ANALYTIC HYPOELLIPTICITY OF KELDYSH OPERATORS

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Abstract. We consider Keldysh-type operators, $P = x_1 D_{x_1}^2 + a(x) D_{x_1} + Q(x, D_{x'})$, $x = (x_1, x')$ with analytic coefficients, and with $Q(x, D_{x'})$ second order, principally real and elliptic in $D_{x'}$ for x near zero. We show that if $Pu = f$, $u \in C^\infty$, and f is *analytic* in a neighbourhood of 0 then u is *analytic* in a neighbourhood of 0. This is a consequence of a microlocal result valid for operators of any order with Lagrangian radial sets. Our result proves a generalized version of a conjecture made in [Zw17], [LeZw19] and has applications to scattering theory.

1. Introduction

We consider analytic regularity for generalizations of the Keldysh operator [Ke51],

$$P := x_1 D_{x_1}^2 + D_{x_2}^2. \tag{1.1}$$

The operator P has the feature of changing from an elliptic to a hyperbolic operator at $x_1 = 0$. It appears in various places including the study of transsonic flows, see for instance Čanić–Keyfitz [CaKe96] or population biology – see Epstein–Mazzeo [EpMa13]. Our interest in such operators comes from the work of Vasy [Va13] where the transition at $x_1 = 0$ corresponds to the boundary at infinity for asymptotically hyperbolic manifolds (see [Zw16]), crossing the event horizons of Schwartzschild black holes (see [DyZw19a, §5.7]) or the cosmological horizon for de Sitter spaces. The Vasy operator in the asymptotically hyperbolic setting is given by

$$P(\lambda) = 4(x_1 D_{x_1}^2 - (\lambda + i) D_{x_1}) - \Delta_{h(x_1)} + i\gamma(x) \left(2x_1 D_{x_1} - \lambda - i\tfrac{n-1}{2} \right), \tag{1.2}$$

where $h(x_1)$ is a smooth family of Riemannian metrics in x', $x = (x_1, x') \in \mathbb{R}^n$ and $\gamma \in C^\infty(\mathbb{R}^n)$. The resonant states at resonant frequencies λ (see [DyZw19a, Chapter 5]) are the smooth solutions of $P(\lambda)u = 0$.

For various reasons reviewed in §1.3 it is interesting to ask if in the case of analytic coefficients the resonant states are real analytic across $x_1 = 0$. That lead to [Zw17, Conjecture 2] which asked if $P(\lambda)u = f$ with u smooth and f analytic near $x_1 = 0$ implies that u is analytic near $x_1 = 0$. For $\gamma(x) \equiv 0$ and h independent of x_1, this was shown by Lebeau–Zworski [LeZw19] under the assumption that $\lambda \notin -i\mathbb{N}^*$.

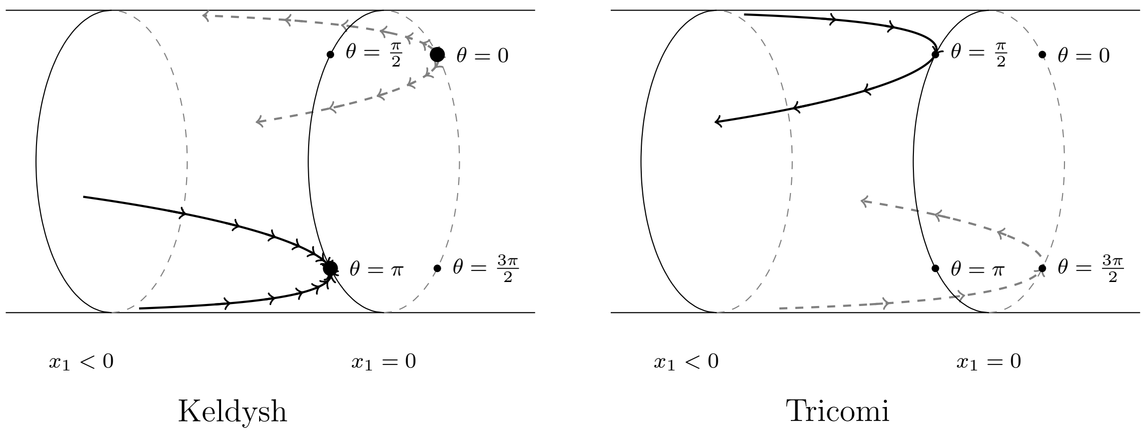

FIGURE 1. A comparison of the Keldysh operator (1.1) and the Tricomi operator (1.5). The figures show the cylinder $\mathbb{R}_{x_1} \times \mathbb{S}^1_\theta$ where $(\xi_1, \xi_2) = |\xi|(\cos\theta, \sin\theta)$ (this is the boundary of the fiber compactified cotangent bundle $\overline{T}^*\mathbb{R}^n$ – see [DyZw19a, §E.1.3] – with the x_2 variable omitted). The characteristic varieties, $x_1 \cos^2\theta + \sin^2\theta = 0$ and $\cos^2\theta + x_1 \sin^2\theta = 0$, respectively, are shown with the direction of the Hamiltonian flow indicated. In the the Keldysh case, the two radial Lagrangians, $\Lambda_\pm$, correspond to $\theta = \pi$ and $\theta = 0$ respectively.

The general case was proved by Zuily [Zu17] under the same restriction on λ. His proof was an elegant adaptation of the work of Baouendi–Goulaouic [BoGu81], Bolley–Camus [BoCa73] and Bolley–Camus–Hanouzet [BCH74].

In this paper we prove this result for generalized Keldysh operators with analytic coefficients (1.3). In particular, we do not make any assumptions on lower order terms:

Theorem 1. *Suppose that $U \subset \mathbb{R}^n$ is a neighbourhood of* 0*,*

$$P := x_1 D_{x_1}^2 + a(x) D_{x_1} + Q(x, D_{x'}), \quad x = (x_1, x') \in U, \tag{1.3}$$

has analytic coefficients, $Q(x, D_{x'})$ is a second order elliptic operator in $D_{x'}$ with a real valued principal symbol. Then there exists a neighbourhood of 0*, $U' \subset U$, such that*

$$Pu \in C^\omega(U), \quad u \in C^\infty(U) \implies u \in C^\omega(U'). \tag{1.4}$$

We will show in §1.1 that this result follows from a more general microlocal result valid for operators of all orders satisfying a natural geometric condition.

Remarks: 1. In the statement of the theorem 0 can be replaced by any point at which $x_1 \geq 0$ and U' can be replaced by U provided we include a bicharacteristic convexity condition. That follows from propagation of analytic singularities – see [Ma02, Theorem 4.3.7] or [HiSj18, Theorem 2.9.1]: since there are no singularities near $x_1 = 0$ there will be no singularities on trajectories hitting $x_1 = 0$ – see Figure 1.

2. The result is false for the Tricomi operator

$$P := D_{x_1}^2 + x_1 D_{x_2}^2. \tag{1.5}$$

This can be seen using results about propagation of analytic singularities (unlike (1.3) this operator can be microlocally conjugated to D_{y_1} – see Figure 1) but is also easily demonstrated by the following example:

$$u(x) := \int_0^\infty Ai(\tau^{4/3} x_1) e^{i\tau^2 x_2} e^{-\tau} d\tau, \quad Pu = 0, \quad u \in C^\infty(\mathbb{R}^2). \tag{1.6}$$

Here, Ai is the Airy function which satisfies

$$Ai''(t) + tAi(t) = 0, \quad |\partial_t^\ell Ai(t)| \leq C_\ell \langle t \rangle^{\frac{\ell}{2} - \frac{1}{4}}, \quad t \in \mathbb{R}, \ \ell \in \mathbb{N}, \quad Ai(0) > 0.$$

We then have

$$D_{x_2}^k u(0) = Ai(0) \int_0^\infty \tau^{2k} e^{-\tau} d\tau = Ai(0)(2k)!$$

and u is not analytic at 0.

3. Results similar to (1.4) have been obtained in the setting of other operators. In addition to the works [BoCa73],[BCH74] cited above, we mention the work of Baouendi–Sjöstrand [BaSj76] who considered a class of Fuchsian operators generalizing

$$P = |x|^2 \Delta + \mu \langle x, D_x \rangle + \lambda \tag{1.7}$$

In the case of (1.7), (1.4) holds for any $\lambda, \mu \in \mathbb{C}$ and [BaSj76] established (1.4) for more general operators satisfying appropriate conditions.

4. The operators (1.3), (1.5) and (1.7) are not C^∞ hypoelliptic, that is, $Pu \in C^\infty \not\Rightarrow u \in C^\infty$. The study of operators which are C^∞ hypoelliptic but not analytic hypoelliptic has a long tradition with a simple example [HöI, §8.6, Example 2] given by

$$P = D_{x_1}^2 + x_1^2 D_{x_2}^2 + D_{x_3}^3.$$

For more complicated cases, references, and connections to several complex variables, see Christ [Ch96] and for some recent progress and additional references, Bove–Mughetti [BoMu17].

1.1. **A microlocal result.** We make the following general assumptions. Let P be a differential operator of order m with analytic coefficients:

$$P := \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \quad a_\alpha \in C^\omega(U), \quad p(x,\xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha, \tag{1.8}$$

where U is an open neighbourhood of $x_0 \in \mathbb{R}^n$. We make the following assumptions valid in a conic neighbourhood of $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$: p is *real valued* and there exists a conic Lagrangian submanifold Λ, such that

$$(x_0, \xi_0) \in \Lambda \subset p^{-1}(0), \quad dp|_\Lambda \neq 0, \quad H_p|_\Lambda \parallel \xi \cdot \partial_\xi|_\Lambda. \tag{1.9}$$

Here $\|$ means that the two vector fields are *positively* proportional, that is the Lagrangian is *radial* (the positivity assumptions can be achieved by multiplying P by ± 1). Except for the analyticity assumption in (1.8) these are the assumptions made in Haber [Ha14] and Haber–Vasy [HaVa15].

Theorem 1 follows from the following microlocal result. We denote by WF the C^∞-wave front set and by $\mathrm{WF_a}$ the analytic wave front set – see [HöI, §8.1] and [HöI, §8.5,9.3], respectively.

Theorem 2. *Suppose that P and $(x_0,\xi_0)\in T^*\mathbb{R}^n\setminus 0$ satisfy the assumptions (1.8) and (1.9). Then for $u\in\mathscr{D}'(\mathbb{R}^n)$,*

$$(x_0,\xi_0)\notin \mathrm{WF}(u),\quad (x_0,\xi_0)\notin \mathrm{WF_a}(Pu) \implies (x_0,\xi_0)\notin \mathrm{WF_a}(u). \tag{1.10}$$

The proof is based on the theory of microlocal symbolic weights developed by Galkowski–Zworski [GaZw19b] and based on the work of Sjöstrand – see [Sj96, §2] (and also [HeSj86] and [Ma02, §3.5]). With this theory in place we can use escape functions, G, $H_pG\geq 0$, which are logarithmically bounded in ξ (hence the C^∞ wave front set assumption on u allows the use of such weights) and which tend to $\langle\xi\rangle$ in a neighbourhood of (x_0,ξ_0). The normal form for p constructed in [Ha14] (following much earlier work of Guillemin–Schaeffer [GuSc77] which was based in turn on Sternberg's linearization theorem [St57]) was helpful in the construction of the specific weights needed here. We indicate the method of the proof in §1.2.

