi^{\mathcal{V}}_0 : \mathcal{H} \to \mathcal{H}$ is bounded by (2.8), (2.9) holds for $j = 1, \ldots, m+1$ and thus Part (i) of Assumption 2.3 holds, as required.

Proof that Part (iv) of Assumption 2.3 holds. We first show that the bound (2.12) follows if we can show that there exists $C > 0$ such that, for $j = 1, \ldots, m+1$,

$$\big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{Z}^j} \leq C\Big(\big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} \mathsf{E} \Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{Z}^j} + \big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{V}}\Big) \quad \text{for all } u \in \mathcal{V}. \tag{11.13}$$

Indeed, since $\Pi^{\mathcal{V}}_{K_N(\Omega)}$ is smoothing (as shown in the proof of Part (i) above), for all $u \in \mathcal{V}$,

$$\big\|\Pi^{\mathcal{V}}_{\nabla H^1(\Omega)} \mathsf{E} \Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{Z}^j} \leq \big\|\Pi^{\mathcal{V}}_0 \mathsf{E} \Pi^{\mathcal{V}}_0 u\big\|_{\mathcal{Z}^j} + C\Big(\big\|\Pi^{\mathcal{V}}_{K_N(\Omega)} u\big\|_{\mathcal{V}} + \big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{V}}\Big).$$

Therefore, by (11.10), (11.13), and the fact that $\Pi^{\mathcal{V}}_{K_N(\Omega)}$ is smoothing, for all $u \in \mathcal{V}$,

$$\begin{aligned}
\big\|\Pi^{\mathcal{V}}_0 u\big\|_{\mathcal{Z}^j} &\leq \big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{Z}^j} + \big\|\Pi^{\mathcal{V}}_{K_N(\Omega)} u\big\|_{\mathcal{V}} \\
&\leq C\Big(\big\|\Pi^{\mathcal{V}}_0 \mathsf{E} \Pi^{\mathcal{V}}_0 u\big\|_{\mathcal{Z}^j} + \big\|\Pi^{\mathcal{V}}_{K_N(\Omega)} u\big\|_{\mathcal{V}} + \big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{V}}\Big),
\end{aligned}$$

and the result (2.12) follows since $\|\Pi^{\mathcal{V}}_{K_N(\Omega)} u\|_{\mathcal{V}} + \|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\|_{\mathcal{V}} = \|\Pi^{\mathcal{V}}_0 u\|_{\mathcal{V}}$ (since $\operatorname{Ker}\mathcal{D} = (\nabla H^1_0(\Omega)) \oplus K_N(\Omega)$ in $\mathcal{V}$).

We now prove (11.13) with $\mathsf{E} = \iota^{-1}\mathcal{E} = \epsilon$; the proof for $\mathsf{E} = \iota^{-1}\mathcal{E}^* = \epsilon^*$ is analogous. By definition, there exists $\phi \in H^1_0(\Omega)$ such that $\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u = \nabla\phi$. Then $\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} (\iota^{-1}\mathcal{E}) \Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u = \nabla w$ where

$$\big(\nabla w, \nabla v\big)_{L^2(\Omega)} = \big(\epsilon\nabla\phi, \nabla v\big)_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega);$$

i.e., $\Delta w = \operatorname{div}(\epsilon\nabla\phi)$. When $j = 1$, since $\mathcal{Z}^1 = \mathcal{H} = H_0(\operatorname{curl}, \Omega)$,

$$\big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{Z}^1} = \big\|\nabla\phi\big\|_{\mathcal{H}} = \big\|\nabla\phi\big\|_{L^2(\Omega)} = \big\|\Pi^{\mathcal{V}}_{\nabla H^1_0(\Omega)} u\big\|_{\mathcal{V}}$$

and thus (11.13) immediately holds when $j = 1$. To prove (11.13) for $j = 2, \ldots, m+1$, we apply the regularity result of Theorem 9.1 with $u = \nabla\phi$, $\zeta = \epsilon$, and $\kappa = m$ (so that the regularity assumptions on $\{\Omega_j\}_{j=1}^n$ and ϵ are satisfied by Assumption 1.2). By the definition of $\mathcal{Z}^j$ (2.17) and the regularity result (9.1) with $\ell + 1 = j - 1$ (so that $\ell = 0, \ldots, k-1$ corresponds to $j = 2, \ldots, m+1$), there exists $C > 0$ such that, for $j = 2, \ldots, m+1$,

$$\|\nabla\phi\|_{\mathcal{Z}^j} = \|\nabla\phi\|_{Z^j_k} = \|\nabla\phi\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \leq C\Big(\|\nabla\phi\|_{L^2(\Omega)} + \big\|k^{-1}\Delta w\big\|_{H^{j-2}_{\mathrm{pw},k}(\Omega)} \Big)$$

$$\leq C\Big(\|\nabla\phi\|_{L^2(\Omega)} + \|\nabla w\|_{H^{j-1}_{\mathrm{pw},k}(\Omega)} \Big)$$
$$\leq C\Big(\|\nabla\phi\|_{L^2(\Omega)} + \|\nabla w\|_{Z^j_k} \Big) = C\Big(\|\nabla\phi\|_{\mathcal{V}} + \|\nabla w\|_{\mathcal{Z}^j} \Big);$$

i.e., (11.13) holds for $j = 2, \ldots, m+1$ and the proof is complete.

11.2 The assumptions of the second part of Lemma 2.10

Lemma 11.3. *Suppose that Ω is C^{m+1} with respect to the partition $\{\Omega_j\}_{j=1}^n$ (in the sense of Definition 1.1) and $\epsilon, \mu \in C^m(\overline{\Omega_j})$ for all $j = 1, \ldots, n$.*

If $\operatorname{div} f = 0$ then there exists $\widetilde{\Pi}_0$ such that (i) $\Pi_0^* f = \widetilde{\Pi}_0^* f$, (ii) $\Pi_0 \widetilde{\Pi}_0 = \widetilde{\Pi}_0$, and (iii) the bound (2.26) holds.

