ARBITRARILY SMALL PERTURBATIONS OF DIRICHLET LAPLACIANS ARE QUANTUM UNIQUE ERGODIC

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ABSTRACT. Given an Euclidean domain with very mild regularity properties, we prove that there exist arbitrarily small perturbations of the Dirichlet Laplacian of the form $-(I + S_\epsilon)\Delta$ with $\|S_\epsilon\|_{L^2\to L^2} \le \epsilon$ whose high energy eigenfunctions are quantum unique ergodic (QUE). Moreover, if we impose stronger regularity on the domain, the same result holds with $\|S_\epsilon\|_{L^2\to H^\gamma} \le \epsilon$ for $\gamma > 0$ depending on the domain. The method of proof is entirely probabilistic. A local Weyl law for domains with rough boundaries is obtained as a byproduct of the proof.

1. INTRODUCTION

In quantum mechanics, the Laplace operator on a manifold describes the behavior of a free particle confined to the manifold. The eigenvalues of the Laplacian (under suitable boundary conditions) are the possible values of the energy of the particle and the eigenfunctions are the energy eigenstates. The square of an energy eigenstate gives the probability density function for the location of a particle with the given energy.

The subject of quantum chaos connects the properties of high energy eigenstates with the chaotic properties of classical particles. The fundamental result is the quantum ergodicity theorem due to Šnirel′man (1974), Colin de Verdière (1985), and Zelditch (1987) on manifolds without boundary and generalized to manifolds with boundary by Gérard and Leichtnam (1993) and Zelditch and Zworski (1993). The theorem states that if the classical dynamics within a manifold are ergodic, then the microlocal lifts of almost all high energy eigenfunctions (in any orthonormal basis of eigenfunctions) equidistribute in phase space. This phenomenon is known as *quantum ergodicity.*

The question of whether all (rather than almost all) high energy eigenfunctions equidistribute in phase space has remained open. This property was christened *quantum unique ergodicity* by Rudnick and Sarnak (1994), who conjectured that the Laplacian on any compact negatively curved manifold is quantum unique ergodic (QUE). The conjecture was motivated by the fact that classical particles on compact negatively curved manifolds are known to be strongly chaotic. Although the Rudnick–Sarnak conjecture is still open, it is now known that quantum unique ergodicity is not always valid, even if classical particles are chaotic; see Faure and Nonnenmacher (2004), Faure, Nonnenmacher and De Bièvre (2003) and Hassell (2010). QUE has been verified in only a handful of cases where exact computations are possible; in particular for the Hecke orthonormal basis by Lindenstrauss (2006), Silberman and Venkatesh (2007) and Holowinsky and Soundararajan (2010). Anantharaman (2008) made partial progress towards the general Rudnick–Sarnak conjecture by showing that high energy Laplace eigenfunctions on compact negatively curved manifolds

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have positive entropy. For a more comprehensive survey of results on quantum unique ergodicity, see Sarnak (2011). For more on quantum ergodicity and semiclassical chaos, see Zelditch (2010).

In spite of the availability of counterexamples to QUE, it is believed that QUE is generically valid for domains with ergodic billiard ball flow (see Sarnak (2011)). In other words, QUE is expected to be true for almost all ergodic domains. There are at present no results like this.

The main result of this paper (Theorem 2.3) says that for any Euclidean domain satisfying some very mild regularity conditions, there exists $S_\epsilon : L^2 \to L^2$ with $\|S_\epsilon\|_{L^2\to L^2} \le \epsilon$ such that the perturbation of the Laplacian (with Dirichlet boundary condition) $-(I + S_\epsilon)\Delta$ has QUE eigenfunctions. In other words, Dirichlet Laplacians lie in the closure (in the $H^2 \to L^2$ norm topology) of the set of operators with QUE eigenfunctions. If we impose more regularity on the domain, then we can take $\|S_\epsilon\|_{L^2\to H^\gamma} \le \epsilon$ for some $\gamma > 0$. Thus, we can take this closure in the $H^2 \to H^\gamma$ norm topology. Furthermore, under a certain dynamical condition, we can take $\gamma = 1$. The required operator is constructed using a probabilistic method (described briefly in Section 2.6) and it is then shown that this random operator satisfies the required property with probability one. Notice that, although we show that Laplacians are close in the operator norm to QUE operators, this is very far from showing that one can perturb the domain to obtain a QUE Laplacian. Indeed, one should probably not expect such a result to hold for arbitrary domains.

Our result is closely related to those in Zelditch (1992, 1996, 2014), Maples (2013) and Chang (2015) where it is shown that certain unitary randomizations of eigenfunctions are quantum ergodic. In effect, this shows that $-U_k\Delta U_k^*$ is quantum ergodic for U_k a random unitary operator. See Section 2.5 for a more detailed comparison of the results.

2. Results

2.1. **Definitions.** Take any $d \ge 2$ and let Ω be a Borel subset of $\mathbb{R}^d$. Let B_t be a standard d-dimensional Brownian motion, started at some point $x \in \mathbb{R}^d$. The exit time of B_t from Ω is defined as

$$\tau_\Omega := \inf\{t > 0 : B_t \notin \Omega\}\,. \tag{2.1}$$

In this paper we will say that Ω is a *regular domain* if it is nonempty, bounded, open, connected, and satisfies the following boundary regularity conditions:

(i) $\mathrm{Vol}(\partial\Omega) = 0$, where $\partial\Omega$ is the boundary of Ω and Vol denotes Lebesgue measure.
(ii) For any $x \in \partial\Omega$, $\mathbb{P}^x(\tau_\Omega = 0) = 1$, where $\mathbb{P}^x$ denotes the law of Brownian motion started at x and τ_Ω is the exit time from Ω.

Condition (ii) may look strange to someone who does not have a background in probabilistic potential theory, but it is actually the well-known sharp condition for the existence of solutions to Dirichlet problems on Ω (see page 225 in Mörters and Peres (2010)). It is not hard to see that if the boundary of Ω is smooth enough, then Ω is a regular domain (see page 224 in Mörters and Peres (2010)). Henceforth, we will assume that Ω is a regular domain and $\overline{\Omega}$ will denote the closure of Ω.

Given any measurable function $f : \overline{\Omega} \to \mathbb{C}$, we denote by $\|f\|$ the $L^2(\Omega)$ norm of f. For such f there is a natural probability measure associated with f that has density $|f(x)|^2$ with respect to Lebesgue measure on $\overline{\Omega}$. We will denote this measure as ν_f. Note that in the definition of $\|f\|$ it does not matter whether we integrate over Ω or $\overline{\Omega}$ since $\mathrm{Vol}(\partial\Omega) = 0$. We will denote the L^2 inner product of two functions f and g by $\langle f, g\rangle$.

Recall that a sequence of probability measures $\{\mu_n\}_{n\ge1}$ on $\overline{\Omega}$ is said to converge weakly to a probability measure μ if

$$\lim_{n\to\infty} \int_{\overline{\Omega}} f d\mu_n = \int_{\overline{\Omega}} f d\mu$$

for every bounded continuous function $f : \overline{\Omega} \to \mathbb{R}$. A probability measure that will be of particular importance in this paper is the uniform probability measure on $\overline{\Omega}$. This is simply the restriction of Lebesgue measure to $\overline{\Omega}$, normalized to have total mass one.

For every bounded sequence of functions $\{f_n\} \in L^2(\mathbb{R}^d)$, we can also associate a family of measures in phase space, $S^*\mathbb{R}^d$ (the cosphere bundle of $\mathbb{R}^d$), called *defect measures*, defined as follows. Recall the notation $\Psi^m(\mathbb{R}^d)$ for the pseudodifferential operators of order m on $\mathbb{R}^d$ and S^m_{phg} for the associated polyhomgeneous symbol classes (see for example Hörmander (2007)). Let

$$\sigma : \Psi^m \to S^m_{\mathrm{phg}}/S^{m-1}_{\mathrm{phg}}$$

be the symbol map from $\Psi^m(\mathbb{R}^d)$ to the set of functions homogeneous in ξ of degree m. Let $\chi \in C^\infty_c(\mathbb{R}^d)$ have $\chi \equiv 1$ in a neighborhood of 0. For $a \in C^\infty_c(S^*\mathbb{R}^d)$, let

$$\tilde{a}(x,\xi) = a(x, \xi/|\xi|)(1-\chi(\xi))\,.$$

Define the distribution $\mu_n \in \mathcal{D}'(S^*\mathbb{R}^d)$ by

$$\mu_n(a) = \langle \tilde{a}(x,D)f_n, f_n\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$ and D is the gradient operator. Not that the L^2 boundedness of $\tilde{a}(x,D)$ implies that for every subsequence of $\{\mu_n\}_{n\geq 1}$ there is a further subsequence that converges in the $\mathcal{D}'(S^*\mathbb{R}^d)$ topology. Moreover, it can be shown that every limit point μ of $\{\mu_n\}_{n\geq 1}$ in the $\mathcal{D}'(S^*\mathbb{R}^d)$ topology is a positive radon measure, with the property that there exists a subsequence $\{f_{n_k}\}_{k\geq 1}$ so that for all $A \in \Psi(\mathbb{R}^d)$ with symbol compactly supported in x,

$$\langle Af_{n_k}, f_{n_k}\rangle \to \int_{S^*\mathbb{R}^d} \sigma(A)d\mu\,,$$

where $\sigma(A)$ is the principal symbol of A. (See for example Burq (1997) or Chapter 5 of Zworski (2012).) The set of such limit points μ is denoted by $\mathcal{M}(\{f_n\}_{n\geq 1})$ and is called the *set of defect measures associated to the family* $\{f_n\}_{n\geq 1}$. We will write $\mathcal{M}(f_n)$ instead of $\mathcal{M}(\{f_n\}_{n\geq 1})$ to simplify notation. Note that while μ_n depends on the exact quantization procedure used to define $a(x,D)$ and the function χ, the set $\mathcal{M}(f_n)$ is independent of such choices.

If H is a positive linear operator from a subspace of $L^2(\overline{\Omega})$ into $L^2(\overline{\Omega})$, we will say that a function f belonging to the domain of H is an eigenfunction of H with eigenvalue λ if $f \neq 0$ and $Hf = \lambda f$. We will say that an eigenfunction f is normalized if $\|f\| = 1$.

Definition 2.1. Let H be a linear operator from some subspace of $L^2(\overline{\Omega})$ into $L^2(\overline{\Omega})$. We say that H has QUE eigenfunctions if for any sequence normalized eigenfunctions $\{f_n\}_{n\geq 1}$ of H,

$$\mathcal{M}(1_{\overline{\Omega}}f_n) = \left\{\frac{1}{\mathrm{Vol}(\Omega)}1_{\overline{\Omega}}dxd\sigma(\xi)\right\} \tag{2.2}$$

where σ is the normalized surface measure on S^{d-1}.

In particular, notice that if (2.2) holds then for all $A \in \Psi^0(\mathbb{R}^d)$,

$$\left\langle A1_{\overline{\Omega}}f_n, 1_{\overline{\Omega}}f_n\right\rangle \to \frac{1}{\mathrm{Vol}(\Omega)}\int_{S^*\mathbb{R}^d}\sigma(A)1_{\overline{\Omega}}dxd\sigma(\xi)$$

and hence that ν_{f_n} converges weakly as a measure to the uniform probability distribution on Ω. With this in mind, we define the weaker notion of equidistribution as follows.

Definition 2.2. Let H be a linear operator from some subspace of $L^2(\overline{\Omega})$ into $L^2(\overline{\Omega})$. We say that H has uniquely equidistributed eigenfunctions if for $\{f_n\}_{n\geq 1}$ any sequence normalized eigenfunctions of H, ν_{f_n} converges weakly to the uniform probability distribution on Ω.

2.2. **The main result.** Let $-\Delta$ be the Dirichlet Laplacian on Ω with domain $\mathcal{F}_\Delta$ (defined in Section 3.3). The following theorem is the main result of this paper.

Theorem 2.3. *Let Ω be a regular domain. Then for any $\epsilon > 0$, there exists a linear operator $S_\epsilon : L^2(\overline{\Omega}) \to L^2(\overline{\Omega})$ such that:*

(i) $\|S_\epsilon\|_{L^2\to L^2} \le \epsilon$.
(ii) $-(I + S_\epsilon)\Delta$ *is a positive operator on* $\mathcal{F}_\Delta$.
(iii) *There is a complete orthonormal basis of $L^2(\overline{\Omega})$ consisting of a sequence of eigenfunctions of $-(I + S_\epsilon)\Delta$ that belong to $\mathcal{F}_\Delta$, and the corresponding eigenvalues can be ordered as $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$.*
(iv) $-(I + S_\epsilon)\Delta$ *has QUE eigenfunctions in the sense of Definition* 2.2.