Proof of Theorem 1. Under the assumptions of Theorem 1 the characteristic set of P over $x_1=0$ is given by (in $T^*\mathbb{R}^n\setminus 0$)

$$p^{-1}(0)\cap\{x_1=0\}=\{(0,x_2,\xi_1,0):\xi_1\in\mathbb{R}\setminus 0; x_2\in \mathrm{neigh}_{\mathbb{R}^{n-1}}(0)\}=\Lambda_+\sqcup\Lambda_-,$$

where $\pm\xi_1>0$ on $\Lambda_\pm$. These two components are Lagrangian and conic and $H_p|_{\Lambda_\pm}=-\xi_1^2\partial_{\xi_1}|_{\Lambda_\pm}$ is radial. Since $Pu\in C^\omega(U)$ we have $\mathrm{WF_a}(Pu)\cap\{x\in U: x_1=0\}=\emptyset$ and hence Theorem 2 shows that $\mathrm{WF_a}(u)\cap\Lambda_\pm=\emptyset$. On the other hand, ([HöI, Theorem 8.6.1]), $\mathrm{WF_a}(u)\cap\{x_1=0\}\subset p^{-1}(0)\cap\{x_1=0\}=\Lambda_+\sqcup\Lambda_-$. Hence $\mathrm{WF_a}(u)\cap\{x_1=0\}=\emptyset$ and, since $\mathrm{singsupp_a}\, u=\pi\,\mathrm{WF_a}(u)$, u is analytic near $x_1=0$. $\square$

1.2. **A proof in a special case.** To indicate the ideas behind the proof we consider P given by

$$P=x_1D_{x_1}^2+D_{x_2}^2+aD_{x_1},\quad a\in\mathbb{C},$$

and a very special u:

$$u=e^{i\tau x_2}v(x_1),\quad v\in\mathscr{S}(\mathbb{R}),\quad Pu=e^{i\tau x_2}f(x_1),\quad e^{|\xi_1|}\widehat{f}\in L^2(\mathbb{R}). \tag{1.11}$$

This assumption is a stronger version of the assumption that f is analytic. We consider a family of smooth functions $G_\epsilon(\xi_1)$ satisfying

$$0\leq G_\epsilon(\xi_1)\leq\min(\tfrac{1}{\epsilon}\log(1+|\xi_1|),|\xi_1|) \tag{1.12}$$

In view of (1.11),

$$\|v_\epsilon\|_{L^2(\mathbb{R})} \le C_\epsilon, \quad \|f_\epsilon\|_{L^2(\mathbb{R})} \le C_0 \quad v_\epsilon := e^{G_\epsilon(D_x)} v, \quad f_\epsilon := e^{G_\epsilon(D_x)} f.$$

where C_0 is independent of ϵ. We then consider

$$P_\epsilon := e^{G_\epsilon(D_x)}(x_1 D_{x_1}^2 + aD_{x_1} + \tau^2)e^{-G_\epsilon(D_x)} = x_1 D_{x_1}^2 + iG'_\epsilon(D_{x_1})D_{x_1}^2 + aD_{x_1} + \tau^2.$$

We have $P_\epsilon v_\epsilon = f_\epsilon$, and

$$\begin{aligned} \operatorname{Im}\langle P_\epsilon v_\epsilon, v_\epsilon\rangle_{L^2(\mathbb{R})} &= \langle G'_\epsilon(D_{x_1})D_{x_1}^2 v_\epsilon, v_\epsilon\rangle_{L^2(\mathbb{R})} + \langle(\operatorname{Im} a + 1)D_{x_1} v_\epsilon, v_\epsilon\rangle_{L^2(\mathbb{R})} \\ &= \langle(\xi_1^2 G'_\epsilon(\xi_1) + (\operatorname{Im} a + 1)\xi_1)\widehat{v}_\epsilon, \widehat{v}_\epsilon\rangle_{L^2(\mathbb{R}_{\xi_1})}, \end{aligned}$$

where we took $d\xi_1/(2\pi)$ as the measure on $L^2(\mathbb{R}_{\xi_1})$. Let $\chi \in C^\infty(\mathbb{R};[0,1])$ satisfy $\chi|_{t\le 1} = 1$, $\chi|_{t\ge 2} = 0$ and $\chi' \le 0$. We define

$$G_\epsilon(\xi_1) = (1-\chi(\xi_1))\int_0^{\xi_1} (\chi(\epsilon t) + (1-\chi(\epsilon t))(\epsilon t)^{-1})dt,$$

which satisfies (1.12) and $G'_\epsilon \ge 0$. Moreover, for $\xi_1 \ge M \ge 2$ and $\epsilon < 1/M$,

$$\xi_1^2 G'_\epsilon(\xi_1) \ge \xi_1^2 \chi(\epsilon\xi_1) + \epsilon^{-1}\xi_1(1-\chi(\epsilon\xi_1)) \ge M\xi_1.$$

Hence, by taking $M = \max(-\operatorname{Im} a + 1, 2)$, and $\epsilon < 1/M$,

$$\begin{aligned} \|f_\epsilon\|\|\widehat{v}_\epsilon\| \ge \operatorname{Im}\langle P_\epsilon v_\epsilon, v_\epsilon\rangle &= \langle(\xi_1^2 G'_\epsilon(\xi_1) + (\operatorname{Im} a + 1)\xi_1)\widehat{v}_\epsilon, \widehat{v}_\epsilon\rangle \\ &\ge \|\widehat{v}_\epsilon\|^2 - \|(1+|\xi_1|(|\operatorname{Im} a|+1))\widehat{v}_\epsilon|_{\xi_1\le M}\|\|\widehat{v}_\epsilon\| \ge \|\widehat{v}_\epsilon\|^2 - C_1\|\widehat{v}_\epsilon\|, \end{aligned}$$

where $C_1 := (|\operatorname{Im} a| + 1)e^M\|v\|_{H^1}$ is independent of ϵ. This implies that

$$\|\widehat{v}_\epsilon\| \le \|f_\epsilon\| + C_1 \le C_0 + C_1.$$

Letting $\epsilon \to 0$ gives $\|e^{\xi_1}\widehat{v}|_{\xi_1\ge 0}\| \le C$. A similar argument applies to $\xi_1 \le 0$ which shows that

$$e^{|\xi_1|}\widehat{v} \in L^2,$$

and consequently that $u(x) = e^{ix_2\tau}v(x_1)$ is analytic.

In the actual proof, the Fourier transform is replaced by the FBI transform (2.1) and its deformation (2.5) defined using a suitably chosen G_ϵ satisfying (1.12) (see Lemma 3.1 which is the heart of the argument). One difficulty not present in the simple one dimensional case is the localization in other variables. It is here that the C^∞ normal forms of [St57],[GuSc77] and [Ha14] are particularly useful. It is essential that no analyticity is needed in the construction of G_ϵ.

1.3. **Applications to scattering theory.** As already indicated in [Zu17] analyticity of smooth solution to the Vasy operator (1.2) implies analyticity of resonant states and of their radiation patterns. We review this here and, in Theorem 3, present a slightly stronger result.

For a detailed presentation of scattering on asymptotically hyperbolic manifolds we refer to [DyZw19a, Chapter 5]. To state Theorem 3, let $\overline{M}$ be a compact $n+1$ dimensional manifold with boundary $\partial M \neq \emptyset$ and let $M := \overline{M} \setminus \partial M$. We assume that $\overline{M}$ is a *real analytic* manifold near ∂M. A metric g on M is called *asymptotically hyperbolic* and *analytic near infinity* if there exist functions $y' \in C^\infty(\overline{M}; \partial M)$ and $y_1 \in C^\infty(\overline{M}; (0,2))$, $y_1|_{\partial M} = 0$, $dy_1|_{\partial M} \neq 0$, such that

$$\overline{M} \supset y_1^{-1}([0,1)) \ni m \mapsto (y_1(m), y'(m)) \in [0,1) \times \partial M \tag{1.13}$$

is a real analytic diffeomorphism, and near ∂M the metric has the form,

$$g|_{y_1 \leq \epsilon} = \frac{dy_1^2 + h(y_1)}{y_1^2}, \tag{1.14}$$

where $[0,1) \ni t \mapsto h(t)$, is an analytic family of real analytic Riemannian metrics on ∂M.

Let

$$R_g(\lambda) = (-\Delta_g - \lambda^2 - (n/2)^2)^{-1} : L^2(M, d\,\mathrm{vol}_g) \to H^2(M, d\,\mathrm{vol}_g), \quad \operatorname{Im} \lambda > 0.$$

Mazzeo–Melrose [MM87] and Guillarmou [Gu05] proved that

$$R_g(\lambda) : C_{\mathrm{c}}^\infty(M) \to C^\infty(M), \tag{1.15}$$

continues to a meromorphic family of operators for $\lambda \in \mathbb{C} \setminus i(-\frac{1}{2} - \mathbf{N})$. In addition, Guillarmou [Gu05] showed that if the metric is *even*, that is,

$$g|_{y_1 \leq \epsilon} = \frac{dy_1^2 + h(y_1^2)}{y_1^2}, \tag{1.16}$$

(see [DyZw19a, Theorem 5.6] for an invariant formulation), then $R_g(\lambda)$ is meromorphic in $\mathbb{C}$. In particular, for $\lambda \neq 0$ we have the following Laurent expansion

$$R_g(\zeta) = \sum_{j=1}^{J(\lambda)} \frac{(-\Delta_g - \lambda^2 - (n/2)^2)^{j-1} \Pi(\lambda)}{(\zeta^2 - \lambda^2)^j} + A(\zeta, \lambda), \quad \Pi(\lambda) := \frac{1}{2\pi i} \oint_\lambda R_g(\zeta) 2\zeta d\zeta,$$

where $\zeta \mapsto A(\zeta, \lambda)$ is holomorphic near λ. For $\lambda = 0$ we have a Laurent expansions in powers of ζ^{-j}.

The operator $\Pi(\lambda)$ has finite rank and its range consists of *generalized resonant states*. We then have

Theorem 3. *Suppose that (M, g) is an even asymptotically hyperbolic manifold (in the sense of (1.16)) analytic near conformal infinity ∂M. Then for $\lambda \in \mathbb{C} \setminus 0$,*

$$u \in \Pi(\lambda) C_c^\infty(M) \implies u = y_1^{-i\lambda + \frac{n}{2}} F, \quad F|_{\partial M} \in C^\omega(\partial M). \tag{1.17}$$

Moreover, in coordinates of (1.16), $F(y) = f(y_1^2, y')$, $y' \in \partial M$ where $f \in C^\omega((-\delta, \delta) \times \partial M)$.

Proof. The metric (1.14) (in the coordinates valid near the boundary) gives the following Laplace operator:

$$\begin{aligned} -\Delta_g &= (y_1 D_{y_1})^2 + i(n + y_1 \gamma_0(y_1^2, y')) y_1 D_{y_1} - y_1^2 \Delta_{h(y_1)}, \\ \gamma_0(t, y') &:= -\tfrac{1}{2} \partial_t \bar{h}(t) / \bar{h}(t), \quad \bar{h}(t) := \det h(t), \quad D := \tfrac{1}{i} \partial. \end{aligned} \tag{1.18}$$

Following Vasy [Va13] we change the variables to $x_1 = y_1^2$, $x' = y'$ so that

$$y_1^{i\lambda - \frac{n}{2}} (-\Delta_g - \lambda^2 - (\tfrac{n}{2})^2) y_1^{-i\lambda + \frac{n}{2}} = x_1 P(\lambda), \tag{1.19}$$

where, near ∂M, $P(\lambda)$ is given by (1.2). This operator is considered on $X := ((-\delta, 0]_{x_1} \times \partial M) \sqcup M$. The key fact is that $P(\lambda)$ is a Fredholm family operators on suitable spaces, $P(\lambda)^{-1}$ is meromorphic and its poles can be studied using microlocal methods – see [Va13], [DyZw19a, Chapter 5] and also [Zw16, §2] for a short self-contained presentation.

