WEYL REMAINDERS: AN APPLICATION OF GEODESIC BEAMS

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Abstract. We obtain new *quantitative* estimates on Weyl Law remainders under dynamical assumptions on the geodesic flow. On a smooth compact Riemannian manifold (M, g) of dimension n, let Π_λ denote the kernel of the spectral projector for the Laplacian, $\mathbb{1}_{[0,\lambda^2]}(-\Delta_g)$. Assuming *only* that the set of near periodic geodesics over $W \subset M$ has small measure, we prove that as $\lambda \to \infty$

$$\int_W \Pi_\lambda(x,x)dx = (2\pi)^{-n}\operatorname{vol}_{\mathbb{R}^n}(B)\operatorname{vol}_g(W)\,\lambda^n + O\Big(\frac{\lambda^{n-1}}{\log\lambda}\Big),$$

where B is the unit ball. One consequence of this result is that the improved remainder holds on *all* product manifolds, in particular giving improved estimates for the eigenvalue counting function in the product setup. Our results also include logarithmic gains on asymptotics for the off-diagonal spectral projector $\Pi_\lambda(x, y)$ under the assumption that the set of geodesics that pass near both x and y has small measure, and quantitative improvements for Kuznecov sums under non-looping type assumptions. The key technique used in our study of the spectral projector is that of geodesic beams.

1. Introduction

Let (M, g) be a smooth compact Riemannian manifold of dimension n, Δ_g be the negative definite Laplace-Beltrami operator acting on $L^2(M)$, and $\{\lambda_j^2\}_{j=0}^\infty$ be the eigenvalues of $-\Delta_g$, repeated with multiplicity, $0 = \lambda_0^2 < \lambda_1^2 \le \lambda_2^2 \le \dots$. In this article we obtain improved asymptotics for both pointwise and integrated Weyl Laws. That is, we study asymptotics for the Schwartz kernel of the spectral projector

$$\Pi_\lambda : L^2(M, g) \to \bigoplus_{\lambda_j\le\lambda} \ker(-\Delta_g - \lambda_j^2),$$

i.e. Π_λ is the orthogonal projection operator onto functions with frequency at most λ. If $\{\phi_{\lambda_j}\}_{j=1}^\infty$ is an orthonormal basis of eigenfunctions, $-\Delta_g\phi_{\lambda_j} = \lambda_j^2\phi_{\lambda_j}$, the Schwartz kernel of Π_λ is

$$\Pi_\lambda(x, y) = \sum_{\lambda_j\le\lambda} \phi_{\lambda_j}(x)\overline{\phi_{\lambda_j}}(y), \qquad (x, y) \in M \times M.$$

Asymptotics for the spectral projector play a crucial role in the study of eigenvalues and eigenfunctions for the Laplacian, with applications to the study of physical phenomena such as wave propagation and quantum evolution. One of the oldest problems in spectral theory is to understand how eigenvalues distribute on the real line. Let $N(\lambda) := \#\{j : \lambda_j \le \lambda\}$ be the eigenvalue counting function. Motivated by black body radiation, Hilbert conjectured that, as $\lambda \to \infty$,

$$N(\lambda) = (2\pi)^{-n}\operatorname{vol}_{\mathbb{R}^n}(B)\operatorname{vol}_g(M)\lambda^n + E(\lambda), \qquad E(\lambda) = o(\lambda^n).$$

Here, $\operatorname{vol}_{\mathbb{R}^n}(B)$ is the volume of the unit ball $B \subset \mathbb{R}^n$, $\operatorname{vol}_g(M)$ is the Riemannian volume of M, and dv_g is the volume measure induced by the Riemannian metric. The conjecture was proved by Weyl [44] and is known as the Weyl Law. We refer to $E(\lambda)$ as a *Weyl remainder*. In 1968, Hörmander [24], provided a framework for the study of $E(\lambda)$ and generalized the works of Avakumović [1] and Levitan [34], who proved $E(\lambda) = O(\lambda^{n-1})$; a result that is sharp on the round sphere and is thought of as the standard remainder.