If Ω has smooth boundary, then for all $\gamma < 1$, there exist such an $S_\epsilon : L^2(\overline{\Omega}) \to H^\gamma(\overline{\Omega})$ with $\|S_\epsilon\|_{L^2\to H^\gamma} \le \epsilon$. Moreover, if Ω has smooth boundary and the set of periodic billiards trajectories has measure zero (see Section 3.4*), then this holds for $\gamma \le 1$.*

It would be interesting to see if a different version of this theorem can be proved, where instead of perturbing the Laplacian, it is the domain Ω that is perturbed. Alternatively, one can try to perturb the Laplacian by some explicit kernel rather than saying that 'there exists S_ϵ'. Yet another possible improvement would be to show that a generic perturbation, rather than a specific one, results in an operator with QUE eigenfunctions. Indeed, the proof of Theorem 2.3 gets quite close to this goal.

2.3. **Additional results.** The techniques of this paper yield the following version of the local Weyl law for regular domains.

Theorem 2.4. *Suppose that $\Omega \subset \mathbb{R}^d$ is a regular domain, where regularity is defined at the beginning of this section. Let $\{(u_j, \mu_j)\}_{j\ge1}$ be a complete orthonormal basis of eigenfunctions of the Dirichlet Laplacian on Ω. Then for $A \in \Psi(\mathbb{R}^d)$ with $\sigma(A)$ supported in a compact subset of Ω and any $E > 1$,*

$$\sum_{\mu_j\in[\mu,\mu E]} \langle A 1_{\overline{\Omega}} u_j, 1_{\overline{\Omega}} u_j\rangle = \frac{\mu^d}{(2\pi)^d} \iint_{1\le|\xi|\le E} \sigma(A) 1_{\overline{\Omega}} dx d\xi + o(\mu^d).$$

In order to state the next theorem, we need the following definition.

Definition 2.5. Let $C_0(S^*\Omega)$ be the set of continuous functions on $S^*\mathbb{R}^d$ vanishing on $(\mathbb{R}^d \setminus \Omega) \times S^{d-1}$. Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be nonincreasing. Let $\{(u_j, \mu_j)\}_{j\ge1}$ be a complete orthonormal basis of eigenfunctions of the Dirichlet Laplacian on Ω. Suppose that there exists $\mathcal{A} \subset \Psi(\mathbb{R}^d)$ with

$$\sigma(\mathcal{A}) := \{\sigma(A)|_{S^*\mathbb{R}^d} : A \in \mathcal{A}\}$$

dense in $C_0(S^*\Omega)$, such that for each $A \in \mathcal{A} \subset \Psi(\mathbb{R}^d)$,

$$\sum_{\mu_j\in[\mu,\mu(1+\alpha(\mu))]} \langle A 1_{\overline{\Omega}} u_j, 1_{\overline{\Omega}} u_j\rangle = \frac{\mu^d}{(2\pi)^d} \iint_{1\le|\xi|\le 1+\alpha(\mu)} \sigma(A) 1_{\overline{\Omega}} dx d\xi + o(\alpha(\mu)\mu^d)$$

where $\sigma(A)$ is the principal symbol of A. In this circumstance, we will say that the domain Ω is *average quantum ergodic (AQE) at scale α.*

Theorem 2.4 implies that regular domains Ω are AQE at scale E for any $E > 0$. In Section 3.1, we recall Weyl laws holding on smoother domains which imply that domains with smoother boundaries are AQE at scale $\alpha(\mu) = o(1)$. For $\gamma \in [0, 2]$, let $\mathcal{F}_\Delta^\gamma$ denote the complex interpolation space $(L^2(\Omega), \mathcal{F}_\Delta)_{\gamma/2}$. Then the following theorem implies Theorem 2.3.

Theorem 2.6. *Suppose that Ω is a regular domain that is AQE at scale $\alpha(\mu) = \mathrm{O}(\mu^{-\gamma})$ for some $2 \geq \gamma \geq 0$. Then for any $\epsilon > 0$, there exists a linear operator $S_\epsilon : L^2(\overline{\Omega}) \to \mathcal{F}^\gamma_\Delta$ such that:*

- (i) $\|S_\epsilon\|_{L^2 \to \mathcal{F}^\gamma_\Delta} \leq \epsilon$.
- (ii) $-(I + S_\epsilon)\Delta$ *is a positive operator on* $\mathcal{F}_\Delta$.
- (iii) *There is a complete orthonormal basis of $L^2(\overline{\Omega})$ consisting of a sequence of eigenfunctions of $-(I + S_\epsilon)\Delta$ that belong to $\mathcal{F}_\Delta$, and the corresponding eigenvalues can be ordered as $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$.*
- (iv) $-(I + S_\epsilon)\Delta$ *has QUE eigenfunctions in the sense of Definition* 2.2.

A consequence of Theorem 2.3 is that $-\Delta$ has a sequence of 'almost-eigenfunctions' that are equidistributed in the limit. This is the content of the following corollaries.

Corollary 2.7. *Let all notation be as in Theorem* 2.3. *Suppose that Ω is AQE at scale $\alpha(\mu) = \mathrm{O}(\mu^\gamma)$ for some $\gamma \geq 0$. Then there is a sequence of functions $\{f_n\}_{n\geq 1}$ belonging to $\mathcal{F}_\Delta$ and a sequence of positive real numbers $\{\alpha_n\}_{n\geq 1}$ such that $\|f_n\| = 1$, $\alpha_n \to \infty$,*

$$(-\alpha_n^{-2}\Delta - 1)f_n = \mathrm{o}_{L^2}(\alpha_n^{-\gamma}),$$

and

$$\mathcal{M}(f_n) = \left\{\frac{1}{\mathrm{Vol}(\Omega)} 1_{\overline{\Omega}} dx d\sigma(\xi)\right\}.$$

Moreover, when Ω is AQE at some scale $\alpha(\mu) = \mathrm{o}(1)$, then there is a full orthonormal basis of (slightly weaker) quasimodes that are QUE. In particular,

Corollary 2.8. *Let all notation be as in Theorem* 2.3. *Suppose that Ω is AQE at scale $\alpha(\mu) = \mathrm{O}(\mu^\gamma)$ for some $\gamma > 0$. Then there is an orthonormal basis of L^2, $\{f_n\}_{n\geq 1}$, belonging to $\mathcal{F}_\Delta$ and a sequence of positive real numbers $\{\alpha_n\}_{n\geq 1}$ such that $\|f_n\| = 1$, $\alpha_n \to \infty$,*

$$(-\alpha_n^{-2}\Delta - 1)f_n = \mathrm{O}_{L^2}(\alpha_n^{-\gamma}),$$

and

$$\mathcal{M}(f_n) = \left\{\frac{1}{\mathrm{Vol}(\Omega)} 1_{\overline{\Omega}} dx d\sigma(\xi)\right\}.$$

2.4. **Improvements on closed manifolds.** Together with the analog of Theorem 2.3, a stronger version of the Weyl law valid on compact manifolds without boundary (see Section 3.1), implies the following corollary.

Corollary 2.9. *Let (M, g) be a compact Riemannian manifold without boundary so that the set of closed geodesics has measure 0. Then there is an orthonormal basis of functions $\{f_n\}_{n\geq 1}$ belonging to $C^\infty(M)$ and a sequence of positive real numbers $\{\alpha_n\}_{n\geq 1}$ such that $\|f_n\| = 1$ for each n, $\alpha_n \to \infty$,*

$$(-\alpha_n^{-2}\Delta_g - 1)f_n = \mathrm{o}_{L^2}(\alpha_n^{-1}),$$

and $\nu_{f_n} \to \frac{1}{\mathrm{Vol}(M)}dx$. That is, f_n are uniquely equidistributed.

Remark 2.10. Notice that if M has ergodic geodesic flow, then the set of periodic geodesics has measure zero and hence Corollary 2.9 applies and there is an orthonormal basis of $\mathrm{o}_{L^2}(\alpha^{-1})$ quasimodes that are equidistributed. In particular, being $\mathrm{o}_{L^2}(\alpha^{-1})$ quasimodes implies that these functions respect the dynamics at the level of defect measures, that is, defect measures associated to the family of quasimodes are invariant under the geodesic flow. See Burq (1997) or Chapter 5 of Zworski (2012) for details.

2.5. **Comparison with previous results.** One can view the results here as a companion to those in Zelditch (1992, 1996, 2014) and Maples (2013). In these papers, the authors work on a compact manifold M and fix a basis of eigenfunctions of the Laplacian, $\{u_n\}_{n=1}^{\infty}$. Their results then show that for almost every block diagonal unitary operator

$$U = \oplus_{k=1}^{\infty} U_k$$

(with respect to the product Haar measure) such that for all k $\dim \operatorname{Ran} U_k < \infty$ and $\dim \operatorname{Ran} U_k \to \infty$ at least polynomially in k, the basis $\{Uu_n\}_{n=1}^{\infty}$ has

$$\mathcal{M}(Uu_n) = \left\{ \frac{1}{\operatorname{Vol}(M)} dx d\sigma(\xi) \right\}.$$

The recent work of Chang (2015) adapts the results of Bourgade and Yau (2013) in order to generalize the measure used in the construction of the U_k (from Haar measure to Wigner meaures) when $M = S^2$ is the sphere.

One reformulation considers a certain basis of eigenfunctions for the operator $-U\Delta U^*$. By taking U_k close to the identity, we may write

$$\tilde{P} := -U\Delta U^* = -(I + \tilde{S})\Delta$$

where $\tilde{S}$ is small in $L^2 \to L^2$ norm. While $\tilde{P}$ is quantum ergodic, it may not be QUE if there is high multiplicity in the spectrum of $-\Delta$.

One can think of the results in the present paper as replacing the U_k by some nearly unitary operator. By choosing these operators carefully, and employing the Hanson–Wright inequality in place of the law of large numbers, we are able to use smaller windows than those in previous work, to prove that the perturbation is regularizing under various conditions, and to show that the resulting operator is QUE.

2.6. **Outline of the proof and organization of the paper.** In order to prove Theorem 2.3, we first prove the local Weyl law for regular domains (Theorem 2.4). The key ingredient here is to compare the heat trace for the Dirichlet Laplacian on Ω with the heat trace for the Laplacian on $\mathbb{R}^d$ as in Gérard and Leichtnam (1993). Let $k(t,x,y)$ and $k_D(t,x,y)$ be respectively the kernels of $e^{t\Delta}$ and $e^{t\Delta_D}$, where Δ is the free Laplacian and Δ_D the Dirichlet Laplacian. The key estimate in proving Theorem 2.4 is

$$|\partial_x^\alpha (k(t,x,y) - k_\Delta(t,x,y))| \le C_\delta t^{-N_\alpha} e^{-c_\delta / t}, \qquad d(x, \partial\Omega) > \delta.$$

We prove this estimate using the relationship between killed Brownian motion on Ω with the Dirichlet heat Laplacian together with the fact that Brownian motion has independent increments. Because of this approach, we are able to complete the proof on domains which are only regular.

The next step is to show that a local Weyl law with a certain window implies the existence of the desired perturbation S_ϵ. The local Weyl law essentially says that when averaged over a certain size window, say $\lambda^{-\gamma}$, eigenfunctions are uniquely ergodic. In Section 5 we give a rigorous meaning to this statement. In particular, we use a modern version of the Hanson–Wright inequality from Rudelson and Vershynin (2013) (see Hanson and Wright (1971) for the original) to show that random rotations (with respect to Haar measure) of small groups of eigenfunctions are uniquely ergodic. Here, the size of the group allowed depends on the remainder in the local Weyl law. Thus, the smaller the remainder, the smaller the required group of eigenfunctions.

In Section 6, we obtain the perturbation, S_ϵ. In order to do this, we make a two scale partition of the eigenvalues, $\lambda_i = \mu_i^2$, of the Laplacian. In particular, we divide the eigenvalues of the Laplacian into

$$L_{n,j} := \left\{ \lambda_i \mid (1+\epsilon)^n \left(1 + \frac{\epsilon j}{\lceil (1+\epsilon)^{n\gamma} \rceil}\right) \leq \lambda_i < (1+\epsilon)^n \left(1 + \frac{\epsilon (j+1)}{\lceil (1+\epsilon)^{n\gamma} \rceil}\right) \right\},$$
$$0 \leq n, \quad 0 \leq j \leq \lceil (1+\epsilon)^{n\gamma} \rceil - 1$$

where γ is determined by the remainder in the local Weyl law. For each $L_{n,j}$ we then make a random rotation of the corresponding eigenfunctions and reassign the eigenvalues so that each new eigenvalue, λ_i' is simple and has

$$(1+\epsilon)^n \left(1 + \frac{\epsilon j}{\lceil (1+\epsilon)^{n\gamma} \rceil}\right) \leq \lambda_i' \leq (1+\epsilon)^n \left(1 + \frac{\epsilon (j+1)}{\lceil (1+\epsilon)^{n\gamma} \rceil}\right).$$

Because of the fact that random rotations of eigenfunctions on the scale $\lambda^{-\gamma}$ are QUE, this results in an operator that is almost surely QUE. The regularizing nature of the perturbation results from the second scale in $L_{n,j}$. In particular, the larger γ, the more regularizing the perturbation.