From meromorphy of $P(\lambda)^{-1}$ we obtain meromorphy of (1.15) using (1.19):

$$R_g(\lambda) f := y_1^{\frac{n}{2} - i\lambda} \big(P(\lambda)^{-1} y_1^{i\lambda - \frac{n+2}{2}} f \big) \big|_M \in C^\infty(M). \tag{1.20}$$

Here we make $y_1^{i\lambda - \frac{n+2}{2}} f$ into an element of $C_c^\infty(X)$ by extending it by zero outside of M. Near any λ, $P(\zeta)^{-1} = \sum_{k=1}^{K(\lambda)} Q_j(\lambda)(\zeta - \lambda)^{-j} + Q_0(\zeta, \lambda)$, with $Q_j(\lambda)$ operators of finite rank and $\zeta \mapsto Q_0(\zeta, \lambda)$ is analytic near λ. We then have

$$\Pi(\lambda) = \tfrac{1}{2\lambda} y_1^{\frac{n}{2} - i\lambda} Q_1(\lambda) y_1^{i\lambda - \frac{n+2}{2}}.$$

Hence, the claim about the range of $\Pi(\lambda)$ follows from analyticity of functions in the range of $Q_1(\lambda)$. This follows from Theorem 1. In fact, $P(\zeta) = P(\lambda) + (\zeta - \lambda) V$, $V := -4 D_{x_1} + i\gamma(x)$, and hence

$$P(\lambda) Q_k(\lambda) = -V Q_{k+1}(\lambda), \quad Q_{K+1}(\lambda) := 0.$$

Since we already know that the ranges of Q_k's are in C^∞ (see [DyZw19a, (5.6.10)]) we inductively conclude that the ranges are in C^ω. ☐

Remark. Vasy's adaptation of Melrose's radial estimates [Me94] shows that to conclude that $u \in C^\infty$ when $P(\lambda) u \in C^\infty$ (see (1.2)), we only need to assume that $u \in H^{s+1}$ near m_0, where $s + \frac{1}{2} > -\operatorname{Im} \lambda$, see [Zw16, §4, Remark 3].

2. Preliminaries on FBI transforms and their deformations

We will use the FBI transform defined in [GaZw19b] in its $\mathbb{R}^n$ (rather than $\mathbb{T}^n$) version. Since the weights we use will be compactly supported in x the same theory applies. The constructions there are inspired by the works of Boutet de Monvel–Sjöstrand [BoSj76], Boutet de Monvel–Guillemin [BoGu81], Helffer–Sjöstrand [HeSj86] and Sjöstrand [Sj96]. An alternative approach to using the classes of weights we need here was developed independently and in greater generality by Guedes Bonthonneau–Jézéquel [GuJe20].

2.1. **Deformed FBI transforms.** We define

$$Tu(x,\xi) := h^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\langle x-y,\xi\rangle + \frac{i}{2}\langle\xi\rangle(x-y)^2)} \langle\xi\rangle^{\frac{n}{4}} u(y)dy, \quad u \in C_c^\infty(\mathbb{R}^n), \tag{2.1}$$

recalling that the left inverse of T is given by

$$Sv(y) = \frac{2^{\frac{n}{2}} h^{-\frac{3n}{4}}}{(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}2n} e^{-\frac{i}{h}(\langle x-y,\xi\rangle - \frac{i}{2}\langle\xi\rangle(x-y)^2)} \langle\xi\rangle^{\frac{n}{4}} (1 + \tfrac{i}{2}\langle x-y, \xi/\langle\xi\rangle\rangle) v(x,\xi) dx d\xi, \tag{2.2}$$

see [GaZw19b, Proposition 2.2].

The first fact we need is the characterization of Sobolev spaces and of the C^∞ wave front set using the FBI transform (2.1). To formulate it we use semiclassical Sobolev spaces H_h^s (see for instance [Zw12, §7.1] or [DyZw19a, Definition E.18]) but we should in general think of h as being fixed.

Proposition 2.1. *There exists a constant C such that for $u \in \mathscr{S}'(\mathbb{R}^n)$,*

$$\|u\|_{H_h^s} \le C \|\langle\xi\rangle^s Tu\|_{L^2(T^*\mathbb{R}^n)} \le C^2 \|u\|_{H_h^s}. \tag{2.3}$$

Moreover,

$$(x_0,\xi_0) \notin \mathrm{WF}(u) \;\Leftrightarrow\; \begin{cases} \exists\, \chi \in S^0(T^*\mathbb{R}^n), \ \chi \equiv 1 \text{ in a conic neighbourhood of } (x_0,\xi_0), \\ \forall\, N \ \exists\, C_N \quad \|\langle\xi\rangle^N \chi Tu\|_{L^2(T^*\mathbb{R}^n)} \le C_N. \end{cases}$$

Proof. This follows from the characterization of the H^s based wave front sets in Gérard [Gé90] as stated in [De, Theorem 1.2]. Since the arguments are similar to the more involved analytic case presented in Proposition 2.3 we omit the details. □

As in [Sj96, §2] and [GaZw19b, §3] we introduce a geometric deformation of $\mathbb{R}^{2n}$, $\Lambda = \Lambda_G$:

$$\begin{gathered} \Lambda := \{(x - iG_\xi(x,\xi), \xi + iG_x(x,\xi)) \mid (x,\xi) \in \mathbb{R}^{2n}\} \subset \mathbb{C}^{2n}, \\ \operatorname{supp} G \subset K \times \mathbb{R}^n, \quad K \Subset \mathbb{R}^n, \\ \sup_{|\alpha|+|\beta|\le 2} \langle\xi\rangle^{-1+|\beta|} |\partial_x^\alpha \partial_\xi^\beta G(x,\xi)| \le \epsilon_0, \quad |\partial_x^\alpha \partial_\xi^\beta G(x,\xi)| \le C_{\alpha\beta} \langle\xi\rangle^{1-|\beta|}, \end{gathered} \tag{2.4}$$

where ϵ_0 is small and fixed (so that the constructions below remain valid as in [GaZw19b]). For convenience, we change here the convention from [GaZw19b]: it amounts to to replacing G by $-G$ everywhere.

This provides us with the following new objects: the deformed FBI transform (see [GaZw19b, §4]),

$$\begin{aligned} T_\Lambda u(x,\xi) &:= Tu(x - iG_\xi(x,\xi), \xi + iG_x(x,\xi)), \quad u \in \mathscr{B}_\delta, \\ \mathscr{B}_\delta &:= \{u \in \mathscr{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\widehat{U}(\xi)|^2 e^{4\delta|\xi|} d\xi < \infty\}, \end{aligned} \tag{2.5}$$

the the spaces H^s_Λ, defined as in [GaZw19b, §4],

$$H^s_\Lambda := \overline{\mathscr{B}_{\delta_0}}^{\|\bullet\|_{H^s_\Lambda}}, \quad \|u\|^2_{H^s_\Lambda} := \int_\Lambda \langle \operatorname{Re}\alpha_\xi\rangle^{2s} |T_\Lambda u(\alpha)|^2 e^{-2H(\alpha)/h} d\alpha, \tag{2.6}$$

and the orthogonal projector

$$\Pi_\Lambda : L_\Lambda := L^2(\Lambda, e^{-2H(\alpha)/h} d\alpha) \to T_\Lambda H_\Lambda, \quad H_\Lambda := H^0_\Lambda,$$

described asymptotically (as $h \to 0$ and as $\xi \to \infty$) in [GaZw19b, §5]. The weight H appears naturally in this subject and is given by [GaZw19b, (3.3),(3.4)] i.e. $H(x,\xi) = \xi \cdot G_\xi(x,\xi) - G(x,\xi)$. The deformed FBI transform T_Λ has an exact left inverse S_Λ obtained by deforming S in (2.2).

We now prove a slightly modified version of [GaZw19b, Proposition 6.2]:

Proposition 2.2. *Suppose that $P = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ is a differential operator with $a_\alpha \in C^\infty_{\rm c}(\mathbb{R}^n)$ satisfying,*

$$a_\alpha \in C^\omega(U), \quad K \Subset U,$$

for an open set U and K as in (2.4). Then

$$\Pi_\Lambda T_\Lambda h^m P S_\Lambda = \Pi_\Lambda b_P \Pi_\Lambda + \mathcal{O}(h^\infty)_{H^{-N}_\Lambda \to H^N_\Lambda},$$

where

$$\begin{aligned} b_P(x,\xi) &\sim \sum_{j=0}^\infty h^j b_j(x,\xi), \quad b_j \in S^{m-j}(\mathbb{R}^{2n}), \\ b_0 &= p|_\Lambda := p(x - iG_\xi(x,\xi), \xi + iG_x(x,\xi)). \end{aligned} \tag{2.7}$$

We remark that the expansion remains valid when h is fixed. We can use smallness of h to dominate the lower order terms and then keep it fixed.

Proof. The result follows from the analogue of [GaZw19b, Lemma 6.1] where the operator $T_\Lambda h^m P S_\Lambda$ is described in the case where the coefficients of P are globally analytic. Here we point out that the analyticity of the coefficients is only needed in the neighbourhood U of $K \Subset \mathbb{R}^n$ such that in (2.4) $\operatorname{supp} G \subset K \times \mathbb{R}^n$ and ϵ_0 is small enough depending on the size of the complex neighbourhood to which the coefficients extend holomorphically.

In fact, arguing as in the proof of [GaZw19b, Proposition 6.2] all we need is that for $a \in C_c^\infty(\mathbb{R}^n)$ and $a \in C^\omega(U)$, the Schwartz kernel of $T_\Lambda M_a S_\Lambda$, $M_a f(x) := a(x)f(x)$, is given by

$$\begin{gathered} K_a(\alpha,\beta) = c_0 h^{-n} e^{\frac{i}{h}\Psi(\alpha,\beta)} A(\alpha,\beta) + r(\alpha,\beta), \quad \alpha,\beta \in \Lambda = \Lambda_G, \\ r(\alpha,\beta) \text{ is the kernel of an operator } R = O(h^\infty) : H_\Lambda^{-N} \to H_\Lambda^N. \end{gathered} \tag{2.8}$$

The phase in (2.8) is given by

$$\Psi(\alpha,\beta) = \frac{i}{2}\frac{(\alpha_\xi - \beta_\xi)^2}{\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle} + \frac{i}{2}\frac{\langle\beta_\xi\rangle\langle\alpha_\xi\rangle(\alpha_x - \beta_x)^2}{\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle} + \frac{\langle\beta_\xi\rangle\alpha_\xi + \langle\alpha_\xi\rangle\beta_\xi}{\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle}\cdot(\alpha_x - \beta_x), \tag{2.9}$$

and the amplitude satisfies

$$A \sim \sum_{j=0}^\infty h^j \langle\alpha_\xi\rangle^{-j} A_j, \quad A_0(\alpha,\alpha) = a|_\Lambda(\alpha),$$

and A_j are supported in a small conic neighbourhood of the diagonal in $\Lambda \times \Lambda$. We note that if ϵ_0 is small enough, a extends to some neighbourhood of K in $\mathbb{C}^n$ and hence $a|_\Lambda = a(x - iG_\xi(x,\xi))$ is well defined.

To see (2.8) we use the definitions of T_Λ and S_Λ to write

$$K_a(\alpha,\beta) = c_n \langle\beta_\xi\rangle^{\frac{n}{4}} \langle\alpha_\xi\rangle^{\frac{n}{4}} h^{-\frac{3n}{2}} \int e^{\frac{i}{h}(\varphi_G(\alpha,y) + \varphi_G^*(\beta,y))} a(y)\,(1 + \langle\beta_x - y, \beta_\xi/\langle\beta_\xi\rangle\rangle)\,dy, \tag{2.10}$$

where

$$\begin{gathered} \varphi_G(\alpha,y) = \Phi(z,\zeta,y)|_{z=\alpha_x,\zeta=\alpha_\xi}, \quad \varphi_G^*(\alpha,y) = -\bar\Phi(z,\zeta,y)|_{z=\alpha_x,\zeta=\alpha_\xi}, \\ \alpha_x = x - iG_\xi(x,\xi), \quad \alpha_\xi = \xi + iG_x(x,\xi), \\ \Phi(z,\zeta,y) = \langle z - y, \zeta\rangle + \tfrac{i}{2}\langle\zeta\rangle(z-y)^2, \quad \bar\Phi(z,\zeta,y) := \overline{\Phi(\bar z,\bar\zeta,y)}. \end{gathered} \tag{2.11}$$

Let V, V_1 open such that $K \subset V_1 \Subset V \Subset U$. We start by showing that the contribution to K_a away from the diagonal is negligible. For that let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0. Then for all $\delta > 0$ small enough, the operator R_1 with kernel

$$\begin{gathered} R_1(\alpha,\beta) = K_a(\alpha,\beta)\tilde\chi_\delta(\alpha,\beta), \\ \tilde\chi_\delta(\alpha,\beta) := (1 - \chi(\delta^{-1}|\alpha_x - \beta_x|))\left(1 - \chi\Big(\frac{|\alpha_\xi - \beta_\xi|}{\delta\langle|\alpha_\xi - \beta_\xi|\rangle}\Big)\right) \end{gathered}$$

satisfies $R_1 = O_{H_\Lambda^{-N} \to H_\Lambda^N}(h^\infty)$. This amounts to showing that the operator with kernel $R_1(\alpha,\beta)e^{\frac{1}{h}(H(\beta) - H(\alpha))}\langle\alpha_\xi\rangle^N\langle\beta_\xi\rangle^N$ is bounded on $L^2(\mathbb{R}^{2n})$ with $O(h^\infty)$ norm.