Proof. Let

$$\widetilde{\Pi}_0 := \Pi^{\mathcal{V}}_{K_N(\Omega)} \Pi_0, \tag{11.14}$$

where (as above) $\Pi^{\mathcal{V}}_{K_N(\Omega)}$ is the $\mathcal{V}$-orthogonal projection onto $K_N(\Omega)$. Then Point (ii) holds since $\widetilde{\Pi}_0$ maps into the kernel. Furthermore, $\Pi_0^* f = \widetilde{\Pi}_0^* f$ if and only if, for all $v \in \mathcal{H}$,

$$\langle f, \Pi_0 v\rangle_{\mathcal{H}^*\times\mathcal{H}} = \langle f, \widetilde{\Pi}_0 v\rangle_{\mathcal{H}^*\times\mathcal{H}}; \qquad \text{i.e.,} \qquad \big\langle f, (I - \Pi^{\mathcal{V}}_{K_N(\Omega)})\Pi_0 v\big\rangle_{\mathcal{H}^*\times\mathcal{H}}.$$

On the kernel, $I - \Pi^{\mathcal{V}}_{K_N(\Omega)}$ projects to $\operatorname{grad} H_0^1(\Omega)$, so $(f, \nabla p)_{L^2(\Omega)} = 0$ for all $p \in H_0^1(\Omega)$, i.e., $\operatorname{div} f = 0$, so that Point (i) holds.

For (iii), we first observe that, by (11.14), the fact that $(\Pi^{\mathcal{V}}_{K_N(\Omega)})^* = \iota \Pi^{\mathcal{V}}_{K_N(\Omega)} \iota^{-1}$, and (3.11),

$$\big\|\iota^{-1}\widetilde{\Pi}_0^*\iota\big\|_{\mathcal{Z}^{m-1}\to\mathcal{Z}^{m+1}} = \big\|\iota^{-1}\Pi_0^*\iota\Pi^{\mathcal{V}}_{K_N(\Omega)}\big\|_{\mathcal{Z}^{m-1}\to\mathcal{Z}^{m+1}} \leq C\big\|\Pi^{\mathcal{V}}_{K_N(\Omega)}\big\|_{\mathcal{Z}^{m-1}\to\mathcal{Z}^{m+1}}. \tag{11.15}$$

The proof above of Part (i) of Assumption 2.3 showed that $\Pi^{\mathcal{V}}_{K_N(\Omega)}$ smooths to the maximal extent possible given the spaces $\{\mathcal{Z}^j\}_{j=0}^{m+1}$; in particular, $\|\Pi^{\mathcal{V}}_{K_N(\Omega)}\|_{\mathcal{Z}^{m-1}\to\mathcal{Z}^{m+1}} \leq C$, and this combined with (11.15) gives the result (2.26). □

11.3 Proof that $\gamma_{\rm dv}(P), \gamma_{\rm dv}(P^*) \leq Ckh(1 + kh)$

Lemma 11.4 (Bound on $\gamma_{\rm dv}(P), \gamma_{\rm dv}(P^*)$)**.** *Let $P = \mathcal{D} - \mathcal{E}$ with $\mathcal{D}$ and $\mathcal{E}$ defined by (2.16) with μ and ϵ satisfying (2.15). Suppose that Ω is C^2 with respect to the partition $\{\Omega_j\}_{j=1}^n$ (in the sense of Definition 1.1) and $\epsilon, \mu \in C^1(\overline{\Omega_j})$ for all $j = 1, \ldots, n$. Then*

$$\max\big\{\gamma_{\rm dv}(P), \gamma_{\rm dv}(P^*)\big\} \leq Ckh(1 + kh).$$

Lemma 11.4 is a consequence of (i) the following abstract bound on $\gamma_{\rm dv}$ and (ii) properties of the interpolation operator $\mathcal{J}_h$ recapped in §10.3.

Lemma 11.5. *Suppose there exists $J : \mathcal{H}_h \to \mathcal{H}_h$ such that (i) $J w_h = w_h$ for all $w_h \in \mathcal{H}_h$, (ii) $J : \Pi_1 \mathcal{H}_h \to \mathcal{H}_h$, and (iii) $J\Pi_0 w_h \in \mathcal{H}_h \cap \operatorname{Ker}\mathcal{D}$ for all $w_h \in \mathcal{H}_h$. Then there exists $C > 0$ such that, if*

$$w_h \in \mathcal{H}_h \ \text{satisfies} \ \big\langle \mathcal{E} w_h, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0 \ \text{ for all } v_h \in \mathcal{H}_h \cap \operatorname{Ker}\mathcal{D} \tag{11.16}$$

then

$$\|\Pi_0 w_h\|_{\mathcal{V}} \leq C\, \|(I - J)\Pi_1 w_h\|_{\mathcal{V}}.$$

The following result then holds immediately from the definition of $\gamma_{\rm dv}(P)$ (2.22).

Corollary 11.6. *Under the assumptions of Lemma 11.5, there exists $C>0$ such that*

$$\gamma_{\rm dv}(P)\leq C\sup\bigg\{\frac{\|(I-J)\Pi_1 w_h\|_{\mathcal{V}}}{\|w_h\|_{\mathcal{H}}}\;:\; w_h\in\mathcal{H}_h \text{ satisfies } \big\langle \mathcal{E}w_h, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}}=0 \text{ for all } v_h\in\mathcal{H}_h\cap \operatorname{Ker}\mathcal{D}\bigg\}.$$

Proof of Lemma 11.4 using Corollary 11.6. We apply Corollary 11.6 with $J=\mathcal{J}_h$. The assumptions (i) and (iii) on J in Lemma 11.5 are satisfied by the properties of $\mathcal{J}_h$ recapped in Theorem 10.2 and (10.4).

To show that $J:\Pi_1\mathcal{H}_h\to\mathcal{H}_h$ (i.e., the assumption (ii) in Lemma 11.5), we apply the regularity result of Theorem 9.1 with $\kappa=1$ and $\ell=0$. As in the proof of Lemma 2.4, for the operator P, $\operatorname{div}(\epsilon\Pi_1 w_h)=0$. Similarly, for the operator P^*, $\operatorname{div}(\epsilon^*\Pi_1 w_h)=0$. Therefore, in both cases, by Theorem 9.1 (with ζ equal either ϵ or ϵ^*) and (2.15),

$$\begin{aligned}\|\Pi_1 w_h\|_{H^1_{{\rm pw},k}(\Omega)} &\leq C\Big(\|\Pi_1 w_h\|_{L^2(\Omega)}+\big\|k^{-1}\operatorname{curl}(\Pi_1 w_h)\big\|_{L^2(\Omega)}\Big)\\ &= C\Big(\|\Pi_1 w_h\|_{L^2(\Omega)}+\big\|k^{-1}\operatorname{curl} w_h\big\|_{L^2(\Omega)}\Big),\end{aligned}\tag{11.17}$$

since $\operatorname{curl}(\Pi_0 w_h)=0$ and thus $\operatorname{curl}(\Pi_1 w_h)=\operatorname{curl} w_h$. Therefore, given $w_h\in\mathcal{H}_h$, $\Pi_1 w_h\in H_0(\operatorname{curl},\Omega)\cap H^1_{\rm pw}(\Omega)$, and thus $J\Pi_1 w_h$ is well-defined by Theorem 10.2.