The paper is organized as follows. Section 3 recalls local Weyl laws valid for more regular domains, the functional analytic definition of the Dirichlet Laplacian, and some geometric preliminaries. Section 4 contains the proof of Theorem 2.4. Section 5 presents the results on random rotations of eigenfunctions. Finally, Section 6 finishes the proof of Theorems 2.3, 2.6 and Corollary 2.7. Section 7 contains the adjustments necessary to obtain the improvements on manifolds without boundary.

3. Preliminaries

3.1. **Local Weyl Laws.** We first recall some now classical local Weyl laws for domains Ω more regular than those in Theorem 2.4. In this setting, we have the following version of the local Weyl law (Duistermaat and Guillemin (1975), Safarov and Vassiliev (1997)).

Theorem 3.1. *Suppose that Ω has smooth boundary. Let $\{(u_j, \mu_j)\}_{j\geq 1}$ be a complete orthonormal basis of eigenfunctions of the (Dirichlet) Laplacian on Ω. Then for $A \in \Psi(\mathbb{R}^d)$ with A having kernel supported in a compact subset of $\Omega \times \Omega$,*

$$\sum_{\mu_j \in [\mu, \mu E]} \langle A 1_{\overline{\Omega}} u_j, 1_{\overline{\Omega}} u_j \rangle = \frac{\mu^d}{(2\pi)^d} \iint_{1 \leq |\xi| \leq E} \sigma(A) 1_{\overline{\Omega}} dx d\xi + \mathrm{O}(\mu^{d-1}).$$

In particular, Ω is AQE at scale $\mu^{-\gamma}$ for any $\gamma < 1$. Moreover if the set of closed trajectories for the billiard flow has measure zero, then

$$\sum_{\mu_j \in [\mu, \mu(1+\mu^{-1})]} \langle A 1_{\overline{\Omega}} u_j, 1_{\overline{\Omega}} u_j \rangle = \frac{\mu^d}{(2\pi)^d} \iint_{1 \leq |\xi| \leq 1+\mu^{-1}} \sigma(A) 1_{\overline{\Omega}} dx d\xi + \mathrm{o}(\mu^{d-1}).$$

In particular, Ω is AQE at scale μ^{-1}.

3.2. **Manifolds without boundary.** Let (M, g) be compact Riemannian manifold without boundary. Then the Laplace operator is given in local coordinates by

$$-\Delta_g := \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial j)$$

where $|g| = \det g_{ij}$ and $g(\partial_{x_i}, \partial_{x_j}) = g_{ij}$ with inverse g^{ij}. The operator $-\Delta_g$ has domain $H^2(M)$ and is invertible as an operator $L^2_m(M) \to H^2_m(M)$ where $B_m(M)$ is the set of functions in B with 0 mean. In this setting, we have the following version of the local Weyl law Safarov and Vassiliev (1997).

Theorem 3.2. *Let $\{(\phi_j,\mu_j)\}_{j\geq 1}$ be the eigenfunctions of $-\Delta_g$. Then*

$$\sum_{\mu_j\leq\mu}|\phi_j(x)|^2=\frac{\mu^d}{(2\pi)^d}\int_{S^*_xM}d\xi+\mathrm{O}(\mu^{d-1})$$

and if the set of closed geodesics has zero measure, then $\mathrm{O}(\mu^{d-1})$ can be replaced by $\mathrm{o}(\mu^{d-1})$. Moreover, the asymptotics are uniform for $x\in M$.

3.3. **Functional Analysis.** Recall our convention that $\|f\|$ denotes the L^2 norm of a function f and $\langle f,g\rangle$ denotes the L^2 inner product of f and g. We now recall the definition of the Dirichlet Laplacian as a self adjoint unbounded operator on $L^2(\Omega)$. Let $H_0^1(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ with respect to the H^1 norm where for $k\in\mathbb{N}$,

$$\|u\|^2_{H^k(\Omega)}:=\sum_{|\alpha|\leq k}\|\partial^\alpha u\|^2.$$

Here for a multiindex $\alpha\in\mathbb{N}^d$,

$$\partial^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\dots\partial_{x_d}^{\alpha_d},\qquad |\alpha|=\alpha_1+\alpha_2+\dots\alpha_d.$$

Then $H_0^1(\Omega)$ is a Hilbert space with inner product

$$(u,v)=\langle u,v\rangle+\langle\nabla u,\nabla v\rangle\,.$$

Define the quadratic form $Q:H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{C}$ by

$$Q(u,v)=\langle\nabla u,\nabla v\rangle\,.$$

Then Q is a symmetric, densely defined quadratic form and for $u,v\in H_0^1(\Omega)$,

$$|Q(u,v)|\leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)},\qquad c\|u\|^2_{H^1(\Omega)}\leq Q(u,u)+C\|u\|^2.$$

Therefore by Theorem VIII.15 of Reed and Simon (1980), Q is defined by a unique self-adjoint operator $-\Delta$ with domain

$$\mathcal{F}_\Delta:=\{u\in H_0^1: Q(u,w)\leq C_u\|w\|\text{ for all }w\in H_0^1(\Omega)\}\,.$$

This operator is called the *Dirichlet Laplacian.* We recall that:

Lemma 3.3. *Suppose that Ω has C^2 boundary. Then $\mathcal{F}_\Delta=H_0^1(\Omega)\cap H^2(\Omega)$ and in particular $(L^2,\mathcal{F}_\Delta)_\theta\subset H^{2\theta}(\Omega)$.*

3.4. **The billiard flow.** Let Ω be a regular domain with smooth boundary. We now define the billiard flow. Let $S^*\mathbb{R}^d$ be the unit sphere bundle of $\mathbb{R}^d$. We write

$$S^*\mathbb{R}^d|_{\partial\Omega}=\partial\Omega_+\sqcup\partial\Omega_-\sqcup\partial\Omega_0$$

where $(x,\xi)\in\partial\Omega_+$ if ξ is pointing out of Ω, $(x,\xi)\in\partial\Omega_-$ if it points inward, and $(x,\xi)\in\partial\Omega_0$ if $(x,\xi)\in S^*\partial\Omega$. The points $(x,\xi)\in\partial\Omega_0$ are called *glancing* points. Let $B^*\partial\Omega$ be the unit coball bundle of $\partial\Omega$ and denote by $\pi_\pm:\partial\Omega_\pm\to B^*\partial\Omega$ and $\pi:S^*\mathbb{R}^d|_{\partial\Omega}\to\overline{B^*\partial\Omega}$ the canonical projections onto $\overline{B^*\partial\Omega}$. Then the maps $\pi_\pm$ are invertible. Finally, write

$$t_0(x,\xi)=\inf\{t>0:\exp_t(x,\xi)\in T^*\mathbb{R}^d|_{\partial\Omega}\}$$

where $\exp_t(x,\xi)$ denotes the lift of the geodesic flow to the cotangent bundle. That is, t_0 is the first positive time at which the geodesic starting at (x,ξ) intersects $\partial\Omega$.

We define the billiard flow as in Appendix A of Dyatlov and Zworski (2013). Without loss of generality, we assume $t_0>0$. Fix $(x,\xi)\in S^*\mathbb{R}^d$ and denote $t_0=t_0(x,\xi)$. If $\exp_{t_0}(x,\xi)\in\partial\Omega_0$, then the billiard flow cannot be continued past t_0. Otherwise there are two cases: $\exp_{t_0}(x,\xi) \in \partial\Omega_+$ or $\exp_{t_0}(x,\xi) \in \partial\Omega_-$. We let

$$(x_0,\xi_0) = \begin{cases} \pi_-^{-1}(\pi_+(\exp_{t_0}(x,\xi))) \in \partial\Omega_-\,, & \text{if } \exp_{t_0}(x,\xi) \in \partial\Omega_+ \\ \pi_+^{-1}(\pi_-(\exp_{t_0}(x,\xi))) \in \partial\Omega_+\,, & \text{if } \exp_{t_0}(x,\xi) \in \partial\Omega_- \end{cases}.$$

We then define $\varphi_t(x,\xi)$, the *billiard flow*, inductively by putting

$$\varphi_t(x,\xi) = \begin{cases} \exp_t(x,\xi) & 0 \le t < t_0, \\ \varphi_{t-t_0}(x_0,\xi_0) & t \ge t_0. \end{cases}$$

We say that the trajectory starting at $(x,\xi) \in S^*\mathbb{R}^d$ is *periodic* if there exists $t > 0$ such that $\varphi_t(x,\xi) = (x,\xi)$.

4. A local Weyl law on regular domains

Throughout this section and all subsequent sections, we will adopt the notation that C denotes any positive constant that may depend only on the set Ω, the dimension d, and nothing else. The value of C may change from line to line. In case we need to deal with multiple constants, they will be denoted by $C_1, C_2, \ldots$. From this point forward we will assume that

$$\mathrm{Vol}(\Omega) = 1\,.$$

This does not result in any loss of generality since we may always rescale Ω with positive volume to have unit volume. Let B_t be a standard d-dimensional Brownian motion, starting at some point $x \in \Omega$. Recall the definition (2.1) of the exit time τ_Ω from the domain Ω. We will need a few well-known facts about this exit time, summarized in the following theorem.

Theorem 4.1 (Compiled from Proposition 4.7 and Theorems 4.12 and 4.13 of Chapter II in Bass (1995) and Section 4 of Chapter 2 in Port and Stone (1978))**.** *For any regular domain Ω (as defined in Section 2), there exists a unique function $p : (0,\infty) \times \overline{\Omega} \times \overline{\Omega} \to [0,\infty)$ such that:*

(i) *For any bounded Borel measurable $f : \Omega \to \mathbb{R}$ and $x \in \Omega$,*

$$\mathbb{E}^x(f(B_t); t < \tau_\Omega) = \int_\Omega p(t,x,y) f(y)\, dy\,,$$

where $\mathbb{E}^x$ denotes expectation with respect to the law of Brownian motion started at x.

(ii) *$p(t,x,y)$ is jointly continuous in (x,y).*

(iii) *There is a complete orthonormal basis $(\phi_i)_{i\ge1}$ of $L^2(\overline{\Omega})$ such that each ϕ_i is C^∞ in Ω, vanishes continuously at the boundary, and there are numbers $0 < \mu_1^2 \le \mu_2^2 \le \cdots$ tending to infinity such that*

$$p(t,x,y) = \sum_{i=1}^\infty e^{-\frac{1}{2}\mu_i^2 t}\phi_i(x)\phi_i(y)\,,$$

where the right side converges absolutely and uniformly on $\overline{\Omega} \times \overline{\Omega}$. Moreover, $-\Delta\phi_i = \mu_i^2\phi_i$ for each i.

Let μ_i be as in the above theorem. For each $\epsilon > 0$ and $\lambda > 0$ define a set of indices $J_{\epsilon,\lambda}$ as

$$J_{\epsilon,\lambda} := \{i : \lambda \le \mu_i < \lambda(1+\epsilon)\}\,.$$

Let $|J_{\epsilon,\lambda}|$ denote the size of the set $J_{\epsilon,\lambda}$. The following theorem is the main result of this section.

Theorem 4.2. *For any fixed $\epsilon > 0$, $J_{\epsilon,\lambda}$ is nonempty for all large enough λ and for $A \in \Psi(\mathbb{R}^d)$, with symbol $\sigma(A)(x,\xi)$ supported in $K_x \times \mathbb{R}^d$ with $K_x \subset \Omega$ compact,*

$$\lim_{\lambda\to\infty}\left|\frac{1}{|J_{\epsilon,\lambda}|}\sum_{i\in J_{\epsilon,\lambda}}\left\langle (A-\bar{A})1_{\overline{\Omega}}\phi_j, 1_{\overline{\Omega}}\phi_j\right\rangle\right| dx = 0\,.$$

where

$$\bar{A} = \int_{S^*\mathbb{R}^d}\sigma(A)(x,\xi)d\lambda$$

with $d\lambda = 1_{\overline{\Omega}}dxd\sigma(\xi)$ where σ is the normalized surface measure on S^{d-1}. Moreover,

$$\lim_{\lambda\to\infty}\lambda^{-d}|J_{\epsilon,\lambda}| = \frac{(1+\epsilon)^d-1}{(4\pi)^{d/2}\Gamma(d/2+1)}.$$

This theorem implies Theorem 2.4 since $\sigma(A)$ is homogeneous of degree 0 and is a variant of results that are sometimes called 'local Weyl laws', as in Zelditch (2010). However, we are not aware of a local Weyl law in the literature that applies for a domain as general as the one considered here. Our proof follows closely that in Gérard and Leichtnam (1993), but by using probabilistic methods to obtain estimates on the kernel of $e^{t\Delta}$, we are able to weaken the regularity assumptions on the domain.