To see this, we first integrate by parts K times in y, using that

$$|\partial_y \Psi| = |\beta_\xi - \alpha_\xi + i(\langle\alpha_\xi\rangle(y - \alpha_x) + \langle\beta_\xi\rangle(y - \beta_x))| \geq c(1 + |\alpha_\xi| + |\beta_\xi|)$$

on $\operatorname{supp}\tilde\chi_\delta$. This reduces the analysis to the case of (2.10) with a is replaced by $b(\cdot,\alpha,\beta) \in C^\omega(U) \cap C_c^\infty(\mathbb{R}^n)$ with $|b| \leq h^K(\langle|\alpha_\xi|\rangle + \langle|\beta_\xi|\rangle)^{-K}$.

Next, we choose $\psi \in C_c^\infty(\mathbb{R}^n; [0,1])$ with $\psi \equiv 1$ on V and $\operatorname{supp}\psi \subset U$, and $\psi_1 \in C_c^\infty(\mathbb{R}^n; [0,1])$ with $\psi_1 \equiv 1$ on V_1 and $\operatorname{supp}\psi_1 \subset V$. We then deform the contour

$$y \mapsto y + i\epsilon\psi(y)\frac{\overline{\beta_\xi - \alpha_\xi}}{\langle|\beta_\xi - \alpha_\xi|\rangle}.$$

This contour deformation is justified since $a \in C^\omega(U)$. The phase in the integrand of (2.10) becomes

$$\begin{aligned}\Psi =& \langle \alpha_x - y, \alpha_\xi\rangle + \langle y - \beta_x, \beta_\xi\rangle + \frac{i\langle\alpha_\xi\rangle}{2}(\alpha_x - y)^2 + \frac{i\langle\beta_\xi\rangle}{2}(\beta_x - y)^2 \\ &+ i\epsilon\psi(y)\frac{|\beta_\xi - \alpha_\xi|^2}{\langle|\beta_\xi - \alpha_\xi|\rangle} + \frac{i\langle\alpha_\xi\rangle}{2}\Big[2\epsilon\psi(y)\langle\alpha_x - y, \frac{\overline{\alpha_\xi - \beta_\xi}}{\langle|\beta_\xi - \alpha_\xi|\rangle}\rangle - \epsilon^2\psi^2(y)\frac{|\beta_\xi - \alpha_\xi|^2}{\langle|\beta_\xi - \alpha_\xi|\rangle^2}\Big] \\ &\frac{i\langle\beta_\xi\rangle}{2}\Big[2\epsilon\psi(y)\langle\beta_x - y, \frac{\overline{\alpha_\xi - \beta_\xi}}{\langle|\beta_\xi - \alpha_\xi|\rangle}\rangle - \epsilon^2\psi^2(y)\frac{|\beta_\xi - \alpha_\xi|^2}{\langle|\beta_\xi - \alpha_\xi|\rangle^2}\Big]\end{aligned}$$

In particular, for $y \in V$, and $(\alpha, \beta) \in \operatorname{supp}\tilde{\chi}_\delta$, the integrand is bounded by

$$e^{-c(\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle)\langle\alpha_x - \beta_x\rangle/h}$$

which is negligible (even after multiplication by $e^{\frac{1}{h}(H(\beta) - H(\alpha))}\langle\alpha_\xi\rangle^N\langle\beta_\xi\rangle^N$).

For the integral over $y \notin V$, we consider three cases. First, if both $\operatorname{Re}\alpha_x \in K$ and $\operatorname{Re}\beta_x \in K$, then it is easy to see that the integrand is bounded by

$$e^{-c(\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle)(\langle\alpha_x - \beta_x\rangle + |y|)/h}$$

and hence produces a negligible contribution. Next, if $\operatorname{Re}\alpha_x \notin K$ and $\operatorname{Re}\beta_x \notin K$, then $H(\alpha) = H(\beta) = 0$, α, β are real, and integration by parts in y shows that the contribution is negligible.

Finally, we consider the case $\operatorname{Re}\alpha_x \in K$, $\operatorname{Re}\beta_x \notin K$, (the case $\operatorname{Re}\beta_x \in K$ and $\operatorname{Re}\alpha_x \notin K$ being similar). In this case, we have $H(\beta) = 0$ and β real. Since $y \notin V$, we have that the integrand is bounded by $e^{-c\langle\alpha_\xi\rangle\langle\alpha_x - y\rangle/h}h^K\langle\beta_\xi\rangle^{-K}$ and hence this term is also negligible.

Since R is negligible, we may assume from now on that

$$|\alpha_x - \beta_x| \ll 1 \ \text{ and } \ |\alpha_\xi - \beta_\xi| \ll \langle|\alpha_\xi|\rangle + \langle|\beta_\xi|\rangle.$$

In particular, there are three cases: $\operatorname{Re}\alpha_x \in K$ and $\operatorname{Re}\beta_x \in V_1$, $\operatorname{Re}\beta_x \in K$ and $\operatorname{Re}\alpha_x \in V_1$, or $\operatorname{Re}\alpha_x \notin K$ and $\operatorname{Re}\beta_x \notin K$.

The first two cases are similar, so we consider only one of them. Since $\operatorname{Re}\alpha_x \in K$ and $\operatorname{Re}\beta_x \in V_1$, the contribution from $y \notin V$ is negligible. Therefore, we may deform the contour to

$$y \mapsto y + \psi(y)y_c(\alpha, \beta), \qquad y_c(\alpha, \beta) = \frac{i(\beta_\xi - \alpha_\xi) + \langle\alpha_\xi\rangle\alpha_x + \langle\beta_\xi\rangle\beta_x}{\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle}.$$

The proof in this case then follows from the method of complex stationary phase.

When, both $\operatorname{Re}\alpha_x \notin K$ and $\operatorname{Re}\beta_x \notin K$, $\alpha = \operatorname{Re}\alpha$, $\beta = \operatorname{Re}\beta$, and $H(\alpha) = H(\beta) = 0$. In order to handle this situation, we will Taylor expand $a(y)$ around $y = \alpha_x$. For that we first consider (2.10) with $a = O(|y-\alpha_x|^{2N})$. In that case, we consider the integral

$$\begin{aligned} K_N(\alpha,\beta) := h^{-\frac{3n}{2}} \int e^{\frac{i}{h}(\langle \alpha_x - y, \alpha_\xi\rangle + \frac{i}{2}(\langle\alpha_\xi\rangle(\alpha_x-y)^2 + \langle\beta_\xi\rangle(\beta_x-y)^2))} \\ O(|y-\alpha_x|^{2N})\langle\alpha_\xi\rangle^{\frac{n}{4}}\langle\beta_\xi\rangle^{\frac{n}{4}}(1-\tilde\chi_\delta(\alpha,\beta))dy. \end{aligned} \tag{2.12}$$

Changing variables $y \mapsto y + \alpha_x$,

$$\begin{aligned} |K_N(\alpha,\beta)| &\le \int \langle\alpha_\xi\rangle^{\frac{n}{4}}\langle\beta_\xi\rangle^{\frac{n}{4}} \frac{h^{N-\frac{3n}{2}}}{\langle\alpha_\xi\rangle^N} e^{-\frac{\langle\beta_\xi\rangle}{2h}(\beta_x-\alpha_x-y)^2}(1-\tilde\chi_\delta)dy \\ &\le C\frac{h^{N-n}}{(\langle\alpha_\xi\rangle + \langle\beta_\xi\rangle)^N} e^{-c\frac{\langle\alpha_\xi\rangle+\langle\beta_\xi\rangle}{h}(\alpha_x-\beta_x)^2}(1-\tilde\chi_\delta(\alpha,\beta)). \end{aligned}$$

Therefore, using the Schur test for boundedness, the operator K_N with kernel $K_N(\alpha,\beta)$ satisfies

$$K_N = O(h^{N-\frac{n}{2}}) : H_\Lambda^{-N+\frac{n}{4}+0} \to H_\Lambda^{N-\frac{n}{4}-0}$$

Now, observe that for any $N > 0$,

$$a(y) = a_N(y) + O(|y-\alpha_x|^{2N})$$

where $a_N(y)$ is a polynomial of order $2N-1$ in $(y-\alpha_x)$. In particular,

$$K_a(\alpha,\beta) = K_{a_N}(\alpha,\beta) + K_N(\alpha,\beta)$$

Since a_N is analytic and the integrand is exponentially decaying in y, we may deform the contour with $y \mapsto y + y_c(\alpha,\beta)$ in the integral forming the kernel of K_{a_N} and apply complex stationary phase as in the case where $\operatorname{Re}\alpha_x \in K$ or $\operatorname{Re}\beta_x \in K$. This finishes the proof of the proposition after taking N large enough. □

2.2. **Analytic wave front set.** We now relate weighted estimates to analyticity.

Proposition 2.3. *Let T be the FBI transform defined in (2.1) for some fixed h, and let $\psi \in S^1(T^*\mathbb{R}^n)$ satisfy*

$$\psi(x,\xi) \ge |\xi|/C, \quad (x,\xi) \in U \times \Gamma, \tag{2.13}$$

where $U \subset \mathbb{R}^n$ and $\Gamma \subset \mathbb{R}^n \setminus 0$ is an open cone. Then, for $u \in H^{-N}(\mathbb{R}^n)$,

$$e^{\psi}\langle\xi\rangle^{-N} Tu \in L^2(T^*\mathbb{R}^n) \implies \mathrm{WF}_{\mathrm{a}}(u) \cap (U \times \Gamma) = \emptyset. \tag{2.14}$$

Conversely, suppose $u \in H^{-N}(\mathbb{R}^n)$, $\Gamma_0 \subset \mathbb{R}^n$ is a conic open set such that $\Gamma_0 \cap \mathbb{S}^{n-1} \Subset \Gamma \cap \mathbb{S}^{n-1}$, $U_0 \Subset U$. Then for any $\psi \in S^1(\mathbb{R}^n \times \mathbb{R}^n)$ with $\operatorname{supp}\psi \subset U_0 \times V_0$,

$$\mathrm{WF}_{\mathrm{a}}(u) \cap (U \times \Gamma) = \emptyset \implies \exists\, \theta > 0 \ \ \langle\xi\rangle^{-N} e^{\theta\psi} Tu \in L^2(T^*\mathbb{R}^n). \tag{2.15}$$

Remark: Here we do not consider uniformity in h in the L^2 bounds. If we demanded that, than we would only need $\psi \in C^\infty_{\rm c}(T^*\mathbb{R}^n)$, $\psi > 0$ on $U \times (\Gamma \cap \mathbb{S}^{n-1})$.

The proof is based on the following

Lemma 2.4. *Let T and S be given by (2.1) and (2.2), respectively, with h fixed. Suppose that $\chi, \tilde\chi \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$ and $\operatorname{supp}\chi, \operatorname{supp}\chi_1 \subset K \times \mathbb{R}^n$, $K \Subset \mathbb{R}^n$. Then for any $a > 0$ there exists $b > 0$ such that*

$$\chi e^{b\langle\xi\rangle} T S \chi_1 e^{-a\langle\xi\rangle} = \mathcal{O}_N(1) : L^2(\mathbb{R}^{2n}) \to H^N(\mathbb{R}^{2n}), \tag{2.16}$$

for any N.