By (10.2) with $r=1$, the definition of $H^1_k(K)$, and the fact that $\operatorname{curl}(\Pi_1 w_h)=\operatorname{curl} w_h$, given $k_0>0$ there exists $C>0$ such that, for all $k\geq k_0$,

$$\|(I-\mathcal{J}_h)\Pi_1 w_h\|_{L^2(K)}\leq Ckh_K\Big(\|\Pi_1 w_h\|_{H^1_k(K)}+kh_K\|k^{-1}\operatorname{curl} w_h\|_{H^1_k(K)}\Big).$$

By a standard inverse inequality (see, e.g., [20, §12.1]),

$$\|k^{-1}\operatorname{curl} w_h\|_{H^1_k(K)}\leq C\big(1+(kh_K)^{-1}\big)\|k^{-1}\operatorname{curl} w_h\|_{L^2(K)},$$

so that

$$\|(I-\mathcal{J}_h)\Pi_1 w_h\|_{L^2(K)}\leq Ckh_K\Big(\|\Pi_1 w_h\|_{H^1_k(K)}+\big(kh_K+1\big)\|k^{-1}\operatorname{curl} w_h\|_{L^2(K)}\Big).$$

Summing over $K\in\mathcal{T}_h$, recalling that $\mathcal{T}_h$ is conforming with the partition $\{\Omega_i\}_{i=1}^n$ of Ω from Assumption 1.2, using that $h_K\leq h$, and using the definition of $\|\cdot\|_{H^1_{{\rm pw},k}(\Omega)}$ (2.14) gives

$$\|(I-\mathcal{J}_h)\Pi_1 w_h\|_{L^2(\Omega)}\leq Ckh\Big(\|\Pi_1 w_h\|_{H^1_{{\rm pw},k}(\Omega)}+\big(kh+1\big)\|k^{-1}\operatorname{curl} w_h\|_{H^1_{{\rm pw},k}(\Omega)}\Big).$$

Therefore, by (11.17) and the boundedness of $\Pi_1:L^2(\Omega)\to L^2(\Omega)$,

$$\|(I-\mathcal{J}_h)\Pi_1 w_h\|_{L^2(\Omega)}\leq Ckh\Big(\|w_h\|_{L^2(\Omega)}+\big(kh+1\big)\big\|k^{-1}\operatorname{curl} w_h\big\|_{L^2(\Omega)}\Big),$$

and the result follows. □

Proof of Lemma 11.5. We prove that, for w_h as in (11.16),

$$C_{P_2}\|\Pi_0 w_h\|^2_{\mathcal{V}}\leq\big|\big\langle\mathcal{E}\Pi_0 w_h,(I-J)\Pi_1 w_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}}\big|,\tag{11.18}$$

and the result then follows from the boundedness of $\mathcal{E}:\mathcal{V}\to\mathcal{V}^*$.

Since J is well defined on both $\mathcal{H}_h$ and $\Pi_1\mathcal{H}_h$, it is well defined on $\Pi_0\mathcal{H}_h$. Now, by assumption $(I-J)w_h = 0$; therefore $(I-J)\Pi_0 w_h = -(I-J)\Pi_1 w_h$ and

$$\Pi_0 w_h = (I-J)\Pi_0 w_h + J\Pi_0 w_h = -(I-J)\Pi_1 w_h + J\Pi_0 w_h. \tag{11.19}$$

By (2.2) with $v = \Pi_0 w_h$ and (11.19), to prove (11.18) it is sufficient to prove that

$$\big\langle \mathcal{E}\Pi_0 w_h, J\Pi_0 w_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} = 0. \tag{11.20}$$

If $v_h \in \operatorname{Ker}\mathcal{D}$ then $v_h = \Pi_0 v_h$ and, by (3.2),

$$\begin{aligned}\big\langle \mathcal{E}w_h, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} = \big\langle \mathcal{E}w_h, \Pi_0 v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} = \big\langle \Pi_0^*\mathcal{E}w_h, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \big\langle \Pi_0^*\mathcal{E}\Pi_0 w_h, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &= \big\langle \mathcal{E}\Pi_0 w_h, v_h\big\rangle_{\mathcal{V}^*\times\mathcal{V}}. \end{aligned} \tag{11.21}$$

Since $J\Pi_0 w_h \in \operatorname{Ker}\mathcal{D}$, (11.20) immediately follows from (11.16) and (11.21), and the proof is complete. $\square$

A The Maxwell radial PML problem

This section recaps the definition of the Maxwell radial PML problem from [17], [51, §13.5.3.2, Page 378] (using slightly different notation) and shows that the coeffcients μ and ϵ in this case satisfy (2.15) (see Lemma A.1 below).

The scattering problem. Let $\Omega_- \subset \mathbb{R}^3$ be such that its open complement $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega_-}$ is connected. Let n be the outward-pointing unit normal vector to Ω_-. Let $\epsilon_{\rm scat}, \mu_{\rm scat}$ be real-valued symmetric positive definite matrix functions on Ω_+ such that $\operatorname{supp}(\epsilon_{\rm scat} - I), \operatorname{supp}(\mu_{\rm scat} - I) \subset B_{R_{\rm scat}}$ for some $R_{\rm scat} > 0$. The scattering problem is then: given $f \in L^2_{\rm comp}(\mathbb{R}^3)$, find $E_{\rm scat} \in H_{\rm loc}(\operatorname{curl}, \Omega)$ with $E_{\rm scat} \times n = 0$ on $\partial\Omega_-$ such that

$$k^{-2}\operatorname{curl}(\mu_{\rm scat}^{-1}\operatorname{curl} E_{\rm scat}) - \epsilon_{\rm scat}E_{\rm scat} = f \quad \text{ in } \Omega_+, \tag{A.1}$$

and $E_{\rm scat}$ satisfies the Silver-Müller radiation condition (see, e.g., [51, Equation 1.29]).