Since we will have occasion to refer to both the Laplace operator on $L^2(\mathbb{R}^d)$ and the Dirichlet Laplacian in this section, we will denote them respectively by $-\Delta_{\mathbb{R}^d}$ and $-\Delta_D$. Theorem 4.2 will follow from the following lemma

Lemma 4.3. *Take $A \in \Psi^0(\mathbb{R}^d)$ with symbol $a(x,\xi)$ supported in $K_x \times \mathbb{R}^d$ where $K_x \subset \Omega$ is compact. Then for all $t > 0$, $1_{\overline{\Omega}}A1_{\overline{\Omega}}e^{t\Delta_D}$ is trace class as an operator on $L^2(\overline{\Omega})$ and*

$$\frac{\operatorname{Tr}(1_{\overline{\Omega}}A1_{\overline{\Omega}}e^{t\Delta_D})}{\operatorname{Tr}(e^{t\Delta_D})} \to \int_{S^*\mathbb{R}^d} a(x,\xi)d\lambda$$

where $\lambda = 1_{\overline{\Omega}}dxd\sigma(\xi)$ and σ is the normalized surface measure on S^{d-1}.

We first show how Theorem 4.2 follows from Lemma 4.3. We will need the following classical Tauberian theorem (see for example Taylor (1981)). We give a probabilstic proof for completeness.

Lemma 4.4. *Suppose that $F : [0,\infty) \to \mathbb{R}$ is nondecreasing and for some $A, \gamma > 0$,*

$$\int_0^\infty e^{-t\alpha}dF(\alpha) \sim At^{-\gamma} \quad \text{as } t \to 0^+\,.$$

Then

$$F(\tau) \sim \frac{A\tau^\gamma}{\Gamma(\gamma+1)} \quad \text{as } \tau \to \infty\,.$$

Proof. Define

$$G(t) = \int_0^\infty e^{-t\tau}dF(\tau)$$

and let Y_t be a random variable with

$$P(Y_t \le \alpha) = \frac{\int_0^\alpha e^{-t\tau}dF(\tau)}{G(t)}.$$

Then,

$$\mathbb{E}\left[e^{-t\theta Y_t}\right] = \frac{\int_0^\infty e^{-t\theta\tau}e^{-t\tau}dF(\tau)}{G(t)} = \frac{G(t(1+\theta))}{G(t)}.$$

Next, let $Z_t = e^{-tY_t}$ and W be a Gamma random variable with shape parameter γ and rate parameter 1. That is a random variable with density function

$$\frac{w^{\gamma-1}e^{-w}}{\Gamma(\gamma)}, \qquad w \in [0,\infty)$$

and $V = e^{-W}$ so that V has values in $(0,1]$ with density function

$$\frac{(-\log v)^{\gamma-1}}{\Gamma(\gamma)}.$$

In the remainder of this proof we will have two occasions to use Carleman's condition (proved by Carleman (1922)) for the uniqueness of the solution to the moment problem. Carleman's condition says that if $(m_{2k})_{k\geq 1}$ are nonnegative real numbers such that

$$\sum_{k=1}^{\infty} m_{2k}^{-1/2k} = \infty\,, \tag{4.1}$$

then there can be at most one probability measure P on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^{2k} dP(x) = m_{2k} \ \text{ for all } k.$$

A consequence of Carleman's condition is that if $\{X_n\}_{n\geq 1}$ is a sequence of random variables and X is a random variable such that $\lim_{n\to\infty} \mathbb{E}(X_n^k) = \mathbb{E}(X^k)$ for every integer $k \geq 1$, and the numbers $m_{2k} := \mathbb{E}(X^{2k})$ satisfy Carleman's condition (4.1), then X_n converges in distribution to X. In particular, this is true whenever X is a bounded random variable, which is the case that we will need. For a simple proof of Carleman's theorem under a slightly stronger condition that suffices for our purposes, see page 110 of Durrett (1996).

Notice that

$$\mathbb{E}[V^k] = (1+k)^{-\gamma}$$

and

$$\lim_{t\to 0^+} \mathbb{E}[Z_t^k] = \lim_{t\to 0^+} \mathbb{E}[e^{-tkY_t}] = \lim_{t\to 0^+} \frac{G((1+k)t)}{G(t)} = (1+k)^{-\gamma}.$$

Hence, by Carleman's theorem, Z_t converges in distribution to V.

Now, let $t = \lambda^{-1}$

$$\frac{F(\lambda)}{G(t)} = \mathbb{E}(Z_t^{-1}\,;\, e^{-1} \leq Z_t \leq 1).$$

Then, as $\lambda \to \infty$,

$$\mathbb{E}(Z_t^{-1}\,;\, e^{-1} \leq Z_t \leq 1) \to \mathbb{E}(V^{-1}\,;\, e^{-1} \leq V \leq 1) = \frac{1}{\Gamma(\gamma)} \int_0^1 w^{\gamma-1} dw = \frac{1}{\Gamma(\gamma+1)}.$$

So, using that $G(\lambda^{-1}) \sim A\lambda^{\gamma}$ as $\lambda \to \infty$, we have

$$F(\lambda) \sim \frac{A\lambda^{\gamma}}{\Gamma(\gamma)}, \qquad \lambda \to \infty$$

as desired. □

The rest of this section is devoted to the proof of Lemma 4.3 and Theorem 4.2. We will freely use the notation introduced in the statements of Theorem 4.1 and Theorem 4.2 without explicit reference. First, note that the following corollary of Theorem 4.1 is immediate from the continuity of p.

Lemma 4.5. *Take any $x, y \in \Omega$ and let $A_{y,r}$ be the closed ball of radius r centered at x. Then*

$$p(t,x,y) = \lim_{r\to 0} \frac{\mathbb{P}^x(B_t \in A_{y,r},\, t < \tau_\Omega)}{\mathrm{Vol}(A_{y,r})}\,.$$

Proof. By assertion (i) of Theorem 4.1,

$$\mathbb{P}^x(B_t \in A_{y,r},\, t < \tau_\Omega) = \int_{A_{y,r}} p(t,x,z)\, dz\,.$$

By assertion (ii) of Theorem 4.1,

$$\lim_{r\to 0} \frac{1}{\mathrm{Vol}(A_{y,r})} \int_{A_{y,r}} p(t,x,z)\, dz = p(t,x,y)\,.$$

The proof is completed by combining the two displays. □

The following lemma compares the transition density of killed Brownian motion with the transition density of unrestricted Brownian motion when t is small.

Lemma 4.6. *Let*

$$\rho(t,x,y) := \frac{1}{(2\pi t)^{d/2}} e^{-\|x-y\|^2/2t}$$

be the transition density of Brownian motion. Take any $x, y \in \Omega$ and let δ_y, δ_x be respectively the distance of y and x from $\partial\Omega$. Then

$$|\partial_y^\alpha(\rho(t,x,y) - p(t,x,y))| \le \frac{Ce^{-\delta_y^2/2t}}{t^{d/2+|\alpha|}}\,, \qquad 0 < t < \delta_y^2/(d+2|\alpha|),$$

$$|\partial_x^\alpha(\rho(t,x,y) - p(t,x,y))| \le \frac{Ce^{-\delta_x^2/2t}}{t^{d/2+|\alpha|}}\,, \qquad 0 < t < \delta_x^2/(d+2|\alpha|),$$

where C is a finite constant that depends only on d, $|\alpha|$ and the diameter of the domain Ω.

Proof. Since τ_Ω is a stopping time, the strong Markov property of Brownian motion implies that $X_s := B_{s+\tau_\Omega}$ is a standard Brownian motion started from B_{τ_Ω} that is independent of the stopped sigma algebra of τ_Ω, which we will denote by $\mathcal{F}_{\tau_\Omega}$. Consequently, if $A_{y,r}$ is the closed ball of radius $r < \delta_y/2$ centered at y, then for any $s \ge 0$,

$$\mathbb{P}^x(X_s \in A_{y,r} \mid \mathcal{F}_{\tau_\Omega}) = \frac{1}{(2\pi s)^{d/2}} \int_{A(y,r)} e^{-\|z-B_{\tau_\Omega}\|^2/2s}\, dz\,.$$

Consequently,

$$\begin{aligned}\mathbb{P}^x(B_t \in A_{y,r},\, t \ge \tau_\Omega) &= \mathbb{P}^x(X_{t-\tau_\Omega} \in A_{y,r},\, t \ge \tau_\Omega)\\ &= \mathbb{E}^x(\mathbb{P}^x(X_{t-\tau_\Omega} \in A_{y,r} \mid \mathcal{F}_{\tau_\Omega})\,;\, t \ge \tau_\Omega)\\ &= \mathbb{E}^x\left(\frac{1}{(2\pi(t-\tau_\Omega))^{d/2}} \int_{A(y,r)} e^{-\|z-B_{\tau_\Omega}\|^2/2(t-\tau_\Omega)}\, dz\,;\, t \ge \tau_\Omega\right),\end{aligned}$$

where the term inside the expectation is interpreted as zero if $t = \tau_\Omega$. Dividing both sides by $\mathrm{Vol}(A_{y,r})$, sending r to zero, and observing that the term inside the above expectation after division by $\mathrm{Vol}(A_{y,r})$ is uniformly bounded by a deterministic constant, we get

$$\lim_{r\to 0} \frac{\mathbb{P}^x(B_t \in A_{y,r},\, t \ge \tau_\Omega)}{\mathrm{Vol}(A_{y,r})} = \mathbb{E}^x\left(\frac{1}{(2\pi(t-\tau_\Omega))^{d/2}} e^{-\|y-B_{\tau_\Omega}\|^2/2(t-\tau_\Omega)}\,;\, t \ge \tau_\Omega\right).$$

Now note that

$$\mathbb{P}^x(B_t \in A_{y,r}) - \mathbb{P}^x(B_t \in A_{y,r},\, t < \tau_\Omega) = \mathbb{P}^x(B_t \in A_{y,r},\, t \geq \tau_\Omega)$$

and

$$\lim_{r\to 0} \frac{\mathbb{P}^x(B_t \in A_{y,r})}{\mathrm{Vol}(A_{y,r})} = \rho(t,x,y)\,,$$

and by Lemma 4.5,

$$\lim_{r\to 0} \frac{\mathbb{P}^x(B_t \in A_{y,r},\, t < \tau_\Omega)}{\mathrm{Vol}(A_{y,r})} = p(t,x,y)\,.$$

Combining all of the above observations, we gt

$$\rho(t,x,y) - p(t,x,y) = \mathbb{E}^x\left(\frac{1}{(2\pi(t-\tau_\Omega))^{d/2}} e^{-\|y-B_{\tau_\Omega}\|^2/2(t-\tau_\Omega)}\,;\, t \geq \tau_\Omega\right).$$

Now note that any derivative of the term inside the expectation (with respect to y) is uniformly bounded by a deterministic constant that does not depend on y or t. Therefore derivatives with respect to y can be carried inside the expectation. Consequently,

$$T\rho(t,x,y) - Tp(t,x,y) = \mathbb{E}^x\left(\frac{1}{(2\pi(t-\tau_\Omega))^{d/2}} T(e^{-\|y-B_{\tau_\Omega}\|^2/2(t-\tau_\Omega)})\,;\, t \geq \tau_\Omega\right).$$

If $t \leq \delta_y^2/(d+2|\alpha|)$, an easy verification shows that

$$\left|\frac{1}{(2\pi(t-\tau_\Omega))^{d/2}} T(e^{-\|y-B_{\tau_\Omega}\|^2/2(t-\tau_\Omega)})\right| \leq \frac{C}{(t-\tau_\Omega)^{d/2+|\alpha|}} e^{-\|y-B_{\tau_\Omega}\|^2/2(t-\tau_\Omega)}\,,$$

where C depends only on d, $|\alpha|$ and the diameter of the domain Ω. Another easy calculation shows that the map $u \mapsto (2\pi u)^{-d/2-|\alpha|}e^{-\beta^2/2u}$ is increasing in u when $0 < u \leq \beta^2/(d+2|\alpha|)$. Therefore if $\tau_\Omega < t \leq \delta_y^2/(d+2|\alpha|)$, then

$$\frac{1}{(t-\tau_\Omega)^{d/2+|\alpha|}} e^{-\|y-B_{\tau_\Omega}\|^2/2(t-\tau_\Omega)} \leq \frac{e^{-\delta_y^2/2t}}{t^{d/2+|\alpha|}}\,.$$