If in addition $\chi_1 \equiv 1$ on a a conic neighbourhood of the support of χ, then there exists $b > 0$ such that

$$\chi e^{b\langle\xi\rangle} T S (1-\chi_1)\langle\xi\rangle^M = \mathcal{O}_{N,M}(1) : L^2(\mathbb{R}^{2n}) \to H^N(\mathbb{R}^{2n}), \tag{2.17}$$

for any N.

Proof. We analyse the Schwartz kernel of the operator in (2.16), $K(x,\xi,y,\eta)$. As in the proofs of [GaZw19b, Lemma 2.1, Proposition 4.5] (the phase of resulting operator can be computed by completion of squares and is given by [GaZw19b, (4.10)] with $\Lambda = T^*\mathbb{R}^n$) we see that

$$\begin{gathered} |(hD)^\alpha_{x,\xi} K(x,\xi,y,\eta)| \le C_\alpha e^{b\langle\xi\rangle - a\langle\eta\rangle - \psi(x,\xi,y,\eta)}, \\ \psi := c(\langle\xi\rangle + \langle\eta\rangle)^{-1}\left(|\xi-\eta|^2 + \langle\xi\rangle\langle\eta\rangle|x-y|^2\right). \end{gathered} \tag{2.18}$$

We have

$$b < \tfrac{1}{8}\min(a,c) \ \Rightarrow \ b\langle\xi\rangle - a\langle\eta\rangle - c(\langle\xi\rangle + \langle\eta\rangle)^{-1}|\xi-\eta|^2 \le -\tfrac{1}{2}(b\langle\xi\rangle + a\langle\eta\rangle),$$

if b is sufficiently small. (By taking $b < a/8$ we can assume that $|\eta| \le |\xi|/2$. But then $|\xi - \eta| \ge \frac{1}{2}|\xi|$ and $\langle\xi\rangle + \langle\eta\rangle \le 2\langle\eta\rangle$.) This proves (2.16) as we can use the Schur criterion.

To see (2.17) we note that we can now assume that $|\xi/\langle\xi\rangle - \eta/\langle\eta\rangle| > \delta$ or $|x-y| > \delta$. But then if the kernel of the operator in (2.17) is given by $K_M(x,\xi,y,\eta)$ where

$$|(hD_{x,\xi})^\alpha K_N(x,\xi,y,\eta)| \le C_{\alpha,N} e^{b\langle\xi\rangle - M\log\langle\eta\rangle - \psi(x,\xi,y,\eta)}.$$

Now, fix $0 < \delta < 1$ small. Then, when $|\xi/\langle\xi\rangle - \eta/\langle\eta\rangle| > \delta$ or $|x-y| > \delta$,

$$|\xi-\eta|^2 + \langle\xi\rangle\langle\eta\rangle|x-y|^2 \ge \frac{\delta^2}{16}(\langle\xi\rangle + \langle\eta\rangle)^2. \tag{2.19}$$

To see this, observe that on

$$\left|\frac{\langle\xi\rangle - \langle\eta\rangle}{\langle\xi\rangle + \langle\eta\rangle}\right| \ge \frac{\delta}{4},$$

we have

$$\frac{\delta}{4} \le \Big|\frac{\langle\xi\rangle^2-\langle\eta\rangle^2}{(\langle\xi\rangle+\langle\eta\rangle)^2}\Big| \le \frac{|\xi-\eta|}{\langle\xi\rangle+\langle\eta\rangle}.$$

On the other hand, when

$$\Big|\frac{\langle\xi\rangle-\langle\eta\rangle}{\langle\xi\rangle+\langle\eta\rangle}\Big| \le \frac{\delta}{4},$$

we have

$$\frac{2\langle\xi\rangle\langle\eta\rangle}{\langle\xi\rangle+\langle\eta\rangle} = \frac{\langle\xi\rangle+\langle\eta\rangle}{2}\Big(1-\Big[\frac{\langle\eta\rangle-\langle\xi\rangle}{\langle\xi\rangle+\langle\eta\rangle}\Big]^2\Big) \ge \frac{1}{4}(\langle\xi\rangle+\langle\eta\rangle)$$

Therefore, if $|x-y|\ge\delta$, (2.19) follows. If instead, $|\xi/\langle\xi\rangle-\eta/\langle\eta\rangle|\ge\delta$, then

$$\frac{|\xi-\eta|}{\langle\xi\rangle+\langle\eta\rangle} \ge \frac{1}{2}\Big[\Big|\frac{\xi}{\langle\xi\rangle}-\frac{\eta}{\langle\eta\rangle}\Big| - \Big(\frac{|\xi|}{\langle\xi\rangle}+\frac{|\eta|}{\langle\eta\rangle}\Big)\Big|\frac{\langle\xi\rangle-\langle\eta\rangle}{\langle\xi\rangle+\langle\eta\rangle}\Big|\Big] \ge \frac{\delta}{4}$$

and (2.19) follows.

From (2.19), we have that there is $C_{M,\delta}>0$ such that if $|\xi/\langle\xi\rangle-\eta/\langle\eta\rangle|>\delta$ or $|x-y|>\delta$,

$$\begin{aligned} b\langle\xi\rangle - &c(\langle\xi\rangle+\langle\eta\rangle)^{-1}\left(|\xi-\eta|^2+\langle\xi\rangle\langle\eta\rangle|x-y|^2\right) + M\log\langle\eta\rangle \\ &\le b\langle\xi\rangle - \tfrac{1}{64}c\delta^2(\langle\xi\rangle+\langle\eta\rangle) - \tfrac{1}{2}c(\langle\xi\rangle+\langle\eta\rangle)^{-1}\left(|\xi-\eta|^2+\langle\xi\rangle\langle\eta\rangle|x-y|^2\right) + C_{M,\delta}, \end{aligned}$$

and the Schur criterion and gives (2.17) for $b\le\frac{c\delta^2}{64}$. □

Proof of Proposition 2.3. We start by recalling the characterization of the analytic wave front set using the standard FBI/Bargmann–Segal transform:

$$\mathscr{T}u(x,\xi;h) := c_n h^{-\frac{3n}{4}}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\langle x-y,\xi\rangle+\frac{i}{2}(x-y)^2)}u(y)dy, \quad u\in\mathscr{S}'(\mathbb{R}^n).$$

Then

$$(x_0,\xi_0)\notin \mathrm{WF_a}(u) \iff \begin{cases} \exists\,\delta,\ U=\mathrm{neigh}((x_0,\xi_0)) \\ |\mathscr{T}u(x,\xi,h)|\le Ce^{-\delta/h}, \quad (x,\xi)\in U, \quad 0<h<h_0. \end{cases} \tag{2.20}$$

see [HöI, Theorem 9.6.3] for a textbook presentation; note the somewhat different convention: $\mathscr{T}u(x,\xi;h)=e^{-\frac{1}{2h}\xi^2}T_{1/h}u(x-i\xi)$.

We first prove (2.14). Hence suppose that $(x_0,\xi_0)\in U\times\Gamma$. Let $\chi\in S^0$ be supported in a small conic neighbourhood, $U_0\times\Gamma_0$, of (x_0,ξ_0) and choose $\chi_1\in S^0$ which is supported in $U\times\Gamma$ and is equal to 1 on a conic neighbourhood of the support of χ and $\chi_2\in S^0$ supported in $U\times\Gamma$ and equal to 1 on a conic neighborhood of the support of χ_1. Our assumptions then show that $e^{a\langle\xi\rangle/h}\chi_2Tu\in L^2(\mathbb{R}^{2n})$ for some $a>0$. We now write

$$\chi e^{b\langle\xi\rangle}Tu = \chi e^{b\langle\xi\rangle}TS\left(\chi_1 e^{-a\langle\xi\rangle}e^{a\langle\xi\rangle}\chi_2Tu + (1-\chi_1)\langle\xi\rangle^N\langle\xi\rangle^{-N}Tu\right).$$

Since $u \in H^{-N}$, $\langle\xi\rangle^{-N}Tu \in L^2(\mathbb{R}^{2n})$ and (2.16), (2.17), now show that $e^{b\langle\xi\rangle}\chi Tu \in H^K$ for some $b > 0$ and any K. By taking $K > n$ and applying [HöI, Corollary 7.9.4] we obtain a uniform bound

$$|Tu(x,\xi)| \leq Ce^{-b\langle\xi\rangle}, \quad (x,\xi) \in U_0 \times \Gamma_0.$$

Let h_1 be the fixed h in the definition of T. Then,

$$\mathscr{T}(x,\xi/\langle\xi\rangle; h_1/\langle\xi\rangle) = Tu(x,\xi) = \mathcal{O}(e^{-b\langle\xi\rangle}), \quad (x,\xi) \in U_0 \times \Gamma_0. \tag{2.21}$$

Putting $\omega_0 := \xi_0/\langle\xi_0\rangle$, it follows that $\mathscr{T}(x,\omega,h) = \mathcal{O}(e^{-\delta/h})$ for (x,ω) in a small neighbourhood of (x_0,ω_0). But then (2.20) shows that $(x_0,\omega_0) \notin \mathrm{WF_a}(u)$. Since $\mathrm{WF_a}(u)$ is a closed conic set, we conclude that $(x_0,\xi_0) \notin \mathrm{WF_a}(u)$.

Now suppose that $\mathrm{WF_a}(u) \cap (U \times \Gamma) = \emptyset$. Then for (x,ω) near $U_0 \times (\Gamma_0 \cap \mathbb{S}^{n-1})$ (with U_0 and Γ_0, as in the statement of the theorem), $\mathscr{T}(x,\omega,h) = \mathcal{O}(e^{-\delta/h})$. Reversing the argument in (2.21) we see that

$$|Tu(x,\xi)| \leq Ce^{-b\langle\xi\rangle}, \quad (x,\xi) \in U_0 \times \Gamma_0.$$

Now, since $u \in H^{-N}(\mathbb{R}^n)$, $\langle\xi\rangle^{-N}Tu \in L^2(\mathbb{R}^{2n})$. In particular, since $|\psi| \leq C\langle\xi\rangle$ and the support of ψ is contained in $U_0 \times \Gamma_0$, (2.15) follows. □

The next proposition relates weighted estimates to deformed FBI transform:

Proposition 2.5. *Suppose that H_Λ, $\Lambda = \Lambda_G$, is defined in* [GaZw19b, (4.7)] *with G satisfying* (2.4) *with ϵ_0 chosen as in the definition of H_Λ.*

Then there exists $\psi \in S^1(T^*\mathbb{R}^n)$ such that $T : \mathscr{B}_\delta \to L^2(T^*\mathbb{R}^n, e^{\delta\langle\xi\rangle/Ch}dxd\xi)$ extends to

$$T = \mathcal{O}(1) : H_\Lambda \to L^2(T^*\mathbb{R}^n, e^{2\psi(x,\xi)/h}dxd\xi), \tag{2.22}$$

and $S : L^2(T^*\mathbb{R}^n, e^{-C\delta\langle\xi\rangle/h}dxd\xi) \to \mathscr{B}_\delta$, extends to

$$S = \mathcal{O}(1) : L^2(T^*\mathbb{R}^n, e^{2\psi(x,\xi)/h}dxd\xi) \to H_\Lambda. \tag{2.23}$$

In addition,

$$\psi(x,\xi) = G(x,\xi) + \mathcal{O}(\epsilon_0^2)_{S^1(T^*\mathbb{R}^n)}. \tag{2.24}$$

For a simpler version of this result in the case of compactly supported weights see [GaZw19a, §8].