PML definition. Let $R_{\rm tr} > R_{\rm PML,-} > R_{\rm scat}$ and let $\Omega_{\rm tr} \subset \mathbb{R}^d$ be a bounded Lipschitz open set with $B_{R_{\rm tr}} \subset \Omega_{\rm tr} \subset B_{CR_{\rm tr}}$ for some $C > 0$ (i.e., $\Omega_{\rm tr}$ has characteristic length scale $R_{\rm tr}$). Let $\Omega := \Omega_{\rm tr} \cap \Omega_+$. For $0 \le \theta < \pi/2$, let the PML scaling function $f_\theta \in C^1([0,\infty);\mathbb{R})$ be defined by $f_\theta(r) := f(r)\tan\theta$ for some f satisfying

$$\big\{f(r) = 0\big\} = \big\{f'(r) = 0\big\} = \big\{r \le R_{\rm PML,-}\big\}, \quad f'(r) \ge 0, \quad f(r) \equiv r \text{ on } r \ge R_{\rm PML,+}; \tag{A.2}$$

i.e., the scaling "turns on" at $r = R_{\rm PML,-}$, and is linear when $r \ge R_{\rm PML,+}$. Note that $R_{\rm tr}$ can be $< R_{\rm PML,+}$, i.e., truncation can occur before linear scaling is reached. Given $f_\theta(r)$, let

$$\alpha(r) := 1 + {\rm i} f'_\theta(r) \quad \text{ and } \quad \beta(r) := 1 + {\rm i} f_\theta(r)/r,$$

and let

$$\mu := \begin{cases} \mu_{\rm scat} & \text{in } B_{R_{\rm PML,-}}, \\ HDH^T & \text{in } (B_{R_{\rm PML,-}})^c \end{cases} \quad \text{and } \epsilon := \begin{cases} \epsilon_{\rm scat} & \text{in } B_{R_{\rm PML,-}}, \\ HDH^T & \text{in } (B_{R_{\rm PML,-}})^c \end{cases} \tag{A.3}$$

where, in spherical polar coordinates (r, φ, ϕ),

$$D = \begin{pmatrix} \beta(r)^2\alpha(r)^{-1} & 0 & 0 \\ 0 & \alpha(r) & 0 \\ 0 & 0 & \alpha(r) \end{pmatrix} \text{ and } H = \begin{pmatrix} \sin\varphi\cos\phi & \cos\varphi\cos\phi & -\sin\phi \\ \sin\varphi\sin\phi & \cos\varphi\sin\phi & \cos\phi \\ \cos\varphi & -\sin\varphi & 0 \end{pmatrix} \tag{A.4}$$

(observe that $\mu_{\text{scat}} = \epsilon_{\text{scat}} = I$ when $r = R_{\text{PML},-}$ and thus μ and ϵ are continuous at $r = R_{\text{PML},-}$).

The perfectly-matched-layer approximation to E_{scat} is then the solution of (A.1) in Ω with coefficients (A.3).

We highlight that, in other papers on PMLs, the scaled variable, which in our case is $r + \mathrm{i}f_\theta(r)$, is often written as $r(1+\mathrm{i}\widetilde{\sigma}(r))$ with $\widetilde{\sigma}(r) = \sigma_0$ for r sufficiently large; see, e.g., [35, §4], [7, §2]. Therefore, to convert from our notation, set $\widetilde{\sigma}(r) = f_\theta(r)/r$ and $\sigma_0 = \tan\theta$.

Lemma A.1. *Given $\epsilon_{\text{scat}}, \mu_{\text{scat}}$ as above and a scaling function $f(r)$ satisfying (A.2), let ϵ, μ be defined by (A.3). Given $\varepsilon > 0$, the following is true.*

(i) There exists $C > 0$ such that, for all $\varepsilon \le \theta \le \pi/2 - \varepsilon$, $x \in \Omega$, and $\xi, \zeta \in \mathbb{C}^d$,

$$\max\left\{\left|\left(\mu^{-1}(x)\xi, \zeta\right)_2\right|, \left|\left(\epsilon(x)\xi, \zeta\right)_2\right|\right\} \le C\|\xi\|_2\|\zeta\|_2.$$

(ii) If, additionally, $f(r)/r$ is nondecreasing, then there exists $c > 0$ such that, for all $\varepsilon \le \theta \le \pi/2 - \varepsilon$, $x \in \Omega$, and $\xi, \zeta \in \mathbb{C}^d$,

$$\min\left\{\left(\mathrm{Re}(\mu^{-1}(x))\xi, \xi\right)_2, \left(\mathrm{Re}(\epsilon(x))\xi, \xi\right)_2\right\} \ge c\|\xi\|_2^2.$$

Sketch proof. Part (i) follows in a straightforward way from the definitions of μ and ϵ. The proof of Part (ii) is very similar to the proof of the analogous Helmholtz result in [28, Lemma 2.3]. □

We highlight that the assumption in Part (ii) of Lemma A.1 that $f(r)/r$ is nondecreasing is standard in the literature; e.g., in the alternative notation described above it is that $\widetilde{\sigma}$ is non-decreasing – see [7, §2].

B Proof of Theorem 10.2 (interpolation results in $\mathcal{H}_h$)

Recall that $D\mathscr{F}_K$ is the Jacobian matrix of $\mathscr{F}_K$. By the first bound in (10.1) with $|\alpha| = 1$ and the fact that $d = 3$, there exists $C > 0$ such that, for all $K \in \mathcal{T}_h$,

$$\frac{1}{C}h_K^3 \le \det(D\mathscr{F}_K) \le Ch_K^3 \quad \text{in } \widehat{K}. \tag{B.1}$$

For $v \in L^2(K)$, we introduce the curl- and divergence-conforming Piola transformations:

$$\mathscr{F}_K^{\mathrm{c}}(v) := (D\mathscr{F}_K)^T(v \circ \mathscr{F}_K), \tag{B.2}$$

$$\mathscr{F}_K^{\mathrm{d}}(v) := \det(D\mathscr{F}_K)(D\mathscr{F}_K)^{-1}(v \circ \mathscr{F}_K); \tag{B.3}$$

see, e.g., [51, §3.9], [20, §9.2.1]. Recall that

$$\mathrm{curl}\left(\mathscr{F}_K^{\mathrm{c}}(v)\right) = \mathscr{F}_K^{\mathrm{d}}(\mathrm{curl}\, v) \tag{B.4}$$

for all $v \in C^1(K)$ by, e.g., [20, Corollary 9.9].