Noticing that $p(t,x,y) = p(t,y,x)$ (for example, by Theorem 4.4 in Chapter II of Bass (1995)) and $\rho(t,x,y) = \rho(t,y,x)$, this completes the proof of the lemma. □

Proof of Theorem 4.2 *from Lemma* 4.3. By Lemma 4.3, we have that

$$\frac{\mathrm{Tr}(1_{\overline{\Omega}} A 1_{\overline{\Omega}} e^{t\Delta_D})}{\mathrm{Tr}(e^{t\Delta_D})} \to \int_{S^*\mathbb{R}^d} a(x,\xi)d\lambda, \qquad t \to 0^+. \tag{4.2}$$

Since $\{\phi_j\}_{j\geq 1}$ is an orthonormal basis of $L^2(\overline{\Omega})$,

$$\mathrm{Tr}(1_{\overline{\Omega}} A 1_{\overline{\Omega}} e^{t\Delta_D}) = \sum_j e^{-t\mu_j^2} \left\langle A 1_{\overline{\Omega}}\phi_j, 1_{\overline{\Omega}}\phi_j\right\rangle. \tag{4.3}$$

By Lemma 4.6 and the assumption that $\mathrm{Vol}(\overline{\Omega}) = 1$,

$$\mathrm{Tr}(e^{t\Delta_D}) = \sum e^{-t\mu_j^2} = \int_\Omega p(2t,x,x) \sim (4\pi t)^{-d/2} \to \infty, \qquad t \to 0^+. \tag{4.4}$$

Putting (4.2), (4.3), and (4.4) together we have that

$$\sum_j e^{-t\mu_j^2} \left\langle A 1_{\overline{\Omega}} u_j, 1_{\overline{\Omega}} u_j\right\rangle \sim (4\pi t)^{-d/2} \int_{S^*\mathbb{R}^d} a(x,\xi)d\lambda.$$

Now, assuming that $\sigma(A) \geq 0$, and adding a regularizing perturbation $C \in \Psi^{-1}$ if necessary, so that $1_{\overline{\Omega}}(A+C)1_{\overline{\Omega}} \geq 0$, we may apply Lemma 4.4 with

$$F_A(\tau) = \sum_j 1_{\mu_j \leq \tau} \left\langle (A+C)1_{\overline{\Omega}}\phi_j, 1_{\overline{\Omega}}\phi_j \right\rangle .$$

More precisely, we apply it with

$$\tilde{F}_A(\tau) = \sum_j 1_{\mu_j \leq \sqrt{\tau}} \left\langle (A+C)1_{\overline{\Omega}}\phi_j, 1_{\overline{\Omega}}\phi_j \right\rangle$$

and rescale so that

$$F_A(\tau) \sim \frac{\tau^d}{(4\pi)^{d/2}\Gamma(d/2+1)} \int_{S^*\mathbb{R}^d} a(x,\xi) d\lambda.$$

Now, $C : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is compact. Therefore, $\left\langle C1_{\overline{\Omega}}\phi_j, 1_{\overline{\Omega}}\phi_j \right\rangle = \mathrm{o}(1)$, and hence

$$\sum 1_{\mu_j \leq \tau} \left\langle C1_{\overline{\Omega}}\phi_j, 1_{\overline{\Omega}}\phi_j \right\rangle = \mathrm{o}(\tau^d)$$

and hence

$$\sum_j 1_{\mu_j \leq \tau} \left\langle A1_{\overline{\Omega}}\phi_j, 1_{\overline{\Omega}}\phi_j \right\rangle \sim \frac{\tau^d}{(4\pi)^{d/2}\Gamma(d/2+1)} \int_{S^*\mathbb{R}^d} a(x,\xi) d\lambda. \tag{4.5}$$

Taking $A_n = \chi_n(x)$ with $\chi_n \in C_c^\infty(\Omega)$ and $\chi_n \to 1$, we see that

$$\#\{\mu_j : \mu_j \leq \tau\} \sim \frac{\tau^d}{(4\pi)^{d/2}\Gamma(d/2+1)}. \tag{4.6}$$

Subtraction of two formulae like (4.5) and (4.6) yields the desired asymptotics. □

The proof of Lemma 4.3 requires one further lemma.

Lemma 4.7. *We have*

$$(4\pi t)^{d/2} \operatorname{Tr}(A\psi e^{t\Delta_{\mathbb{R}^d}}\psi) \to \int_{S^*\mathbb{R}^d} \sigma(A) d\lambda \quad \text{as } t \to 0^+$$

and there exists $\epsilon > 0$, $t_0 > 0$, *so that for* $0 < t < t_0$,

$$|\operatorname{Tr} A\psi(e^{t\Delta_D} - e^{t\Delta_{\mathbb{R}^d}})\psi| \leq \epsilon^{-1}e^{-\epsilon/t} .$$

Proof. The kernel $K(t,x,y)$ of $A\psi e^{t\Delta_{\mathbb{R}^d}}\psi$ is given by

$$\begin{aligned} K(t,x,y) &= (2\pi)^{-d} \int a(x,\xi) \int e^{i\langle x-w,\xi\rangle - |w-y|^2/4t} (4\pi t)^{-d/2} \psi(w)\psi(y) dw d\xi \\ &= (2\pi)^{-2d} \int a(x,\xi) \int e^{i\langle x,\xi\rangle} e^{-|\xi-\eta|^2 t} e^{-i\langle y,\xi-\eta\rangle} \hat{\psi}(\eta)\psi(y) d\eta d\xi. \end{aligned}$$

So, changing variables so that $\xi\sqrt{t} = \zeta$,

$$\begin{aligned} (4\pi t)^{d/2} \operatorname{Tr} \psi e^{it\Delta_{\mathbb{R}^d}}\psi &= \pi^{-d/2} t^{d/2} (2\pi)^{-d} \int a(x,\xi) \int e^{-|\xi-\eta|^2 t} e^{i\langle x,\eta\rangle} \hat{\psi}(\eta)\psi(x) d\eta d\xi dx \\ &= \pi^{-d/2}(2\pi)^{-d} \int a(x,\zeta t^{-1/2}) \int e^{-|\zeta-\eta\sqrt{t}|^2} e^{i\langle x,\eta\rangle} \hat{\psi}(\eta)\psi(x) d\eta d\zeta dx \end{aligned}$$

Now, since $\psi \in \mathcal{S}$, we can use the dominated convergence theorem and let $t \to 0^+$ to obtain

$$\begin{aligned}\lim_{t\to 0^+} (4\pi t)^{d/2} \operatorname{Tr} \psi e^{it\Delta_{\mathbb{R}^d}}\psi &= \pi^{-d/2}(2\pi)^{-d} \int \sigma(A)(x,\zeta) \int e^{-|\zeta|^2} e^{i\langle x,\eta\rangle} \hat{\psi}(\eta)\psi(x) d\eta d\zeta dx \\ &= \pi^{-d/2} \int \sigma(A)(x,\zeta) \int e^{-|\zeta|^2} \psi(x)\psi(x) d\zeta dx \\ &= \pi^{-d/2} \int \sigma(A)(x,\zeta) \int e^{-|\zeta|^2} d\zeta dx\end{aligned}$$

where we have used that $\psi \equiv 1$ on $\operatorname{supp} \sigma(A)$. Now, since $\sigma(A)$ is homogeneous of degree 0 in ζ, this is equal to

$$\pi^{-d/2} \mathrm{Vol}(S^{d-1}) \int_{S^*\mathbb{R}^d} \sigma(A)(x,\zeta) d\lambda \int_0^\infty e^{-r^2} r^{d-1} dr = \int_{S^*\mathbb{R}^d} \sigma(A)(x,\zeta) d\lambda.$$

For the second claim, we use Lemma 4.6. Let $g(t,x,y)$ denote the kernel of $\psi(e^{t\Delta_D} - e^{t\Delta})\psi$. Then

$$\operatorname{Tr}(A\psi(e^{t\Delta_D} - e^{t\Delta_{\mathbb{R}^d}})\psi) = \int_{\overline{\Omega}} (Ag)(t,x,x) dx.$$

Now, $\|A\|_{H^s\to H^s} \le C$ for all s. Taking $m > \frac{d}{2}$, and letting $\delta = \frac{1}{2} d(\operatorname{supp}\psi, \partial\Omega)$,

$$|(Ag)(t,x,x)| \le \|A\|_{H^m\to H^m} \sum_{|\alpha|\le m} \sup_{x,y} |\partial_x^\alpha g(t,x,y)| \le C \frac{e^{-\delta^2/4t}}{t^{d/2+m}}$$

for each $t < \delta^2/2(d+2m)$. □

We are now ready to prove Lemma 4.3.

Proof of Lemma 4.3. Since $1_{\overline{\Omega}} A 1_{\overline{\Omega}} : L^2(\Omega) \to L^2(\Omega)$, and $e^{t\Delta_D}$ is trace class, $1_{\overline{\Omega}} A 1_{\overline{\Omega}} e^{t\Delta_D}$ is trace class. Let $\psi \in C_c^\infty(\Omega)$ with $\psi \equiv 1$ on $\operatorname{supp}\sigma(A)$. Then,

$$\psi A = A - (1-\psi)A, \qquad A\psi = \psi A + [A,\psi].$$

But, $(1-\psi)A$, $[A,\psi] \in \Psi^{-1}$ and hence $1_{\overline{\Omega}}(1-\psi)A$, $1_{\overline{\Omega}}[A,\psi]$ are compact on $L^2(\mathbb{R}^d)$ and have

$$\|1_{\overline{\Omega}}(1-\psi)A 1_{\overline{\Omega}}\phi_k\| + \|1_{\overline{\Omega}}[A,\psi]1_{\overline{\Omega}}\phi_k\| \to 0, \qquad k \to \infty.$$

In particular,

$$\frac{\operatorname{Tr}(1_{\overline{\Omega}} A 1_{\overline{\Omega}} e^{t\Delta_D})}{\operatorname{Tr} e^{t\Delta_D}} \sim \frac{\operatorname{Tr}(\psi A \psi e^{t\Delta_D})}{\operatorname{Tr} e^{t\Delta_D}} = \frac{\operatorname{Tr}(A\psi e^{t\Delta_D}\psi)}{\operatorname{Tr} e^{t\Delta_D}}, \qquad t \to 0^+.$$

By Lemma 4.7, the proof of Lemma 4.3 is now complete. □

5. Concentration of random rotations

Let $u_1, \dots, u_n$ be an orthonormal set of bounded functions belonging to $L^2(\overline{\Omega})$. Let Q be an $n \times n$ Haar-distributed random orthogonal matrix. Let q_{ij} denote the $(i,j)^{\text{th}}$ entry of Q. Define a new set of functions $v_1, \dots, v_n$ as

$$v_i(x) := \sum_{j=1}^n q_{ij} u_j(x)\,.$$

Then $v_1, \dots, v_n$ are also orthonormal, since

$$\begin{aligned}\langle v_i, v_j\rangle &= \left\langle \sum_{k,l=1}^n q_{ik}q_{jl}u_k, u_l \right\rangle = \sum_{k,l=1}^n q_{ik}q_{jl}\langle u_k, u_l\rangle \\ &= \sum_{k=1}^n q_{ik}q_{jk} = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We will refer to $v_1, \dots, v_n$ as a *random rotation* of $u_1, \dots, u_n$. The goal of this section is to prove the following concentration result for random rotations.

Theorem 5.1. *Let u_i and v_i be as above. Let $A : L^2(\Omega) \to L^2(\Omega)$ be a bounded operator. Then for any $1 \le i \le n$ and any $t > 0$,*

$$\mathbb{P}\left(\left|\langle Av_i, v_i\rangle - \frac{1}{n}\sum_{i=1}^n \langle Au_i, u_i\rangle\right| \ge t\right) \le C_1 \exp\bigl(-C_2(\|A\|)\min\{t^2,\, t\}n\bigr)\,,$$

where C_1 depends only of d and Ω, and $C_2(\|A\|)$ depends on d, Ω and the operator norm, $\|A\|$.

The key ingredient in the proof of Theorem 5.1 is the Hanson–Wright inequality, due to Hanson and Wright (1971), for quadratic forms of sub-Gaussian random variables. The original form of the Hanson–Wright inequality does not suffice for our objective. Instead, the following modern version of the inequality, proved recently by Rudelson and Vershynin (2013), is the one that we will use.

Define the ψ_2 norm a random variable X as

$$\|X\|_{\psi_2} := \sup_{p\ge 1} p^{-1/2}(\mathbb{E}|X|^p)^{1/p}\,.$$

The random variable X is called sub-Gaussian if its ψ_2 norm is finite. In particular, Gaussian random variables have this property.