Proof. The statement (2.22) is equivalent to

$$TS_\Lambda = \mathcal{O}(1) : L^2(\Lambda, e^{-2H(\alpha)/h}d\alpha) \to L^2(T^*\mathbb{R}^n, e^{2\psi(\beta)}d\beta)$$

and hence we analyse the kernel of the operator TS_Λ which is given by

$$K(\alpha,\beta) = c_n h^{-\frac{3n}{2}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\varphi_0(\alpha,y)+\varphi_G^*(\beta,y))}\langle\beta_\xi\rangle^{\frac{n}{4}}\langle\alpha_x\rangle^{\frac{n}{4}}(1+\tfrac{i}{2}\langle\alpha_x - y\rangle)dy,$$

where the notation (and also notation for Φ below) comes from (2.11). The integral in y converges and can be evaluated by a completion of squares as in [GaZw19b, Proposition 4.4]. That gives the phase (2.9) with $\alpha \in T^*\mathbb{R}^n$ and $\beta \in \Lambda$. The critical point in y is given by

$$y_c(\alpha, \beta) = \frac{1}{\langle \alpha_\xi \rangle + \langle \beta_\xi \rangle} \left(\langle \alpha_\xi \rangle \alpha_x + \langle \beta_\xi \rangle \beta_x + i(\beta_\xi - \alpha_\xi) \right). \tag{2.25}$$

We then have (2.22) with

$$\psi(\alpha) := \max_{\beta \in \Lambda} \left(-\operatorname{Im} \Psi(\alpha, \beta) + H(\beta) \right). \tag{2.26}$$

We have (see [GaZw19b, (3.3),(3.4)])

$$d_\beta(-\operatorname{Im} \Psi(\alpha, \beta) + H(\beta)) = \operatorname{Im}(-\partial_{z,\zeta} \Psi(\alpha, (z, \zeta)) - \zeta dz|_\Lambda)|_{(z,\zeta)=\beta \in \Lambda}.$$

Now, if $y_c(\alpha, (z, \zeta))$ is the critical point in y, then

$$\begin{aligned}\partial_{z,\zeta} \Psi(\alpha, z) &= \partial_{z,\zeta}(\Phi(\alpha, y_c(\alpha, (z, \zeta))) - \bar{\Phi}((z, \zeta), y_c(\alpha, (z, \zeta)))) = -\partial_{z,\zeta} \bar{\Phi}\big|_{y=y_c(z,\zeta)}(z, \zeta) \\ &= -\zeta \cdot dz + (y_c - z) \cdot d\zeta + i\langle \zeta \rangle (z - y_c) \cdot dz + \tfrac{i}{2}(z - y_c)^2 \zeta \cdot d\zeta / \langle \zeta \rangle.\end{aligned}$$

For $G = 0$ the critical point (see (2.25)) is given by $\alpha = \beta$. Hence

$$\beta_c = \beta_c(\alpha) = (\alpha_x + \mathcal{O}(\epsilon_0)_{S^0}, \alpha_\xi + \mathcal{O}(\epsilon_0)_{S^1}), \tag{2.27}$$

with ϵ_0 as in (2.4).

Hence we obtain ψ by inserting the critical point β_c into the right hand side of (2.26)

$$\psi(\alpha) = -\operatorname{Im} \Psi(\alpha, \beta_c(\alpha)) + H(\beta_c(\alpha)) \in S^1(T^*\mathbb{R}^n). \tag{2.28}$$

(We note that for $G = 0$ the maximum in (2.26) is non-degenerate and unique and it remains such under small symbolic perturbations.) From (2.9) we see that

$$\operatorname{Im} \Psi(\alpha, \beta_c(\alpha)) = \operatorname{Im} \Psi(\alpha, \alpha + \mathcal{O}(\epsilon_0)_{S^0 \times S^1}) = \alpha_\xi \cdot G_\xi(\alpha) + \mathcal{O}(\epsilon_0^2)_{S^1}.$$

Inserting this into (2.28) and recalling that $H = \xi G_\xi - G$ we obtain (2.24).

To obtain (2.23) we apply the same analysis to $T_\Lambda S$ and we need to show that two weights coincide. That is done as in [GaZw19a, §8]. □

3. Proof of Theorem 2

As already indicated in §1.2, to prove the theorem we construct a family of weights $G_\epsilon \in S^1$, uniformly bounded in S^1, supported in a conic neighbourhood of $\Gamma = \{(0, 0, \xi_1, 0) : \xi_1 > M\}$, $M \gg 1$, and satisfying $0 \le G_\epsilon \le C_\epsilon \log\langle \xi \rangle$. In addition,

$$H_p G_\epsilon \ge 0, \qquad G_\epsilon \to \xi_1 \text{ on } \Gamma \text{ (in } S^{1+}), \tag{3.1}$$

with $H_p G_\epsilon \gg \xi_1^{m-1}$ in a suitable sense (see (3.4)) for $\epsilon \ll 1$.

We will then put $\Lambda_\epsilon := \Lambda_{G_\epsilon}$ so that the assumption $u \in C^\infty$ will give $u \in H_{\Lambda_\epsilon}$. On the other hand the assumption that $\Gamma \cap \mathrm{WF_a}(Pu)$ shows that $\|Pu\|_{H_{\Lambda_\epsilon}} \leq C$ with the constant C independent of ϵ. But then [GaZw19b, Proposition 6.2] and the properties of G_ϵ show that $\|u\|_{H_{\Lambda_\epsilon}}$ is bounded independently of ϵ. Propositions 2.3 and 2.5 then show that $\mathrm{WF_a}(u) \cap \Gamma_0 = \emptyset$.

3.1. **Construction of the weight.** We now construct a family of weights, G_ϵ, satisfying (3.1). In fact, we need more precise conditions on G_ϵ given in the following

Lemma 3.1. *Suppose that p satisfies (1.9) at $\rho_0 = (x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ and Γ is an open conic neighbourhood of ρ_0. Then, there exists $G_\epsilon \in S^1(T^*\mathbb{R}^n)$, $\operatorname{supp} G_\epsilon \subset \Gamma$, such that*

$$\begin{gathered} |\partial_x^\alpha \partial_\xi^\beta G_\epsilon| \leq C_{\alpha\beta} \langle \xi \rangle^{1-|\beta|}, \quad 0 \leq G_\epsilon \leq C\epsilon^{-1} \log\langle \xi \rangle, \\ G_\epsilon(x,\xi)|_{1 \leq |\xi| \leq 1/\epsilon} = \Phi(x,\xi)|\xi|, \quad \Phi \in S^0_{\mathrm{phg}}(T^*\mathbb{R}^n), \quad \Phi(x_0, t\xi_0) = 1, \ t \gg 1, \end{gathered} \tag{3.2}$$

$$H_p G_\epsilon(x,\xi) \geq c_0 \left(\langle \xi \rangle^m |\partial_\xi G_\epsilon(x,\xi)|^2 + \langle \xi \rangle^{m-2} |\partial_x G_\epsilon(x,\xi)|^2 \right), \tag{3.3}$$

$$\forall\, M_1,\, \gamma \geq 0 \ \exists\, M_2,\, K,\, \epsilon_0 \,\forall\, 0 < \epsilon < \epsilon_0, \quad H_p G_\epsilon e^{\gamma G_\epsilon} + M_2 \langle \xi \rangle^K \geq M_1 \langle \xi \rangle^{m-1} e^{\gamma G_\epsilon}. \tag{3.4}$$

We stress that the constants $C_{\alpha\beta}$ and c_0 are independent of ϵ and M_1.

Proof. We use the normal form for p constructed in [Ha14, §3]. That means that we take $x_0 = 0$ and $\xi_0 = e_1 := (1, 0, \cdots, 0)$ and can assume that $p(x,\xi) = -\xi_1^m x_1$ in a conic neighbourhood of $\rho = (0, e_1)$. For simplicity we can assume that $m = 1$ as the argument is the same otherwise.

Let $\chi \in C_c^\infty(\mathbb{R}; [0,1])$ satisfy

$$\operatorname{supp} \chi \subset [-2, 2], \quad \chi_{|t| \leq 1} = 1, \quad t\chi'(t) \leq 0. \tag{3.5}$$

and put $\varphi(t) := \chi(t/\delta)$. Here δ will be fixed depending on Γ. Using this function we define $\Phi = \Phi(x,\xi) := \varphi_1 \varphi_2 \varphi_3 \psi$ where

$$\varphi_1 := \varphi(x_1), \quad \varphi_2 := \varphi(|\xi'|/\xi_1) \quad \varphi_3 = \varphi(|x'|), \quad \psi := (1 - \varphi((\xi_1)_+)). \tag{3.6}$$

We choose δ small enough so that $\operatorname{supp} \Phi \subset \Gamma$.

We define G_ϵ as follows

$$G_\epsilon(x,\xi) = \Phi(x,\xi) q_\epsilon(\xi_1), \quad q_\epsilon(t) := \int_0^t \left(\chi(\epsilon s) + (1 - \chi(\epsilon s))(s\epsilon)^{-1} \right) ds. \tag{3.7}$$

We check that

$$\begin{gathered} \xi_1 \partial_{\xi_1} q_\epsilon \geq \min(\xi_1, \epsilon^{-1}), \\ \xi_1 \, \mathbb{1}_{\xi_1 \leq 1/\epsilon} + \epsilon^{-1}(1 + \log(\epsilon\xi_1)) \, \mathbb{1}_{\xi \geq 1/\epsilon} \leq q_\epsilon \leq \xi_1 \, \mathbb{1}_{\xi_1 \leq 1/\epsilon} + \epsilon^{-1}(2 + \log(\epsilon\xi_1)) \, \mathbb{1}_{\xi \geq 1/\epsilon} . \end{gathered} \tag{3.8}$$

Uniform boundedness of G_ϵ in S^1 means that q_ϵ in (3.7) satisfies $|\partial_{\xi_1}^k q_\epsilon| \le C_k \xi_1^{1-k}$ with C_k's independent of ϵ. But this is immediate from the definition. We also easily see that G_ϵ converges to $G := \Phi(x,\xi)\xi_1$ in S^{1+} as $\epsilon \to 0$. This proves (3.2).

To see (3.3), we first note that, since $\Phi \ge 0$, $\Phi \in S^0$, the standard estimate $f(z) \ge 0 \Longrightarrow |df(z)|^2 \le Cf(z)$ gives,

$$\Phi(x,\xi) \ge c_1 \left(\xi_1^2 |\partial_\xi \Phi(x,\xi)|^2 + |\partial_x \Phi(x,\xi)|^2 \right). \tag{3.9}$$

Note also that we have $H_p = \xi_1 \partial_{\xi_1} - x_1 \partial_{x_1}$ and therefore

$$H_p \Phi = -x_1 \varphi'(x_1)\varphi_2\varphi_3\psi - (|\xi'|/\xi_1)\varphi'(|\xi'|/\xi_1)\varphi_1\varphi_3\psi - \varphi_1\varphi_2\varphi_3\xi_1\varphi'((\xi_1)_+) \ge 0. \tag{3.10}$$

Since $q_\epsilon \in S^1$, $\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \ge c_2 \xi_1 (\partial_{\xi_1} q_\epsilon(\xi_1))^2$. We also claim that

$$\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \ge c_2 \xi_1^{-1} q_\epsilon(\xi_1)^2. \tag{3.11}$$

In fact, using (3.8) we see that to prove (3.11) it is enough to have

$$\min(t, \epsilon^{-1}) \ge c_2 t^{-1} \left(t\, \mathbb{1}_{t \le 1/\epsilon}(t) + \epsilon^{-1}(2 + \log(t\epsilon))\, \mathbb{1}_{t \ge 1/\epsilon}(t) \right)^2 .$$

This clearly holds (with $c_2 = 1$) for $t \le 1/\epsilon$ and for $t \ge \epsilon$ is equivalent to $c_2(2+\log s)^2 \le s$, $s = t\epsilon \ge 1$, which holds with $c_2 = \frac{1}{4}$. It follows that

$$\xi_1 \partial_{\xi_1} q_\epsilon(\xi_1) \ge c_2 \left(\xi_1^{-1} q_\epsilon(\xi_1)^2 + \xi_1 (\partial_{\xi_1} q_\epsilon(\xi_1))^2 \right),$$

which combined with (3.9) and (3.10) gives

$$\begin{aligned} H_p G_\epsilon &= \Phi(\xi_1 \partial_{\xi_1} q_\epsilon) + (H_p \Phi) q_\epsilon \\ &\ge \Phi(\xi_1 \partial_{\xi_1} q_\epsilon) \ge c_2 \xi_1 \Phi (\partial_{\xi_1} q_\epsilon)^2 + c_3 \left(\xi_1^2 |\partial_\xi \Phi|^2 + |\partial_x \Phi|^2 \right) \xi_1^{-1} q_\epsilon^2 \\ &\ge c_0 \left(\xi_1 |\partial_\xi G_\epsilon|^2 + \xi_1^{-1} |\partial_x G_\epsilon|^2 \right). \end{aligned}$$

Since $\langle \xi \rangle \sim \xi_1$ on the support of G_ϵ, we obtain (3.3).