In analogue with the definition of the space Z^j (2.17), let

$$Z^j(\mathcal{T}_h) := \left\{u \in H_0(\mathrm{curl}\,, \Omega) : u|_K \in H^{j-1}(K) \text{ and } (\mathrm{curl}\, u)|_K \in H^{j-1}(K)\right\}.$$

We denote by $\widehat{I}^{\mathrm{c}}, \widehat{I}^{\mathrm{d}}$ the canonical Nédélec and Raviart-Thomas interpolants of degree p on the reference element $\widehat{K}$ (see, e.g., [51, §5.4–5.5], [20, Chapter 16]). We then consider the interpolation operators $I_h^{\mathrm{c}} : Z^2(\mathcal{T}_h) \to V_h$ and $I_h^{\mathrm{d}} : H^1_{\mathrm{pw}}(\mathcal{T}_h) \to V_h$ by setting

$$I_h^{\mathrm{c}}|_K := (\mathscr{F}_K^{\mathrm{c}})^{-1} \circ \widehat{I}^{\mathrm{c}} \circ \mathscr{F}_K^{\mathrm{c}} \tag{B.5}$$

and

$$I_h^{\mathrm{d}}|_K := (\mathscr{F}_K^{\mathrm{d}})^{-1} \circ \widehat{I}^{\mathrm{d}} \circ \mathscr{F}_K^{\mathrm{d}}$$

(see, e.g., [20, Proposition 9.3]). By standard commuting properties,

$$(\mathrm{curl} \circ I_h^{\mathrm{c}})|_K = (I_h^{\mathrm{d}} \circ \mathrm{curl}\,)|_K; \tag{B.6}$$

see, e.g., [20, Lemma 16.8].

We prove below that Theorem 10.2 holds with $\mathcal{J}_h = I_h^{\mathrm{c}}$. The following two lemmas are key ingredients in this proof.

Lemma B.1 (Norm bounds on Piola transformations). *With $\mathscr{F}_K^{\mathrm{c}}(v)$ and $\mathscr{F}_K^{\mathrm{d}}(v)$ defined by (B.2) and (B.3), respectively, there exists $C > 0$ such that, for $\ell \in \{1, \dots, p\}$, for all $K \in \mathcal{T}_h$, and for all $v \in H^\ell(K)$,*

$$h_K^{3/2} |\mathscr{F}_K^{\mathrm{c}}(v)|_{H^\ell(\widehat{K})} \leq CL \left(\frac{h_K}{L}\right)^{\ell+1} \sum_{j=1}^{\ell} L^j |v|_{H^j(K)} \tag{B.7}$$

and

$$h_K^{3/2} \big|\mathscr{F}_K^{\mathrm{d}}(v)\big|_{H^\ell(\widehat{K})} \leq CL^2 \left(\frac{h_K}{L}\right)^{\ell+2} \sum_{j=0}^{\ell} L^j |v|_{H^j(K)}. \tag{B.8}$$

Lemma B.2 (Derivative of co-factor matrix). *There exists $C > 0$ such that, for all $K \in \mathcal{T}_h$,*

$$\big\|\partial^\alpha\big(\det(D\mathscr{F}_K)(D\mathscr{F}_K)^{-1}\big)\big\|_{L^\infty(\widehat{K})} \leq CL^2 \left(\frac{h}{L}\right)^{|\alpha|+2} \tag{B.9}$$

for $1 \leq |\alpha| \leq p$.

Proof of Theorem 10.2 using Lemmas B.1 and B.2. Let $\widehat{v} := \mathscr{F}_K^{\mathrm{c}}(v)$, so that, by (B.5),

$$\mathscr{F}_K^{\mathrm{c}}(v - I_h^{\mathrm{c}} v) = \widehat{u} - \widehat{I}^{\mathrm{c}} \widehat{u}.$$

We now claim that

$$\|v - I_h^{\mathrm{c}} v\|_{L^2(K)} \leq C h_K^{-1} h_K^{3/2} \|\widehat{v} - \widehat{I}^{\mathrm{c}} \widehat{v}\|_{L^2(\widehat{K})};$$

indeed, the $h_K^{3/2}$ comes from the Jacobian in the change of variable with (B.1), and the h_K^{-1} comes from the factor $(D\mathscr{F}_K^{-1})$ via (10.1). On the reference element, the proof of [34, Theorem 3.14] implies that there exists $C > 0$ such that, for $\ell \in \{1, \dots, p\}$,

$$\|\widehat{v} - \widehat{I}^{\mathrm{c}} \widehat{v}\|_{L^2(\widehat{K})} \leq C \left(|\widehat{v}|_{H^\ell(\widehat{K})} + |\mathrm{curl}\, \widehat{v}|_{H^\ell(\widehat{K})} \right).$$

so that

$$\|v - I_h^{\mathrm{c}} v\|_{L^2(K)} \leq C h_K^{-1} h_K^{3/2} \left(|\widehat{v}|_{H^\ell(\widehat{K})} + |\mathrm{curl}\, \widehat{v}|_{H^\ell(\widehat{K})} \right). \tag{B.10}$$

By (B.7),

$$h_K^{-1} h_K^{3/2} |\widehat{v}|_{H^\ell(\widehat{K})} \leq C \left(\frac{h_K}{L}\right)^{\ell} \sum_{j=1}^{\ell} L^j |v|_{H^j(K)}. \tag{B.11}$$

We now let $w := \mathrm{curl}\, v$ and $\widehat{w} := \mathrm{curl}\, \widehat{v}$, so that $\widehat{w} = \mathscr{F}_K^{\mathrm{d}}(w)$ by (B.4). By (B.8), (B.1), and (B.9) with $|\alpha| = 1$,

$$h_K^{-1} h_K^{3/2} |\mathrm{curl}\, \widehat{v}|_{H^\ell(\widehat{K})} = h_K^{-1} h_K^{3/2} |\widehat{w}|_{H^\ell(\widehat{K})}$$

$$\leq CL^2 h_K^{-1} \left(\frac{h_K}{L}\right)^{\ell+2} \sum_{j=1}^{\ell} L^j |w|_{H^j(K)} = Ch_K \left(\frac{h_K}{L}\right)^{\ell} \sum_{j=1}^{\ell} L^j |\mathrm{curl}\, v|_{H^j(K)}. \tag{B.12}$$

The bound (10.2) then follows from the combination of (B.10), (B.11), and (B.12).