Let $M = (m_{ij})_{1\le i,j\le n}$ be a square matrix with real entries. The Hilbert–Schmidt norm of M is defined as

$$\|M\|_{\mathrm{HS}} := \left(\sum_{i,j=1}^n m_{ij}^2\right)^{1/2}\,,$$

and the operator norm of M is defined as

$$\|M\| := \sup_{x\in\mathbb{R}^n,\, \|x\|=1} \|Mx\|\,,$$

where the norm on the right side is the Euclidean norm on $\mathbb{R}^n$. Rudelson and Vershynin's version of the Hanson–Wright inequality states that if $X_1, \dots, X_n$ are independent random variables with mean zero and ψ_2 norms bounded by some constant K, and

$$R := \sum_{i,j=1}^n m_{ij}X_iX_j\,,$$

then for any $t \ge 0$,

$$\mathbb{P}(|R - \mathbb{E}(R)| \ge t) \le 2\,\exp\left(-C\min\left\{\frac{t^2}{K^4\|M\|_{\mathrm{HS}}^2}, \frac{t}{K^2\|M\|}\right\}\right)\,, \tag{5.1}$$

where C is a positive universal constant.

Proof of Theorem 5.1. Fix $1 \leq i \leq n$. Define

$$A_i := \langle Av_i, v_i \rangle \,, \qquad B := \frac{1}{n}\sum_{i=1}^{n} \langle Au_i, u_i \rangle \,.$$

By a simple symmetry argument,

$$\mathbb{E}(q_{ij}q_{ik}) = \begin{cases} 1/n & \text{if } j = k\,, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \mathbb{E}(\langle Av_i, v_i \rangle) &= \sum_{jk} \mathbb{E}(\langle Aq_{ij}u_j, q_{ik}u_k \rangle) = \sum_{jk} \mathbb{E}(q_{ik}q_{ij}) \langle Au_j, u_k \rangle \\ &= \frac{1}{n}\sum_{j=1}^{n} \langle Au_j, u_j \rangle = B. \end{aligned} \tag{5.2}$$

Let q_i be the vector whose j^{th} component is q_{ij}. Since Q is a Haar-distributed random orthogonal matrix, symmetry considerations imply that q_i is uniformly distributed on the unit sphere S^{n-1}. Now recall that if z is an n-dimensional standard Gaussian random vector, then $z/\|z\|$ is uniformly distributed on S^{n-1}, and is independent of $\|z\|$. Therefore if r_i is a random variable that has the same distribution as $\|z\|$ and is independent of q_i, then the vector r_iq_i is a standard Gaussian random vector. Let $w_{ij} := r_iq_{ij}$, so that $w_{i1}, \ldots, w_{in}$ are i.i.d. standard Gaussian random variables. Define

$$A_i' := r_i^2 A_i = \sum_{j,k=1}^{n} w_{ij}w_{ik}h_{jk} \,.$$

Let H Be the matrix with $(j,k)^{\text{th}}$ entry

$$h_{jk} = \langle Au_j, u_k \rangle$$

so that

$$A_i = \sum_{j,k} q_{ij}q_{ik}h_{jk}.$$

That is, the operator $\Pi A \Pi$, written in u_j coordinates where Π denotes orthogonal projection onto $\operatorname{span}\{u_j : 1 \leq j \leq n\}$. Then, with the standard Euclidean norm on $\mathbb{R}^n$,

$$\|H\| \leq \|A\|$$

(recall that $\|A\|$ is the operator norm on L^2). Moreover,

$$\|H\|_{\text{HS}} = \sqrt{\|H^*H\|_{\text{Tr}}} \leq \sqrt{\|H^*H\|\|I_{n\times n}\|_{\text{Tr}}} \leq \|A\|\sqrt{n}.$$

Therefore by the Hanson–Wright inequality (5.1),

$$\mathbb{P}(|A_i' - \mathbb{E}(A_i')| \geq t) \leq 2\exp\left(-C(\|A\|)\min\left\{\frac{t^2}{n}, t\right\}\right), \tag{5.3}$$

where $C(\|A\|)$ is a constant that depends only on d, Ω and A. Again note that by the Hanson–Wright inequality,

$$\mathbb{P}(|r_i^2 - n| \geq t) \leq 2\exp\left(-C\min\left\{\frac{t^2}{n}, t\right\}\right). \tag{5.4}$$

Next, note that

$$|A_i| \le \|A\|_{L^2\to L^2}\|v\|^2 = \|A\|_{L^2\to L^2}\sum_{j,k=1}^{n} q_{ij}q_{ik}\langle u_j, u_k\rangle \tag{5.5}$$

$$= \|A\|_{L^2\to L^2}\sum_{j=1}^{n} q_{ij}^2 = \|A\|_{L^2\to L^2}\,. \tag{5.6}$$

Finally, observe that

$$\mathbb{E}(A_i') = n\mathbb{E}(A_i)\,. \tag{5.7}$$

Combining (5.2), (5.3), (5.4), (5.6) and (5.7) we get

$$\begin{aligned}\mathbb{P}(|A_i - B| \ge t) &\le \mathbb{P}(|nA_i - A_i'| \ge nt/2) + \mathbb{P}(|A_i' - \mathbb{E}(A_i')| \ge nt/2)\\ &\le \mathbb{P}(|(r_i^2 - n)A_i| \ge nt/2) + \mathbb{P}(|A_i' - \mathbb{E}(A_i')| \ge nt/2)\\ &\le \mathbb{P}(|r_i^2 - n| \ge nt/2K) + \mathbb{P}(|A_i' - \mathbb{E}(A_i')| \ge nt/2)\\ &\le C_1\exp\bigl(-C_2(\|A\|)\min\{t^2,\, t\}n\bigr)\,,\end{aligned}$$

which concludes the proof of the theorem. □

6. Construction of the perturbed Laplacian

Let $\Psi = (\psi_i)_{i\ge 1}$ be a complete orthonormal basis of $L^2(\overline{\Omega})$. Let $\Lambda = (\lambda_i)_{i\ge1}$ be a sequence of real numbers. For $s \ge 0$, let $\mathcal{F}^s(\Psi,\Lambda)$ be the Hilbert space consisting of all $f \in L^2(\overline{\Omega})$ such that the norm

$$\|f\|^2_{\mathcal{F}^s(\Psi,\Lambda)} := \sum_{i=1}^{\infty} \langle\lambda_i\rangle^{2s}\,|\langle f, \psi_i\rangle|^2 < \infty\,.$$

Here $\langle\lambda\rangle := (1+|\lambda|^2)^{1/2}$. For $s<0$, $\mathcal{F}^s(\Psi,\Lambda) := (\mathcal{F}^{-s}(\Psi,\Lambda))^*$ is the completion of $L^2(\overline{\Omega})$ with respect to $\|\cdot\|_{\mathcal{F}^s(\Psi,\Lambda)}$. For any $f\in\mathcal{F}(\Psi,\Lambda) := \mathcal{F}^1(\Psi,\Lambda)$, the series

$$T_{\Psi,\Lambda} f := \sum_{i=1}^{\infty}\lambda_i\langle f,\psi_i\rangle\,\psi_i$$

converges in $L^2(\overline{\Omega}) = \mathcal{F}^0(\Psi,\Lambda)$. When Ψ and Λ are clear from context, we will sometimes write $\mathcal{F}^s$ instead of $\mathcal{F}^s(\Psi,\Lambda)$.

Lemma 6.1. *Let $T_{\Psi,\Lambda}$ be as above. Let $\Lambda' = (\lambda_i')_{i\ge1}$ be another sequence of real numbers. Let $\epsilon\in(0,1)$ and $\gamma\ge 0$ be numbers such that for all i,*

$$|\lambda_i' - \lambda_i| \le \epsilon\,\langle\lambda_i\rangle^{1-\gamma}\,.$$

Then $\|\cdot\|_{\mathcal{F}^s(\Psi,\Lambda')}$ is equivalent to $\|\cdot\|_{\mathcal{F}^s(\Psi,\Lambda)}$, and for all $s\in\mathbb{R}$, $T_{\Psi,\Lambda'} - T_{\Psi,\Lambda} : \mathcal{F}^s(\Psi,\Lambda)\to\mathcal{F}^{s-1+\gamma}(\Psi,\Lambda)$ with

$$\|T_{\Psi,\Lambda'} - T_{\Psi,\Lambda}\|_{\mathcal{F}^s(\Psi,\Lambda)\to\mathcal{F}^{s-1+\gamma}(\Psi,\Lambda)} \le \epsilon.$$

Proof. Since $\langle\lambda_i'\rangle \le (1+\epsilon)\langle\lambda_i\rangle$, we have $\|\cdot\|_{\mathcal{F}^s(\Psi,\Lambda')} \le C\|\cdot\|_{\mathcal{F}^s(\Psi,\Lambda)}$. On the other hand since $\langle\lambda_i\rangle \le \langle\lambda_i'\rangle/(1-\epsilon)$, so $\|\cdot\|_{\mathcal{F}^s(\Psi,\Lambda)} \le C\|\cdot\|_{\mathcal{F}^s(\Psi,\Lambda')}$.

Next, let $f\in\mathcal{F}^s(\Psi,\Lambda)$ with $s\ge 1$. Then

$$(T_{\Psi,\Lambda} - T_{\Psi,\Lambda'})f = \sum_i(\lambda_i - \lambda_i')\langle f,\psi_i\rangle\,\psi_i.$$

Therefore,

$$\begin{aligned}\|(T_{\Psi,\Lambda} - T_{\Psi,\Lambda'})f\|^2_{\mathcal{F}^{s-1+\gamma}(\Psi,\Lambda)} &= \sum_i \langle\lambda_i\rangle^{2s-2+2\gamma} |\lambda_i - \lambda_i'|^2 | \langle f, \psi_i\rangle |^2 \\ &\le \sum_i \langle\lambda_i\rangle^{2s-2+2\gamma} \epsilon^2 \langle\lambda_i\rangle^{2(1-\gamma)} | \langle f, \psi_i\rangle |^2 \\ &\le \epsilon^2 \sum_i \langle\lambda_i\rangle^{2s} | \langle f, \psi_i\rangle |^2 \le \epsilon^2 \|f\|^2_{\mathcal{F}^s(\Psi,\Lambda)} .\end{aligned}$$

The density of $\mathcal{F}(\Psi, \Lambda)$ in $\mathcal{F}^s(\Psi, \Lambda)$ for $s \le 1$ implies that the result extends to $s \in \mathbb{R}$. This concludes the proof of the lemma. □

Lemma 6.2. *Let Ψ and Λ be as above. Let L be the set of distinct elements of Λ. For each $\ell \in L$, let I_ℓ be the set of all i such that $\lambda_i = \ell$. Assume that $|I_\ell|$ is finite for each ℓ. Let $\Psi' = (\psi_i')_{i\ge1}$ be another complete orthonormal basis, such that for each $\ell \in L$, the span of $(\psi_i')_{i\in I_\ell}$ equals the span of $(\psi_i)_{i\in I_\ell}$. Then for all s, $\mathcal{F}^s(\Psi', \Lambda) = \mathcal{F}^s(\Psi, \Lambda)$ and $T_{\Psi',\Lambda} = T_{\Psi,\Lambda}$.*

Proof. Take some $\ell \in L$. Let $n = |I_\ell|$. Rename the elements of $(\psi_i)_{i\in I_\ell}$ as $\xi_1, \dots, \xi_n$ and the elements of $(\psi_i')_{i\in I_\ell}$ as $\xi_1', \dots, \xi_n'$. By assumption, n is finite. Since the span of $(\xi_i')_{1\le i\le n}$ equals the span of $(\xi_i)_{1\le i\le n}$, there is a matrix $Q = (q_{ij})_{1\le i,j\le n}$ such that for each i,

$$\xi_i' = \sum_{j=1}^n q_{ij}\xi_j .$$

By orthonormality of $\xi_1, \dots, \xi_n$ and $\xi_1', \dots, \xi_n'$,

$$\sum_{k=1}^n q_{ik}q_{jk} = \langle \xi_i', \xi_j'\rangle = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{otherwise.}\end{cases}$$

Therefore Q is an orthogonal matrix. Thus, for any $f \in \mathcal{F}(\Psi, \Lambda)$,

$$\begin{aligned}\sum_{i=1}^n \langle f, \xi_i'\rangle \xi_i' &= \sum_{i=1}^n \left(\sum_{j=1}^n q_{ij} \langle f, \xi_j\rangle\right)\left(\sum_{k=1}^n q_{ik}\xi_k\right) \\ &= \sum_{j,k=1}^n \left(\sum_{i=1}^n q_{ij}q_{ik}\right) \langle f, \xi_j\rangle \xi_k = \sum_{j=1}^n \langle f, \xi_j\rangle \xi_j .\end{aligned} \tag{6.1}$$