Finally we prove (3.4). Since by (3.10) we have $H_p G_\epsilon \ge \Phi H_p q_\epsilon$, we see that (3.4) follows from proving that for any M_1 we can find K, M_2 and ϵ_0 such that for $\xi_1 \ge 1$,

$$\Phi H_p q_\epsilon e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \ge M_1 e^{\gamma \Phi q_\epsilon}. \tag{3.12}$$

Using (3.8), we see that for $\xi_1 \le 1/\epsilon$ we need $G_\epsilon e^{\gamma G_\epsilon} + M_2 \xi_1^K \ge M_1 e^{\gamma G_\epsilon}$. This holds for

$$K = 0, \quad M_2 = 2\gamma^{-1} e^{\gamma M_1 - 1}$$

since for $\gamma > 0$ and $a \ge 0$, $a e^{\gamma a} - M_1 e^{\gamma a} \ge -2\gamma^{-1} e^{\gamma M_1 - 1}$.

For $\xi_1 \ge 1/\epsilon$, we need to find K and M_2 for which

$$\epsilon^{-1} \Phi e^{\gamma \Phi q_\epsilon} + M_2 \xi_1^K \ge M_1 e^{\gamma \Phi q_\epsilon}. \tag{3.13}$$

Using $a e^{ab} + M_1 e^{M_1 b} \ge M_1 e^{ab}$ with $a := \epsilon^{-1}\Phi$ and

$$b := \gamma \epsilon q_\epsilon \le \gamma(2 + \log(\epsilon \xi_1)) \le \gamma(2 + \log \xi_1),$$

we obtain (3.13) with $M_2 = M_1 e^{2\gamma M_1}$ and $K = \gamma M_1$. Hence we obtain (3.12) proving (3.4). ☐

3.2. **Microlocal analytic hypoellipticity.** We will have bounds which are uniform in ϵ but not in h. We start with the following

Lemma 3.2. *Suppose that P is of the form (1.8) with real valued principal symbol p and suppose that $\Gamma \subset U \times \mathbb{R}^n \setminus$ is an open cone, $\Gamma \cap \mathbb{S}^{n-1} \Subset U \times \mathbb{S}^{n-1}$ and*

$$\begin{gathered} G \in S^1(\Gamma;\mathbb{R}), \quad |G| \le C \log\langle\xi\rangle, \\ H_p G(x,\xi) \ge c_0 \left(\langle\xi\rangle^m |\partial_\xi G(x,\xi)|^2 + \langle\xi\rangle^{m-2} |\partial_x G(x,\xi)|^2 \right). \end{gathered} \tag{3.14}$$

Then for T_Λ, H_Λ, $\Lambda = \Lambda_{\theta G}$ defined in (2.4) and (2.6), h and θ sufficiently small, and $u \in H_\Lambda^{-N+m}$,

$$\operatorname{Im}\langle h^m P u, u\rangle_{H_\Lambda^{-N}} \ \ge \tfrac12 \theta \langle H_p G \,\langle\xi\rangle^{-N} T_\Lambda u, \langle\xi\rangle^{-N} T_\Lambda u\rangle_{L^2_\Lambda} - Mh\|u\|^2_{H_\Lambda^{\frac{m-1}{2}-N}}, \tag{3.15}$$

where M depends only on P and the semi-norms of G in S^1.

Proof. We use Proposition 2.2 and [GaZw19b, Proposition 6.3] to see that for any $K > 0$,

$$\begin{aligned} \operatorname{Im}\langle h^m P u, u\rangle_{H_\Lambda^{-N}} &= \operatorname{Im}\langle \langle\xi\rangle^{-2N} T_\Lambda h^m P S_\Lambda T_\Lambda u, T_\Lambda u\rangle_{L^2_\Lambda} \\ &= \operatorname{Im}\langle \Pi_\Lambda \langle\xi\rangle^{-2N} \Pi_\Lambda h^m P S_\Lambda \Pi_\Lambda T_\Lambda u, T_\Lambda u\rangle_{L^2_\Lambda} \\ &= \langle (\operatorname{Im} b_{P,N}) T_\Lambda u, T_\Lambda u\rangle_{L^2_\Lambda} + \mathcal{O}(h^\infty)\|u\|_{H_\Lambda^{-K}} \\ &\ge \langle (\operatorname{Im} p|_\Lambda)\, \langle\xi\rangle^{-N} T_\Lambda u, \langle\xi\rangle^{-N} T_\Lambda u\rangle_{L^2_\Lambda} - Mh\|u\|_{H_\Lambda^{\frac{m-1}{2}-N}}. \end{aligned} \tag{3.16}$$

From (2.7) and (3.14) we obtain

$$\begin{aligned} \operatorname{Im} p|_\Lambda &= \operatorname{Im} p(x - i\theta\partial_\xi G(x,\xi), \xi + i\theta\partial_x G(x,\xi)) \\ &= \theta H_p G(x,\xi) + \theta^2 \mathcal{O}\left(\langle\xi\rangle^m |\partial_\xi G(x,\xi)|^2 + \langle\xi\rangle^{m-2} |\partial_x G(x,\xi)|^2 \right) \\ &\ge \tfrac12 \theta H_p G(x,\xi), \end{aligned}$$

if θ is small enough. ☐

The next lemma allows us to use smoothness of u to obtain weaker weighted estimates:

Lemma 3.3. *Suppose $U \subset \mathbb{R}^n$ is an open set,*

$$G \in S^1(T^*\mathbb{R}^n), \quad G \ge 0, \quad \operatorname{supp} G \subset K \times \mathbb{R}^n, \quad K \Subset U,$$

and T_Λ, H_Λ, $\Lambda = \Lambda_{\theta G}$ are defined in (2.4) and (2.6). Then, there exists $a > 0$ such that for every $\chi, \tilde\chi \in S^1$ with $\tilde\chi \equiv 1$ in a conic neighborhood of $\operatorname{supp}\chi$ and every $K, N > 0$, there exists $c, C > 0$ such that for all $u \in H^{-N}(\mathbb{R}^n)$,

$$\|\langle\xi\rangle^K e^{-aG/h} \chi T_\Lambda u\|_{L^2_\Lambda} \le C(\|\langle\xi\rangle^K \tilde\chi T u\|_{L^2(T^*\mathbb{R}^n)} + e^{-c/h}\|\langle\xi\rangle^{-N} T u\|_{L^2(T^*\mathbb{R}^n)}). \tag{3.17}$$

In particular, if $\chi \equiv 1$ *on* $\operatorname{supp} G$*, then*

$$\begin{aligned}&\|(\langle\xi\rangle^K e^{-a/h}\chi + \langle\xi\rangle^{-N}(1-\chi))T_\Lambda u\|_{L^2_\Lambda}\\ &\qquad \le C(\|\langle\xi\rangle^N \tilde{\chi} T u\|_{L^2(T^*\mathbb{R}^n)} + e^{-C/h}\|\langle\xi\rangle^{-N} T u\|_{L^2(T^*\mathbb{R}^n)}).\end{aligned} \tag{3.18}$$

Proof. First, observe that by [GaZw19b, Lemma 4.5], for any $\delta > 0$,

$$T_\Lambda S = K_\delta + O_{N,\delta}(e^{-c_\delta/h})_{\langle\xi\rangle^N L^2(T^*\mathbb{R}^n)\to\langle\xi\rangle^{-N}L^2_\Lambda},$$

and K_δ has kernel, $K_\delta(\alpha,\beta)$, given by

$$h^{-n}e^{\frac{i}{h}\Psi(\alpha,\beta)}k(\alpha,\beta)\psi(\delta^{-1}|\operatorname{Re}\alpha_x - \beta_x|))\psi(\delta^{-1}\min(\langle\operatorname{Re}\alpha_\xi\rangle,\langle\beta_\xi\rangle)^{-1}|\operatorname{Re}\alpha_\xi - \beta_\xi|),$$

where $(\alpha,\beta)\in\Lambda\times T^*\mathbb{R}^n$ and Ψ is as in (2.9), and $\psi\in C_c^\infty(\mathbb{R})$ is identically 1 near 0. Therefore, we need only consider $K_\delta(\alpha,\beta)$.

To do this, let $\tilde{\chi}\in S^0$ be identically 1 on a conic neighborhood of $\operatorname{supp}\chi$. Then, for $\delta > 0$ small enough,

$$\chi(\operatorname{Re}\alpha)K_\delta(\alpha,\beta)(1-\tilde{\chi})(\beta)\equiv 0.$$

Therefore,

$$\chi e^{-aG/h}\langle\xi\rangle^K T_\Lambda S(1-\tilde{\chi}) = O_N(e^{-c/h})_{\langle\xi\rangle^N L^2(T^*\mathbb{R}^n)\to\langle\xi\rangle^{-N}L^2_\Lambda}.$$

For the mapping properties

$$\chi e^{-aG/h}T_\Lambda S\tilde{\chi} : \langle\xi\rangle^{-K}L^2(T^*\mathbb{R}^n)\to\langle\xi\rangle^{-K}L^2_\Lambda,$$

we consider the operator

$$\chi e^{-aG/h}e^{-H/h}\langle\xi\rangle^K T_\Lambda S\tilde{\chi}\langle\xi\rangle^{-K} : L^2(T^*\mathbb{R}^n)\to L^2(\Lambda; dxd\xi).$$

Modulo negligible terms, the kernel of this operator is given by

$$h^{-n}e^{\frac{i}{h}(\varphi((x,\xi),(y,\eta)))}\tilde{k}((x,\xi),(y,\eta))$$

where $\tilde{k}\in S^0$ has

$$\operatorname{supp}\tilde{k}\subset\{|\xi-\eta|\le C\delta\langle\xi\rangle\}\cap\{|x-y|\le C\delta\}. \tag{3.19}$$

and

$$\varphi = iH(x,\xi) + ia\theta G(x,\xi) + \Psi((x - i\theta G_\xi, \xi + i\theta G_x(x,\xi)),(y,\eta)),$$

with $H(x,\xi) = \theta\langle\xi, G_\xi(x,\xi)\rangle - \theta G(x,\xi)$. Using (3.19), we have

$$\begin{aligned}\operatorname{Im}\varphi &= aG + \theta\xi\cdot G_\xi - \theta G + \frac{\langle\eta\rangle\langle\xi\rangle}{2(\langle\eta\rangle+\langle\xi\rangle)}\left((x-y)^2 - (\theta G_\xi)^2\right) + \frac{(\xi-\eta)^2 - (\theta G_\xi)^2}{2(\langle\eta\rangle+\langle\xi\rangle)}\\ &\quad + \theta\xi\cdot G_\xi + O(\theta(|x-y||G_x| + \langle\xi\rangle^{-1}|\xi-\eta||G_\xi|))\\ &\quad + O(\theta^2(\langle\xi\rangle^{-1}|G_x|^2 + \langle\xi\rangle|G_\xi|^2))\\ &\ge (a-\theta)G - C\theta^2(\langle\xi\rangle^{-1}(G_x)^2 + \langle\xi\rangle|G_\xi|^2) + c\langle\xi\rangle(x-y)^2 + c\langle\xi\rangle^{-1}(\xi-\eta)^2.\end{aligned}$$

In particular, taking a large enough and using that $G \geq 0$, $G \in S^1$, (see the argument for (3.9)), we have

$$\operatorname{Im} \varphi \geq \frac{a}{2} G(x, \xi) + c\langle \xi \rangle (x-y)^2 + c\langle \xi \rangle^{-1} (\xi - \eta)^2.$$

Therefore, applying the Schur test for L^2 boundedness completes the proof that

$$\chi \langle \xi \rangle^K e^{-aG/h} T_\Lambda S \langle \xi \rangle^{-K} = O(1) : L^2(T^*\mathbb{R}^n) \to L^2_\Lambda$$

and the lemma follows. □

With these two lemmas in place we can prove the main result:

Proof of Theorem 2. By multiplying u by a $C^\infty_{\rm c}$-function which is 1 in a neighbourhood of x_0, we can assume that $u \in H^{-N+m}$, for some N, is compactly supported in U and $\rho_0 := (x_0, \xi_0) \notin \mathrm{WF}(u)$. By Proposition 2.1, there exists $\tilde{\chi} \in S^0$ with $\tilde{\chi} \equiv 1$ in an open conic neighborhood, Γ, of ρ_0 such that for any $K > 0$,

$$\|\langle \xi \rangle^K \tilde{\chi} T u\|_{L^2} \leq C_K. \tag{3.20}$$

Also, since $u \in H^{-N+m}$,

$$\|\langle \xi \rangle^{-N+m} T u\|_{L^2} \leq C. \tag{3.21}$$

Let $\Gamma_1 \Subset \Gamma$ be an open conic neighborhood of ρ_0 and $\chi \in S^1$ with $\chi \equiv 1$ on Γ_1 and $\operatorname{supp} \chi \subset \Gamma$.