To prove (10.3), first observe that, by (B.6),

$$\|\mathrm{curl}\,(v - I_h^{\mathrm{c}} v)\|_{L^2(K)} = \|w - I_h^{\mathrm{d}} w\|_{L^2(K)}. \tag{B.13}$$

By the definition of $\mathscr{F}^{\mathrm{d}}$ (B.3), the lower bound in (B.1), and the first bound in (10.1) with $|\alpha| = 1$,

$$\|w - I_h^{\mathrm{d}} w\|_{L^2(K)} \leq h_K^{-2} h_K^{3/2} \|\widehat{w} - \widehat{I}^{\mathrm{d}} \widehat{w}\|_{L^2(\widehat{K})}. \tag{B.14}$$

Since I^{d} is continuous over $H^1(\widehat{K})$ and preserves polynomials of degree $p-1$, the Bramble-Hilbert lemma (see, e.g., [16, Theorem 28.1], [20, §11.3]) implies that there exists $C > 0$ such that, for $\ell \in \{1, \dots, p\}$,

$$\|\widehat{w} - \widehat{I}^{\mathrm{d}} \widehat{w}\|_{L^2(\widehat{K})} \leq C |\widehat{w}|_{H^\ell(\widehat{K})}; \tag{B.15}$$

the result (10.3) then follows from the combination of (B.13), (B.14), (B.15), and (B.8). □

It therefore remains to prove Lemmas B.1 and B.2.

Proof of Lemma B.2. We first observe that

$$\det(D\mathscr{F}_K)(D\mathscr{F}_K)^{-1}$$

is just the cofactor matrix of $D\mathscr{F}_K$. Since this is a 3×3 matrix, its entries are sum of products of pairs of elements of $D\mathscr{F}_K$. As a result, we just need to estimate terms of the form $\partial_m \mathscr{F}_K^r \partial_n \mathscr{F}_K^q$, which easily follows by the product rule:

$$\partial^\alpha (\partial_m \mathscr{F}_K^r \partial_n \mathscr{F}_K^q) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta \partial_m \mathscr{F}_K^r \partial^{\alpha-\beta} \partial_n \mathscr{F}_K^q,$$

leading to

$$\|\partial^\alpha (\det D\mathscr{F}_K (D\mathscr{F}_K)^{-1})\|_{L^\infty(\widehat{K})} \leq C \sum_{\beta \leq \alpha} |\partial^\beta (D\mathscr{F}_K)||\partial^{\alpha-\beta}(D\mathscr{F}_K)|;$$

the result then follows from the first bound in (10.1). □

We now need to describe how partial derivatives of functions are modified under the element mappings.

Lemma B.3 (Sobolev norms of composed functions)**.** *Given $m \geq 1$ there exists $C > 0$ such that if $K \in \mathcal{T}_h$ and $u \in H^\ell(K)$ then*

$$h^{3/2} |u \circ \mathscr{F}_K|_{H^\ell(\widehat{K})} \leq C \left(\frac{h}{L}\right)^{\ell} \sum_{j=1}^{\ell} L^j |u|_{H^j(K)}. \tag{B.16}$$

Proof. In this proof we denote the jth component of $\mathscr{F}_K$ ($j = 1, 2, 3$) by $\mathscr{F}_K^j$. We claim that, for any multi-index $\alpha \geq 0$,

$$\partial^\alpha (u \circ \mathscr{F}_K) = \sum_{\beta \leq \alpha} \Psi_\beta (\partial^\beta u) \circ \mathscr{F}_K, \tag{B.17}$$

where each Ψ_β is of the form

$$\Psi_\beta = \sum_{\ell=1}^{N_\beta} \prod_{j=1}^{|\beta|} \partial^{\gamma_j^\ell} \mathscr{F}_K^{\mu_j^\ell}, \tag{B.18}$$

for some integer N_β, multi-indices γ_j^ℓ with $\sum_{j=1}^{|\beta|} |\gamma_j^\ell| = |\alpha|$, and $\mu_j^\ell \in \{1, 2, 3\}$.

Once (B.17) is established, the result follows, since

$$|\Psi_\beta| \leq C L^{|\beta|} \left(\frac{h}{L}\right)^{|\alpha|}$$

by (i) the first bound in (10.1) and (ii) using (B.1) to take into account the change of variable in the $L^2(K)$ integrals.

We prove (B.17) by induction. When $|\alpha| = 1$,

$$\partial_m (u \circ \mathscr{F}_K) = \sum_{r=1}^{3} (\partial_m \mathscr{F}_K^r) \big((\partial_r u) \circ \mathscr{F}_K\big)$$

for all $m \in \{1, 2, 3\}$, and so (B.17) holds. Suppose that (B.17) holds for all α with $|\alpha| = M \geq 1$. By (B.17), (B.18), and the chain and product rules,

$$\begin{aligned}
&\partial_m \Big(\partial^\alpha (u \circ \mathscr{F}_K)\Big) \\
&= \sum_{\beta \leq \alpha} \left[\sum_{\ell=1}^{N_\beta} \partial_m \left(\prod_{j=1}^{|\beta|} \partial^{\gamma_j^\ell} \mathscr{F}_K^{\mu_j^\ell} \right) (\partial^\beta u) \circ \mathscr{F}_K + \prod_{j=1}^{|\beta|} \partial^{\gamma_j^\ell} \mathscr{F}_K^{\mu_j^\ell} \sum_{r=1}^{3} (\partial_m \mathscr{F}_K^r) \big(\partial_r (\partial^\beta u) \circ \mathscr{F}_K\big) \right] \\
&= \sum_{\beta \leq \alpha + e_m} \Psi_{\beta'} (\partial^{\beta'} u) \circ \mathscr{F}_K,
\end{aligned}$$

with $\Psi_{\beta'}$ of the form (B.18) except now $\sum_{j=1}^{|\beta|} |\gamma_j^\ell| = |\alpha| + 1$. That is, (B.17) holds for all α with $|\alpha| \leq M + 1$ and the proof is complete. □

Proof of Lemma B.1. The bound (B.7) follows from the definition of $\mathscr{F}_K^{\mathrm{c}}$ (B.2), the product rule, the first bound in (10.1), and (B.16). The bound (B.8) follows in a similar way from the definition of $\mathscr{F}_K^{\mathrm{d}}$ (B.3), the product rule, (B.9), and (B.16). □

Acknowledgements

JG was supported by EPSRC grants EP/V001760/1 and EP/V051636/1.