For any function $g : \mathbb{R} \to \mathbb{R}$, (6.1) gives

$$\begin{aligned}\sum_{i\in I_\ell} g(\lambda_i) \langle \psi_i', f\rangle \psi_i' &= g(\ell) \sum_{i=1}^n \langle \xi_i', f\rangle \xi_i' \\ &= g(\ell) \sum_{i=1}^n \langle \xi_i, f\rangle \xi_i = \sum_{i\in I_\ell} g(\lambda_i) \langle \psi_i, f\rangle \psi_i .\end{aligned}$$

Taking L^2 norms of both sides with $g(x) = \langle x\rangle^s$ and $g(x) = x$ respectively we see that $\mathcal{F}^s(\Psi', \Lambda) = \mathcal{F}^s(\Psi, \Lambda)$ and $T_{\Psi',\Lambda} = T_{\Psi,\Lambda}$. □

Lemma 6.3. *Suppose that Λ has $|\lambda_i| > c > 0$ with $|\lambda_i| \to \infty$. Let $\gamma_i := 1/\lambda_i$ and $\Gamma := (\gamma_i)_{i\ge1}$. Then $\mathcal{F}(\Phi, \Gamma) \supset L^2(\overline{\Omega})$ and the range of $T_{\Phi,\Gamma}$ is contained in $\mathcal{F}(\Phi, \Lambda)$. Moreover, $T_{\Phi,\Lambda}T_{\Phi,\Gamma} = I$.*

Proof. If $f \in L^2(\overline{\Omega})$, then clearly $f \in \mathcal{F}(\Phi, \Gamma)$ since $\gamma_i \to 0$ as $i \to \infty$. Thus $\mathcal{F}(\Phi,\Gamma) \supset L^2(\overline{\Omega})$. Next, note that

$$T_{\Phi,\Gamma} f = \sum_{i=1}^{\infty} \gamma_i \langle f, \phi_i \rangle \phi_i \,,$$

which implies that for any i,

$$\langle T_{\Phi,\Gamma} f, \phi_i \rangle = \gamma_i \langle f, \phi_i \rangle \,. \tag{6.2}$$

Therefore

$$\sum_{i=1}^{\infty} \lambda_i^2 |\langle T_{\Phi,\Gamma} f, \phi_i\rangle|^2 = \sum_{i=1}^{\infty} \lambda_i^2 \gamma_i^2 |\langle f, \phi_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle f, \phi_i\rangle|^2 = \|f\|^2 \,.$$

This proves that the range of $T_{\Phi,\Gamma}$ is contained in $\mathcal{F}(\Phi,\Lambda)$. A similar argument using (6.2) shows that for any $f \in L^2(\overline{\Omega})$,

$$\begin{aligned} T_{\Phi,\Lambda} T_{\Phi,\Gamma} f &= \sum_{i=1}^{\infty} \lambda_i \langle T_{\Phi,\Gamma} f, \phi_i \rangle \phi_i \\ &= \sum_{i=1}^{\infty} \lambda_i \gamma_i \langle f, \phi_i \rangle \phi_i = \sum_{i=1}^{\infty} \langle f, \phi_i \rangle \phi_i = f \,. \end{aligned}$$

This completes the proof of the lemma. ☐

Now let $\Phi = (\phi_i)_{i \ge 1}$ be the complete orthonormal basis from Theorem 4.1, and let $\Lambda = (\lambda_i)_{i\ge1}$ be defined as $\lambda_i = \mu_i^2$, where μ_i's are the numbers from part (iii) of Theorem 4.1.

Lemma 6.4. *Let $T_{\Phi,\Gamma}$ be as in Lemma* 6.3*. Then $\mathcal{F}(\Phi,\Lambda) = \mathcal{F}_\Delta$ and $T_{\Phi,\Gamma}\Delta f = -f$.*

Proof. Let $g \in \mathcal{F}_\Delta$ the domain of Δ. Then $g = \sum_{i=1}^{\infty} \langle g, \phi_i\rangle \phi_i$ and

$$\Delta g = -\sum_{i=1}^{\infty} \mu_i^2 \langle g, \phi \rangle \phi_i \in L^2.$$

Therefore $g \in \mathcal{F}(\Phi,\Lambda)$ and $\Delta g = -T_{\Phi,\Lambda} g$. Now, suppose that $f \in \mathcal{F}(\Phi,\Lambda)$. Then,

$$|\langle f, \Delta g\rangle| = \left| \sum_i \langle f, \phi_i \rangle \overline{\mu_i^2 \langle g, \phi_i\rangle} \right| \le \|f\|_{\mathcal{F}(\Phi,\Lambda)} \|g\|.$$

Therefore, since Δ is self-adjoint and $\mathcal{F}_\Delta$ is dense in L^2, $f \in \mathcal{F}_\Delta$. Hence, $\mathcal{F}_\Delta = \mathcal{F}(\Phi,\Lambda)$ and $T_{\Phi,\Lambda} = -\Delta$. Together with Lemma 6.3, this implies the lemma. ☐

Remark 6.5. Lemma 6.4 is also an easy consequence of the spectral theorem applied to the Dirichlet Laplacian.

We are now ready to construct the perturbed Laplacian and finish the proof of Theorem 2.6 (and hence, also of Theorem 2.3).

Proof of Theorem 2.6. Let $\{\mu_i^2\}_{i\ge1}$ be the eigenvalues of $-\Delta$ and let $\Lambda = \{\mu_i^2\}_{i\ge1}$. Let $\tilde{\gamma} = \gamma/2$. Fix $\epsilon \in (0,1)$ and take $i \ge 1$. Then either $\lambda_i < 1+\epsilon$ or there exist positive integers n, $0 \le j \le N_n - 1$ where

$$N_n := \lceil (1+\epsilon)^{n\tilde{\gamma}} \rceil \tag{6.3}$$

such that

$$(1+\epsilon)^n \left(1 + \frac{j\epsilon}{N_n}\right) \le \lambda_i < (1+\epsilon)^n \left(1 + \frac{(j+1)\epsilon}{N_n}\right).$$

In the first case, let $\lambda_i' = \lambda_i$. In the second, let

$$\lambda_i' = (1+\epsilon)^n \left(1 + \frac{j\epsilon}{N_n}\right).$$

Note that

$$|\lambda_i - \lambda_i'| \le \epsilon(1+\epsilon)^{-n\tilde{\gamma}}(1+\epsilon)^n \le \epsilon(1+\epsilon)^{-1+\tilde{\gamma}}|\lambda_i|^{1-\tilde{\gamma}}.$$

Therefore, by Lemma 6.1, for $s \ge 0$

$$\mathcal{F}^s(\Phi, \Lambda') = \mathcal{F}^s(\Phi, \Lambda)$$

and for $s \ge 1 - \tilde{\gamma}$ and ϵ small enough,

$$\|T_{\Phi,\Lambda'} - T_{\Phi,\Lambda}\|_{\mathcal{F}^s \to \mathcal{F}^{s-1+\tilde{\gamma}}} \le 2\epsilon.$$

Let L be the set of distinct eigenvalues in Λ'. For each $l \in L$, let I_l be the set of i such that $\lambda_i' = l$. Then by Definition 2.5, $|I_l| < \infty$ for all l. For each l, let $(\phi_i')_{i \in I_l}$ be a random rotation of $(\phi_i)_{i \in I_l}$. Then, by Lemma 6.2,

$$T_{\Phi,\Lambda'} = T_{\Phi',\Lambda'}, \qquad \mathcal{F}^s(\Phi, \Lambda') = \mathcal{F}^s(\Phi', \Lambda').$$

Now, for each $l \in L$,

$$l = (1+\epsilon)^n \left(1 + \frac{j\epsilon}{N_n}\right)$$

for some n, j or $0 < l < (1+\epsilon)$. Denote this set of l with $0 < l < 1+\epsilon$ by $L_<$ and let $I_< := \cup_{l \in L_<} I_l$. Let $(\lambda_i'')_{i \in I_<}$ be an arbitrary set of distinct real numbers with

$$(1-\epsilon)\lambda_i' \le \lambda_i'' < \lambda_i'.$$

For $l \notin L_<$, let $(\lambda_i'')_{i \in I_l}$ be an arbitrary set of distinct real numbers with

$$(1+\epsilon)^n \left(1 + \frac{j\epsilon}{N_n}\right) \le \lambda_i'' < (1+\epsilon)^n \left(1 + \frac{(j+1)\epsilon}{N_n}\right).$$

Then for any i,

$$|\lambda_i' - \lambda_i''| \le \epsilon |\lambda_i'|^{1-\tilde{\gamma}}$$

and hence

$$\mathcal{F}^s(\Phi', \Lambda'') = \mathcal{F}^s(\Phi', \Lambda') = \mathcal{F}^s(\Phi, \Lambda') = \mathcal{F}^s(\Phi, \Lambda)$$

and

$$\|T_{\Phi',\Lambda''} - T_{\Phi,\Lambda'}\|_{\mathcal{F}^s \to \mathcal{F}^{s-1+\tilde{\gamma}}} \le \epsilon.$$

Thus,

$$\|T_{\Phi',\Lambda''} - T_{\Phi,\Lambda}\|_{\mathcal{F}^s \to \mathcal{F}^{s-1+\tilde{\gamma}}} \le 10\epsilon.$$

Now, let

$$\Gamma = \{\lambda_i^{-1}\}_{i \ge 1}.$$

and $G := T_{\Phi,\Gamma}$. For convenience, write $T = T_{\Phi,\Lambda}$ and $T'' = T_{\Phi',\Lambda''}$.

Then by Lemma 6.3, G is bounded on $L^2(\overline{\Omega})$, has range in $\mathcal{F}(\Phi, \Lambda)$, and satisfies $TG = I$. Therefore, the operator

$$S := (T'' - T)G$$

maps L^2 into $\mathcal{F}^{\tilde{\gamma}}$. We will show that S satisfies the three assertions of the theorem. Note that the construction of S is random; what we will actually show is that S satisfies the conditions with probability one. This will suffice to demonstrate the existence of an S that satisfies the requirements.

First, notice that

$$\|Sf\|_{\mathcal{F}^{\tilde{\gamma}}} = \|(T'' - T)Gf\|_{\mathcal{F}^{\tilde{\gamma}}} \leq 10\epsilon \|Gf\|_{\mathcal{F}} \leq C\epsilon \|f\|.$$

Now, by Lemma 6.4, $\mathcal{F}_\Delta = \mathcal{F}$, therefore $\mathcal{F}^{\tilde{\gamma}} = (L^2(\Omega), \mathcal{F}_\Delta)_{\tilde{\gamma}}$, the complex interpolation space of L^2 and $\mathcal{F}_\Delta$. Hence (i) holds.

Next, note that by Lemma 6.4 for $f \in \mathcal{F}_\Delta$, $-G\Delta f = f$. Therefore, for $f \in \mathcal{F}_\Delta$,

$$(I + S)\Delta f = (I + (T'' - T)G)\Delta f = T'' G\Delta f = -T'' f.$$

That is, $-(I + S)\Delta = T''$ on $\mathcal{F}_\Delta$. This proves part (ii) of the theorem. Part (iii) of the theorem follows from the fact that $\{\phi_i'\}$ is an orthonormal basis for $L^2(\overline{\Omega})$ and each ϕ_i' is a linear combination of finitely many ϕ_i which have $\phi_i \in \mathcal{F}_\Delta^s$ for all s.