We choose θ small enough so that (2.4) and (3.16) hold. We then fix $0 < h \leq 1$ small enough so that (3.16) holds. From now we neglect the dependence on h which is considered to be a fixed parameter. We choose for $G = G_\epsilon$ constructed in Lemma 3.1 and supported in Γ_1. We recall that the estimates depend only on the S^1 seminorms of G and these are uniform in ϵ. We now claim that

$$u \in H^{-N+m}_{\Lambda_\epsilon}, \qquad \Lambda_\epsilon := \Lambda_{\theta G_\epsilon}.$$

In fact, we can use (3.18) together with (3.20) and (3.21), observing that $\exp(aG_\epsilon/h) = \mathcal{O}_\epsilon(\langle \xi \rangle^{Ca/(h\epsilon)})$ and taking $K = Ca/(h\epsilon)$.

Next, note that $Pu \in H^{-N}$ is supported in U and $\rho_0 \notin \mathrm{WF}_a(Pu)$ .Propositions 2.3 and 2.5 (see (2.15) and (2.23) respectively) then show that for G_ϵ satisfying the assumptions of Lemma 3.2 and θ sufficiently small $\|Pu\|_{H^{-N}_{\Lambda_\epsilon}} \leq C_0$, where C_0 depends only on Pu and S^1-seminorms of θG_ϵ.

We now apply (3.15) to obtain with Λ_ϵ as above,

$$\tfrac{1}{2}\|u\|^2_{H^{-N}_{\Lambda_\epsilon}} + 2C_0^2 \geq \langle (\theta H_p G_\epsilon - M\langle \xi \rangle^{m-1}) \langle \xi \rangle^{-N-m} T_{\Lambda_\epsilon} u, \langle \xi \rangle^{-N} T_{\Lambda_\epsilon} u \rangle_{L^2_{\Lambda_\epsilon}}, \tag{3.22}$$

Let a be given by Lemma 3.3 (so that (3.17) holds). Then by (3.4), there exist M_2 and K such that

$$\theta H_p G_\epsilon + M_2 \langle \xi \rangle^{2K} e^{-2aG_\epsilon/h} \geq (M+1)\langle \xi \rangle^{m-1}.$$

From (3.17) we have

$$\begin{aligned}&\|M_2\chi\langle\xi\rangle^K e^{-aG_\epsilon/h}\langle\xi\rangle^{-N}T_\Lambda u\|^2_{L^2_{\Lambda_\epsilon}}\\&\le C(\|\langle\xi\rangle^{K-N}\tilde{\chi}Tu\|^2_{L^2(T^*\mathbb{R}^n)}+\|\langle\xi\rangle^{-N}Tu\|^2_{L^2(T^*\mathbb{R}^n)})\le C_1^2\end{aligned}\tag{3.23}$$

Therefore, adding (3.23) to (3.22), and using that $\operatorname{supp} G_\epsilon \subset \chi\equiv 1$, we have

$$\begin{aligned}&\tfrac{1}{2}\|u\|^2_{H^{-N}_{\Lambda_\epsilon}}+C_1^2+2C_0^2\\&\ge\langle\chi^2\langle\xi\rangle^{m-1}\langle\xi\rangle^{-N}T_{\Lambda_\epsilon}u,\langle\xi\rangle^{-N}T_{\Lambda_\epsilon}u\rangle_{L^2_{\Lambda_\epsilon}}\\&\qquad\qquad-\langle M(1-\chi^2)\langle\xi\rangle^{m-1}\langle\xi\rangle^{-N}T_{\Lambda_\epsilon}u,\langle\xi\rangle^{-N}T_{\Lambda_\epsilon}u\rangle_{L^2_{\Lambda_\epsilon}}\\&\ge\langle\langle\xi\rangle^{m-1}\langle\xi\rangle^{-N}T_{\Lambda_\epsilon}u,\langle\xi\rangle^{-N}T_{\Lambda_\epsilon}u\rangle_{L^2_{\Lambda_\epsilon}}-(M+1)\|u\|_{H^{-N+\frac{m-1}{2}}},\end{aligned}\tag{3.24}$$

where in the last line we use that $\chi\equiv 1$ on $\operatorname{supp} G_\epsilon$.

Using $m\ge 1$ and rearranging, this yields

$$\|u\|^2_{H^{-N}_{\Lambda_\epsilon}}\le 2C_1^2+4C_0^2+2(M+1)\|u\|_{H^{-N+\frac{m-1}{2}}}.$$

where C_1, C_0 and M are constants *independent of* ϵ.

Since $\Lambda_\epsilon\cap\{|\xi|<1/\epsilon\}=\Lambda_0\cap\{|\xi|<1/\epsilon\}$ where $G_0:=\Phi|\xi|$, we have that $H_\epsilon|_{|\xi|<1/\epsilon}=H_0|_{|\xi|<1/\epsilon}$, where $H_\epsilon=\theta\xi\partial_\xi G_\epsilon+\theta G$ is the corresponding weight. Therefore, the monotone convergence theorem implies that $u\in H_{\Lambda_0}$. Since $\Phi(x_0,t\xi_0)=1$, $t\gg 1$, Proposition 2.3 shows that $(x_0,\xi_0)\notin \mathrm{WF_a}(u)$. □

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References

[BaGo71] M.S. Baouendi and C. Goulaouic. *Étude de l'analyticité et de la régularité Gevrey pour une classe d'opérateurs elliptiques dégénérés,* Annales scientifiques de l'École normale supérieure **4**(1971), 31–46.

[BaSj76] M.S. Baouendi and J. Sjöstrand, *Régularité analytique pour des opérateurs elliptiques singuliers en un point,* Ark. Mat. **14**(1976), 9–33.

[BoCa73] P. Bolley and J. Camus. *Sur une classe d'opérateurs elliptiques et dégénérés à plusieurs variables,* Mémoires de la Société Mathématique de France, **34**(1973), 55–140.

[BCH74] P. Bolley, J. Camus, and B. Hanouzet. *Étude de l'analyticité et de la régularité Gevrey pour une classe de problèmes aux limites elliptiques et dégénérés,* Asterisque **19**(1974), 25–48.

[BoSj76] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegö.* Astérisque, **34-35**(1976), 123–164.

[BoGu81] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators.* Annals of Mathematics Studies, **99**(1981). Princeton University Press.

[BoMu17] A. Bove and M. Mughetti, *Analytic hypoellipticity for sums of squares and the Treves conjecture, II,* Anal. PDE **10**(2017), 1613–1635.

[Ch96] M. Christ, *A progress report on analytic hypoellipticity,* in *Geometric complex analysis (Hayama, 1995),* 123–146, World Sci. Publ., River Edge, NJ, 1996.

[CaKe96] S. Čanić and B. Keyfitz, *A smooth solution for a Keldysh type equation,* Comm. P.D.E. **21**(1996), 319–340.

[De] J.-M. Delort, *F.B.I. Transformation: Second Microlocalization and Semilinear Caustics,* Lecture Notes in Mathematics **1522**, Springer 1992.

[DyZw19a] S. Dyatlov and M. Zworski, *Mathematical theory of scattering resonances,* Graduate Studies in Mathematics **200**, AMS 2019, http://math.mit.edu/~dyatlov/res/

[EpMa13] C.L. Epstein and R. Mazzeo, *Degenerate Diffusion Operators Arising in Population Biology,* Annals of Mathematics Studies **186** (2013).

[GaZw19a] J. Galkowski and M. Zworski, *An introduction to microlocal complex deformations,* arXiv:1912.09845.

[GaZw19b] J. Galkowski and M. Zworski, *Viscosity limits for 0th order pseudodifferential operators*, arXiv:1912.09840.

[Gé90] P. Gérard, *Moyennisation et régularit4 deux-microlocale,* Ann. Scient. Ec. Norm. Sup. 4éme série, **23**(1990), 89–121.

[GuJe20] Y. Guedes Bonthonneau and M. Jézéquel, *FBI Transform in Gevrey classes and Anosov flows,* arXiv:2001.03610.

[Gu05] C. Guillarmou, *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds,* Duke Math. J. **129**(2005), 1–37.

[GuSc77] V. Guillemin and D. Schaeffer. *On a certain class of fuchsian partial differential equations,* Duke Mathematical Journal **44**(1977), 157–199.

[Ha14] N. Haber, *A normal form around a Lagrangian submanifold of radial points,* Int. Math. Res. Not. IMRN **2014**, 17, 4804–4821.

[HaVa15] N. Haber and A. Vasy, *Propagation of singularities around a Lagrangian submanifold of radial points,* Bull. Soc. Math. France **143**(2015), 679–726.

[HeSj86] B. Helffer and J. Sjöstrand, *Resonances en limite semiclassique,* Bull. Soc. Math. France **114**, no. 24–25, 1986.

[HiSj18] M. Hitrik and J. Sjöstrand, *Two minicourses on analytic microlocal analysis,* in in "Algebraic and Analytic Microlocal Analysis", M. Hitrik, D. Tamarkin, B. Tsygan, and S. Zelditch, eds. Springer, 2018, arXiv:1508.00649

[HöI] L. Hörmander, *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis,* Springer Verlag, 1983.

[Ke51] M.V. Keldysh, *On some cases of degenerate elliptic equations on the boundary of a domain,* Doklady Acad. Nauk USSR, **77**(1951), 181–183.

[LeZw19] G. Lebeau and M. Zworski, *Remarks on Vasy's operator with analytic coefficients,* Proc. Amer. Math. Soc. **147**(2019), 145–152.

[Ma02] A. Martinez, *An Introduction to Semiclassical and Microlocal Analysis*, Springer, 2002.

[MM87] R.R. Mazzeo and R.B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature,* J. Funct. Anal. **75**(1987), 260–310.

[Me94] R.B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces,* in *Spectral and scattering theory* (M. Ikawa, ed.), Marcel Dekker, 1994, http://www-math.mit.edu/~rbm/papers/sslaes/sslaes.ps

[Sj96] J. Sjöstrand, *Density of resonances for strictly convex analytic obstacles,* Can. J. Math. **48**(1996), 397–447.

[St57] S. Sternberg, *Local contractions and a theorem of Poincaré,* Amer. J. Math. **79**(1957), 809?-824.

[Va13] A. Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces,* with an appendix by Semyon Dyatlov, Invent. Math. **194**(2013), 381–513.

[Zu17] C. Zuily, *Real analyticity of radiation patterns on asymptotically hyperbolic manifold,* Appl. Math. Res. Express. AMRX **2017**, 386–401.

[Zw12] M. Zworski, *Semiclassical analysis,* Graduate Studies in Mathematics **138**, AMS, 2012.

[Zw16] M. Zworski, *Resonances for asymptotically hyperbolic manifolds: Vasy's method revisited,* J. Spectr. Theory. **6**(2016), 1087–1114.

[Zw17] M. Zworski, *Mathematical study of scattering resonances,* Bull. Math. Sci. **7**(2017), 1–85.

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