References

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Mathematical Methods in the Applied Sciences*, 21(9):823–864, 1998.

[2] D. N. Arnold, R. S. Falk, and R. Winther. Multigrid in $H(div)$ and $H(curl)$. *Numerische Mathematik*, 85(2):197–217, 2000.

[3] I. M. Babuška and S. A. Sauter. Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? *SIAM Review*, pages 451–484, 2000.

[4] H. Barucq, T. Chaumont-Frelet, and C. Gout. Stability analysis of heterogeneous Helmholtz problems and finite element solution based on propagation media approximation. *Math. Comp.*, 86(307):2129–2157, 2017.

[5] M. Bernkopf, T. Chaumont-Frelet, and J. M. Melenk. Stability and convergence of Galerkin discretizations of the Helmholtz equation in piecewise smooth media. *Math. Comp.*, to appear, 2024.

[6] J. Bramble and J. Pasciak. Analysis of a finite element PML approximation for the three dimensional time-harmonic Maxwell problem. *Mathematics of Computation*, 77(261):1–10, 2008.

[7] J. H. Bramble and J. Pasciak. Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems. *Mathematics of Computation*, 76(258):597–614, 2007.

[8] J. Céa. Approximation variationnelle des problèmes aux limites. *Annales de l'institut Fourier*, 14(2):345–444, 1964.

[9] T. Chaumont-Frelet and A. Ern. Asymptotic optimality of the edge finite element approximation of the time-harmonic Maxwell's equations. *arXiv preprint arXiv:2309.14189*, 2023.

[10] T. Chaumont-Frelet, S. Gallistl, D.and Nicaise, and J. Tomezyk. Wavenumber explicit convergence analysis for finite element discretizations of time-harmonic wave propagation problems with perfectly matched layers. *Communications in Mathematical Sciences*, 20(1):1–52, 2022.

[11] T. Chaumont-Frelet, A. Moiola, and E. A. Spence. Explicit bounds for the high-frequency time-harmonic Maxwell equations in heterogeneous media. *Journal de Mathématiques Pures et Appliquées*, 179:183–218, 2023.

[12] T. Chaumont-Frelet and S. Nicaise. Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problem. *IMA J. Numer. Anal.*, 40(2):1503–1543, 2020.

[13] T. Chaumont-Frelet and P. Vega. Frequency-explicit approximability estimates for time-harmonic Maxwell's equations. *Calcolo*, 59(2):22, 2022.

[14] S. Christiansen and R. Winther. Smoothed projections in finite element exterior calculus. *Mathematics of Computation*, 77(262):813–829, 2008.

[15] S. H. Christiansen. Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension. *Numerische Mathematik*, 107:87–106, 2007.

[16] P. G. Ciarlet. Basic error estimates for elliptic problems. In *Handbook of numerical analysis, Vol. II*, pages 17–351. North-Holland, Amsterdam, 1991.

[17] F. Collino and P. Monk. The perfectly matched layer in curvilinear coordinates. *SIAM Journal on Scientific Computing*, 19(6):2061–2090, 1998.

[18] M. Costabel, M. Dauge, and S. Nicaise. Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth domains. 2010. https://hal.archives-ouvertes.fr/file/index/docid/453934/filename/CoDaNi_Analytic_Part_I.pdf.

[19] Y. Du and H. Wu. Preasymptotic error analysis of higher order FEM and CIP-FEM for Helmholtz equation with high wave number. *SIAM J. Numer. Anal.*, 53(2):782–804, 2015.

[20] A. Ern and J.-L. Guermond. *Finite elements I: Approximation and interpolation*, volume 72 of *Texts in Applied Mathematics*. Springer Nature, 2021.

[21] S. Esterhazy and J. M. Melenk. On stability of discretizations of the Helmholtz equation. In I. G. Graham, T. Y. Hou, O. Lakkis, and R. Scheichl, editors, *Numerical Analysis of Multiscale Problems*, pages 285–324. Springer, 2012.

[22] X. Feng, P. Lu, and X. Xu. A hybridizable discontinuous Galerkin method for the time-harmonic Maxwell equations with high wave number. *Computational Methods in Applied Mathematics*, 16(3):429–445, 2016.

[23] X. Feng and H. Wu. Discontinuous Galerkin methods for the Helmholtz equation with large wave number. *SIAM J. Numer. Anal.*, 47(4):2872–2896, 2009.

[24] X. Feng and H. Wu. *hp*-Discontinuous Galerkin methods for the Helmholtz equation with large wave number. *Math. Comp.*, 80(276):1997–2024, 2011.

[25] X. Feng and H. Wu. An absolutely stable discontinuous Galerkin method for the indefinite time-harmonic Maxwell equations with large wave number. *SIAM Journal on Numerical Analysis*, 52(5):2356–2380, 2014.

[26] K. Friedrichs. Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren. *Mathematische Annalen*, 109(1):465–487, 1934.

[27] J. Galkowski, D. Lafontaine, E. A. Spence, and J. Wunsch. Decompositions of high-frequency Helmholtz solutions via functional calculus, and application to the finite element method. *SIAM J. Math. Anal.*, 55(4):3903–3958, 2023.

[28] J. Galkowski, D. Lafontaine, E. A. Spence, and J. Wunsch. The *hp*-FEM applied to the Helmholtz equation with PML truncation does not suffer from the pollution effect. *Comm. Math. Sci.*, to appear, 2024.