It remains to show that the eigenvalues of T'' are equidistributed. For this, recall that

$$l = (1+\epsilon)^n \left(1 + \frac{j\epsilon}{N_n}\right)$$

for l large enough and hence

$$I_l = \left\{ i : (1+\epsilon)^n \left(1 + \frac{j\epsilon}{N_n}\right) \leq \lambda_i < (1+\epsilon)^n \left(1 + \frac{(j+1)\epsilon}{N_n}\right) \right\}.$$

Then since $\lambda_i = \mu_i^2$, I_l may be alternately expressed as

$$I_l = \{ i : \mu \leq \mu_i < \mu_+ \} ,$$

where

$$\mu := (1+\epsilon)^{\frac{n}{2}} \sqrt{1 + \frac{j\epsilon}{N_n}}, \qquad \mu_+ := (1+\epsilon)^{\frac{n}{2}} \sqrt{1 + \frac{(j+1)\epsilon}{N_n}} .$$

Now,

$$r_+ := \frac{\mu_+}{\mu} = \frac{\sqrt{1 + \frac{(j+1)\epsilon}{N_n}}}{\sqrt{1 + \frac{j\epsilon}{N_n}}} = 1 + \frac{\epsilon}{2N_n} + \mathrm{O}(\epsilon^2 N_n^{-1}).$$

Then since Ω is AQE at scale $\alpha(\mu) = \mathrm{O}(\mu^{-2\tilde{\gamma}})$ and $N_n^{-1} \geq c\mu^{-2\tilde{\gamma}}$,

$$\lim_{l \in L,\, l \to \infty} \frac{1}{|I_l|} \left| \sum_{i \in I_l} \left\langle (A - \overline{\sigma(A)}) 1_{\overline{\Omega}} \phi_i, 1_{\overline{\Omega}} \phi_i \right\rangle \right| = 0 \tag{6.4}$$

for $A \in \mathcal{A} \subset \Psi(\mathbb{R}^d)$, where

$$\overline{\sigma(A)} = \frac{1}{\mathrm{Vol}(1 \leq |\xi| \leq 1 + r_+)} \iint_{1 \leq |\xi| \leq 1 + r_+} \sigma(A)(x,\xi) 1_{\overline{\Omega}} dx d\xi = \int_{S^*\mathbb{R}^d} \sigma(A)(x,\xi) 1_{\overline{\Omega}} dx d\sigma(\xi) .$$

Note that we have used that $\sigma(A)$ is homogeneous of degree 0.

Now, by Theorem 5.1, for any $A \in \mathcal{A}$ and $t \in (0,1)$,

$$\mathbb{P}\left(\left| \langle A 1_{\overline{\Omega}} \phi_i', 1_{\overline{\Omega}} \phi_i' \rangle - \frac{1}{|I_l|} \sum_{i \in I_l} \langle A 1_{\overline{\Omega}} \phi_i, 1_{\overline{\Omega}} \phi_i \rangle \right| \geq t \right) \leq C_1 \exp(-C_2(\|A\|) \min(t^2, t) |I_l|).$$

So,

$$\mathbb{P}\left(\max_{i\in I_l}\left|\left\langle A1_{\overline{\Omega}}\phi_i', 1_{\overline{\Omega}}\phi_i'\right\rangle - \frac{1}{|I_l|}\sum_{i\in I_l}\left\langle A1_{\overline{\Omega}}\phi_i, 1_{\overline{\Omega}}\phi_i\right\rangle\right| \geq t\right) \leq C_1|I_l|\exp(-C_2(\|A\|)\min(t^2,t)|I_l|).$$

The Weyl law implies that

$$\sum_{l\in L}|I_l|\exp(-C_2(\|A\|)\min(t^2,t)|I_l|) < \infty$$

and hence we have, using the Borel–Cantelli lemma that

$$\mathbb{P}\left(\left|\left\langle A1_{\overline{\Omega}}\phi_i', 1_{\overline{\Omega}}\phi_i'\right\rangle - \frac{1}{|I_l|}\sum_{i\in I_l}\left\langle A1_{\overline{\Omega}}\phi_i, 1_{\overline{\Omega}}\phi_i\right\rangle\right| \geq t \text{ for infinitely many } i \text{ and } l \text{ with } i\in I_l\right) = 0.$$

Thus, by (6.4) for all $\delta > 0$,

$$\mathbb{P}\left(\limsup_{i\to\infty}\left|\left\langle A1_{\overline{\Omega}}\phi_i', 1_{\overline{\Omega}}\phi_i'\right\rangle - \overline{\sigma(A)}\right| \geq \delta\right) = 0.$$

The fact that $\mathcal{A}$ is dense in $C_0(S^*\Omega)$ and $C_0(S^*\Omega)$ is separable then implies that $\mathcal{M}(\phi_i') = \{1_{\overline{\Omega}}dxd\sigma(\xi)\}$.

Now, suppose that $f \in \mathcal{F}_\Delta$ is an L^2 normalized eigenfunction of T''. Then

$$0 = \|T''f - \lambda f\|^2 = \sum_i(\lambda_i'' - \lambda)^2|\left\langle f, \phi_i'\right\rangle|^2.$$

Hence, since $f \neq 0$, $\lambda = \lambda_i''$ for some i. Thus, for any j

$$\left\langle \phi_j', f\right\rangle = \frac{1}{\lambda_i''}\left\langle \phi_j', T''f\right\rangle = \frac{1}{\lambda_i''}\sum_k\left\langle \phi_j', \phi_k'\right\rangle\lambda_k''\left\langle \phi_k', f\right\rangle = \frac{\lambda_j''}{\lambda_i''}\left\langle \phi_j', f\right\rangle.$$

Hence $\langle \phi_j', f\rangle = 0$ or $\lambda_j'' = \lambda_i''$. But for j large enough, $\lambda_i'' \neq \lambda_j''$ for $i \neq j$ and hence $f = \phi_i'$ and T'' has equidistributed eigenfunctions.

Notice also that this implies that for $\{f_n\}_{n=1}^\infty$ the eigenfunctions of $-(I+S)\Delta$ with $-(I+S)\Delta f_n = \alpha_n^2 f_n$, and n large enough, $f_n = \phi_{n_j}'$ and hence

$$-(I+S)\Delta f_n = T''f_n = \alpha_n^2 f_n.$$

Consequently,

$$\|Sf_n\| = \|(T''-T)G\Delta\phi_{n_j}'\| = \|(T''-T)\phi_{n_j}\| \leq C\epsilon\left\langle \alpha_n\right\rangle^{-\gamma}. \tag{6.5}$$

This completes the proof of both Theorem 2.3 and Corollary 2.8. □

Proof of Corollary 2.7. By Theorem 2.3, there exists a sequence of linear operators $\{S_n\}_{n\geq 1}$ such that

$$\|S_n\|_{L^2\to\mathcal{F}_\Delta^{\gamma/2}} \to 0$$

and $-(I+S_n)\Delta$ is positive and has QUE eigenfunctions for each n. This implies the existence of an orthonormal basis of $L^2(\overline{\Omega})$, $\{f_{n,k}\}_{k=1}^\infty$ and $\alpha_{n,k}$ such that $\|f_{n,k}\| = 1$ for each n and k, $\alpha_{n,k}^2 \to \infty$ as $k\to\infty$, and

$$(I+S_n)\Delta f_{n,k} = -\alpha_{n,k}^2 f_n.$$

Without loss of generality, $\|S_n\| < 1$. Then the series

$$(I + S_n)^{-1} = \sum_{k=0}^{\infty} (-1)^k S_n^k$$

converges in the space of bounded linear operators on $L^2(\overline{\Omega})$. Moreover,

$$(I + S_n)^{-1} - I = -(I + S_n)^{-1} S_n$$

Therefore, by (6.5)

$$\begin{aligned} \| - \Delta f_{n,k} - \alpha_{n,k}^2 f_{n,k}\| &= \|\alpha_{n,k}^2 (I + S_n)^{-1} S_n f_{n,k}\| \\ &\leq \alpha_{n,k}^2 \|(I + S_n)^{-1}\|_{L^2 \to L^2} \|S_n f_n\| \\ &\leq C \frac{\langle \alpha_{n,k} \rangle^{2-\gamma}}{1 - \|S_n\|_{L^2 \to L^2}} . \end{aligned}$$

Dividing both sides by $\alpha_{n,k}^2$ completes the proof. □

7. Improvements on closed manifolds

In order to prove Theorem 2.3 on a manifold M with $\mathrm{Vol}(M) = 1$, we work with $L_0^2(M)$, the set of 0 mean functions in L^2 to remove the 0 eigenvalue of the Laplacian. Let $\{(\lambda_i, \phi_i)\}_{i=1}^{\infty}$ be the eigenvalues and eigenfunctions of $-\Delta_g$. Then with $T_{\Phi,\Lambda}$ and $T_{\Phi,\Gamma}$ as above, the proof of Theorem 2.3 for M proceeds as above.

We now prove Corollary 2.9. For this, we need to use the full strength of Theorem 3.2.

Proof of Corollary 2.9. Let $\gamma = 1$, $\tilde{\gamma} = 1/2$. Then return to (6.3), where we replace N_n with

$$N_n := \lceil (1 + \epsilon)^{n/2} \rceil \beta_n$$

where $\beta_n \in \mathbb{N}$ has $\beta_n \to \infty$ slowly enough. We then proceed as in the proof of Theorem 2.3 until (6.4). At this point we need to show that there exists $\beta_n \to \infty$ slowly enough so that for $\|f\|_{L^\infty(M)} \leq 1$,

$$\lim_{l \in L, i \to \infty} \frac{1}{|I_l|} \Big| \sum_{i \in I_l} \big\langle (f - \overline{f}) \phi_i, \phi_i \big\rangle \Big| = 0$$

where

$$\overline{f} = \int_M f d\mathrm{Vol}.$$

First, observe that

$$\lambda_i = \mu_i^2, \qquad \mu := (1 + \epsilon)^{\frac{n}{2}} \sqrt{1 + \frac{j\epsilon}{N_n}}, \qquad \mu_+ := (1 + \epsilon)^{\frac{n}{2}} \sqrt{1 + \frac{(j+1)\epsilon}{N_n}},$$
$$I_l = \{ i \,|\, \mu \leq \mu_i < \mu_+ \} .$$

Note also that by Theorem 3.2,

$$\sum_{\mu_1 \leq \mu_j \leq \mu_2} |\phi_j(y)|^2 = \frac{(\mu_2 - \mu_1)\mu_2^{d-1}}{(2\pi)^d} \mathrm{Vol}(S^{d-1}) + g(\mu_2, \mu, x)$$

where

$$\lim_{\mu_2 \to \infty} \|g(\mu_2, \mu_1, x)\|_{L^\infty_{x,\mu_1}} \mu_2^{-d+1} = 0.$$

Therefore, integrating, we have

$$\#\{\mu_1 \leq \mu_j \leq \mu_2\} = \frac{(\mu_2-\mu_1)\mu_2^{d-1}}{(2\pi)^d}\mathrm{Vol}(S^{d-1}) + \int g(\mu_2,\mu,x)dx$$

and

$$\left|\frac{\sum_{\mu_1\leq\mu_j\leq\mu_2}|\phi_j(y)|^2}{\#\{\mu_1 \leq \mu_j \leq \mu_2\}} - 1\right| \leq C\|g(\mu_2,\mu_1,x)\|_{L^\infty_{x,\mu_1}}\mu_2^{-d+1}(\mu_2-\mu_1)^{-1}$$

Thus, taking $\mu_1 = \mu$ and $\mu_2 = \mu_+$, we have

$$\mu_2 \sim (1+\epsilon)^{n/2}, \qquad \mu_2 - \mu_1 \sim (1+\epsilon)^{n/2}\frac{\epsilon}{(1+\epsilon)^{1/2}\beta_n}.$$

Therefore, taking $\beta_n \to \infty$ slowly enough so that

$$\lim_{n\to\infty}\|g(\mu,\mu_+,x)\|_{L^\infty_{x,\mu_1}}\mu_+^{-d+1}(\mu_+-\mu)^{-1} = 0$$

gives that uniformly for $\|f\|_{L^\infty} \leq 1$,

$$\lim_{l\in L, i\to\infty}\frac{1}{|I_l|}\Big|\sum_{i\in I_l}\left\langle (f-\overline{f})\phi_i,\phi_i\right\rangle\Big| = 0.$$

Then, using the fact that $f \in C^\infty(M)$ with $\|f\|_{L^\infty(M)} \leq 1$ is dense in the unit ball of the dual space to finite radon measures, that this space is separable, and following the proof of Theorem 2.3 from (6.4) shows that for all $\epsilon > 0$, there exists $S : L^2(M) \to H^1(M)$ so that $\|S\|_{L^2\to H^1} \leq \epsilon$, $-(I+S)\Delta_g$ has equidistributed eigenfunctions, $\{(f_n,\alpha_n)\}_{n=1}^\infty$, and by (6.5) $\|Sf_n\| = \mathrm{o}(\alpha_n^{-1})\|f_n\|$.

Therefore,

$$-(I+S)\Delta f_n = \alpha_n^2 f_n.$$

Now,

$$(I+S)^{-1} = \sum_{k=0}^\infty (-1)^k S^k, \qquad (I+S)^{-1} - I = -(I+S)^{-1}S.$$

Therefore,

$$(-\Delta - \alpha_n)f_n = -\alpha_n^2(I+S)^{-1}Sf_n$$

and hence,

$$\begin{aligned}\|(-\Delta-\alpha_n^2)f_n\| &\leq |\alpha_n^2|\|(I+S)^{-1}\mathrm{o}(\alpha_n^{-1})\|f_n\| \\ &= \mathrm{o}(\alpha_n)\|f_n\|\end{aligned}$$

Dividing by α_n^2 completes the proof of the corollary. □

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