[29] J. Galkowski and E. A. Spence. Sharp preasymptotic error bounds for the Helmholtz *h*-FEM. *arXiv preprint arXiv:2301.03574*, 2023.

[30] V. Girault. Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in $\mathbb{R}^3$. *Mathematics of Computation*, 51(183):55–74, 1988.

[31] J. Gopalakrishnan and J. Pasciak. Overlapping Schwarz preconditioners for indefinite time harmonic Maxwell equations. *Mathematics of computation*, 72(241):1–15, 2003.

[32] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman, Boston, 1985.

[33] G. Grubb. *Distributions and operators*, volume 252 of *Graduate Texts in Mathematics*. Springer Science & Business Media, 2008.

[34] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237–339, 2002.

[35] T. Hohage, F. Schmidt, and L. Zschiedrich. Solving time-harmonic scattering problems based on the pole condition II: convergence of the PML method. *SIAM Journal on Mathematical Analysis*, 35(3):547–560, 2003.

[36] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number Part I: The h-version of the FEM. *Comput. Math. Appl.*, 30(9):9–37, 1995.

[37] F. Ihlenburg and I. Babuska. Finite element solution of the Helmholtz equation with high wave number part II: the *hp* version of the FEM. *SIAM J. Numer. Anal.*, 34(1):315–358, 1997.

[38] D. Lafontaine, E. A. Spence, and J. Wunsch. Wavenumber-explicit convergence of the *hp*-FEM for the full-space heterogeneous Helmholtz equation with smooth coefficients. *Comp. Math. Appl.*, 113:59–69, 2022.

[39] P. D. Lax and A. N. Milgram. Parabolic equations. *Annals of Mathematics Studies*, 33, 1954.

[40] R. Leis. *Initial Boundary Value Problems in Mathematical Physics*. Wiley, 1986.

[41] P. Lu, H. Chen, and W. Qiu. An absolutely stable *hp*-HDG method for the time-harmonic Maxwell equations with high wave number. *Mathematics of Computation*, 86(306):1553–1577, 2017.

[42] P. Lu, H. Wu, and X. Xu. Continuous interior penalty finite element methods for the time-harmonic Maxwell equation with high wave number. *Advances in Computational Mathematics*, 45:3265–3291, 2019.

[43] S. Lu and H. Wu. Preasymptotic error estimates of EEM and CIP-EEM for the time-harmonic Maxwell equations with large wave number. *arXiv preprint arXiv:2407.06784*, 2024.

[44] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, 2000.

[45] J. M. Melenk and S. Sauter. Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions. *Math. Comp*, 79(272):1871–1914, 2010.

[46] J. M. Melenk and S. Sauter. Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation. *SIAM J. Numer. Anal.*, 49:1210–1243, 2011.

[47] J. M. Melenk and S. A. Sauter. Wavenumber-explicit *hp*-FEM analysis for Maxwell's equations with transparent boundary conditions. *Foundations of Computational Mathematics*, 21:125–241, 2021.

[48] J. M. Melenk and S. A. Sauter. Wavenumber-explicit *hp*-FEM analysis for Maxwell's equations with impedance boundary conditions. *Foundations of Computational Mathematics*, pages 1–69, 2023.

[49] P. Monk. A finite element method for approximating the time-harmonic Maxwell equations. *Numerische Mathematik*, 63(1):243–261, 1992.

[50] P.. Monk. A simple proof of convergence for an edge element discretization of Maxwell's equations. In *Computational Electromagnetics: Proceedings of the GAMM Workshop on Computational Electromagnetics, Kiel, Germany, January 26–28, 2001*, pages 127–141. Springer, 2003.

[51] P. Monk. *Finite element methods for Maxwell's equations.* Oxford University Press, 2003.

[52] J.-C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numerische Mathematik*, 35:315–341, 1980.

[53] J.-C. Nédélec. A new family of mixed finite elements in $\mathbb{R}^3$. *Numerische Mathematik*, 50:57–81, 1986.

[54] S. Nicaise and J. Tomezyk. The time-harmonic Maxwell equations with impedance boundary conditions in polyhedral domains. *Maxwell's Equations: Analysis and Numerics, Radon Series on Computational and Applied Mathematics*, 24:285–340, 2019.

[55] S. Nicaise and J. Tomezyk. Convergence analysis of a *hp*-finite element approximation of the time-harmonic Maxwell equations with impedance boundary conditions in domains with an analytic boundary. *Numerical Methods for Partial Differential Equations*, 36(6):1868–1903, 2020.

[56] O. R. Pembery. *The Helmholtz Equation in Heterogeneous and Random Media: Analysis and Numerics.* PhD thesis, University of Bath, 2020. https://researchportal.bath.ac.uk/en/studentTheses/the-helmholtz-equation-in-heterogeneous-and-random-media-analysis.

[57] R. Picard. An elementary proof for a compact imbedding result in generalized electromagnetic theory. *Mathematische Zeitschrift*, 187(2):151–164, 1984.

[58] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. I: Functional Analysis.* Academic Press, Inc. New York, second edition, 1980.

[59] S. A. Sauter. A refined finite element convergence theory for highly indefinite Helmholtz problems. *Computing*, 78(2):101–115, 2006.

[60] A. H. Schatz. An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Math. Comp.*, 28(128):959–962, 1974.

[61] J. Schöberl. Commuting quasi-interpolation operators for mixed finite elements. *Preprint ISC-01-10-MATH, Texas A&M University, College Station, TX*, 2001.

[62] C. Weber. A local compactness theorem for Maxwell's equations. *Mathematical Methods in the Applied Sciences*, 2(1):12–25, 1980.

[63] C. Weber. Regularity theorems for Maxwell's equations. *Mathematical Methods in the Applied Sciences*, 3(1):523–536, 1981.

[64] N. Weck. Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries. *Journal of Mathematical Analysis and Applications*, 46(2):410–437, 1974.

[65] H. Wu. Pre-asymptotic error analysis of CIP-FEM and FEM for the Helmholtz equation with high wave number. Part I: linear version. *IMA J. Numer. Anal.*, 34(3):1266–1288, 2014.

[66] L. Zhu and H. Wu. Preasymptotic error analysis of CIP-FEM and FEM for Helmholtz equation with high wave number. Part II: *hp* version. *SIAM J. Numer. Anal.*, 51(3):1828–1852, 2013.