The article [24] provided a framework for the study of Weyl remainders which led to many advances, including the work of Duistermaat–Guillemin [17] who showed $E(\lambda) = o(\lambda^{n-1})$ when the set of periodic geodesics has measure 0. Recently, [26] verified this dynamical condition on all product manifolds. A striking application of our main theorem on Weyl remainders is:

Theorem 1. *Let (M_i, g_i) be smooth compact Riemannian manifolds of dimension $n_i \geq 1$ for $i = 1, 2$. Then, with $M = M_1 \times M_2$, $g = g_1 \oplus g_2$, and $n := n_1 + n_2$,*

$$N(\lambda) = (2\pi)^{-n} \operatorname{vol}_{\mathbb{R}^n}(B) \operatorname{vol}_g(M)\lambda^n + O\big(\lambda^{n-1}/\log\lambda\big), \qquad \lambda \to \infty.$$

For future reference, we note that $N(\lambda) = \int_M \Pi_\lambda(x, x)\, \mathrm{d}v_g(x)$ and thus $N(\lambda)$ can be studied by understanding the kernel of Π_λ restricted to the diagonal. We study both on and off diagonal Weyl remainders in this article. The main idea is to adapt the geodesic beam techniques developed by authors [21, 9, 11] to study Weyl remainders. These techniques were originally used to study averages of quasimodes over submanifolds by decomposing the quasimodes into geodesic beams and controlling the averages in terms of the L^2 norms of these beams. In this work the key point is to study the eigenvalue counting function by viewing it as a sum of quasimodes averaged over the diagonal in $M \times M$. We start our exposition in the setting of the on diagonal estimates.

1.1. **On diagonal Weyl remainders.** The connection between the spectrum of the Laplacian and the properties of periodic geodesics on M has been known since at least the works [15, 16, 43], with their relation to Weyl remainders first explored in the seminal work [17]. To control $E(\lambda)$ we impose dynamical conditions on the periodicity properties of the geodesic flow $\varphi_t : T^*M \setminus \{0\} \to T^*M\setminus\{0\}$, i.e., the Hamiltonian flow of $(x, \xi) \mapsto |\xi|_{g(x)}$. For $t_0 > 0$, $T > 0$, and $R > 0$, define the set of near periodic directions in $U \subset S^*M$ by

$$\mathcal{P}_U^R(t_0, T) := \Big\{\rho \in U \,:\, \bigcup_{t_0 \leq |t| \leq T} \varphi_t(B_{S^*M}(\rho, R)) \cap B_{S^*M}(\rho, R) \neq \emptyset\Big\}. \tag{1.1}$$

Given two sets $U \subset V \subset T^*M$, and $R > 0$, we write $B_V(U, R) := \{\rho \in V :\ d(U, \rho) < R\}$, where d is the distance induced by some fixed metric on T^*M, $B(U, R) = B_{T^*M}(U, R)$, and $B_V(\rho, R) = B_V(\{\rho\}, R)$.

We phrase our dynamical conditions in terms of a resolution function $\mathbf{T} = \mathbf{T}(R)$. This is a function of the scale, R, at which the manifold is resolved, which increases as $R \to 0^+$. We use $\mathbf{T}$ to measure the time for which balls of radius R can be propagated under the geodesic flow while satisfying a given dynamical assumption, e.g. being non periodic.

Definition 1.1. We say a decreasing, continuous function $\mathbf{T} : (0, \infty) \to (0, \infty)$ is a *resolution function*. In addition, we say a resolution function $\mathbf{T}$ is *sub-logarithmic*, if it is differentiable and

$$(\log\log R^{-1})' = -1\big/R\log R^{-1} \leq [\log \mathbf{T}(R)]' \leq 0, \qquad 0 < R < 1.$$

We measure how close $\mathbf{T}$ is to being logarithmic through

$$\Omega(\mathbf{T}) := \limsup_{R\to 0^+} \mathbf{T}(R)\big/\log R^{-1}. \tag{1.2}$$

Simple examples of sub-logarithmic resolution functions are $\mathbf{T}(R) = \alpha(\log R^{-1})^\beta$ for any $\alpha > 0$ and $0 < \beta \leq 1$. See Remark 1.13 for comments on the use of general resolution functions.

For improved integrated Weyl remainders, we need a condition on the geodesic flow.

Definition 1.2. Let $\mathbf{T}$ be a resolution function. Then $U \subset S^*M$ is said to be $\mathbf{T}$ *non-periodic* with constant C_{np} provided there exists $t_0 > 0$ such that

$$\limsup_{R \to 0^+} \mu_{S^*M}\Big(B_{S^*M}\big(\mathcal{P}_U^R(t_0, \mathbf{T}(R)), R\big)\Big)\, \mathbf{T}(R) \le C_{\text{np}}. \tag{1.3}$$

We say U is $\mathbf{T}$ non-periodic if there is such C_{np}, and $W \subset M$ is $\mathbf{T}$ non-periodic if $S^*_W M$ is.

See Appendix A for the notation $\mu_{S^*_x M}$ and $\dim_{\text{box}}$ used below.

Theorem 2. *Let (M, g) be a Riemannian manifold of dimension n, $W \subset M$ be an open subset with $\dim_{\text{box}} \partial W < n$, and $\Omega_0 > 0$. There exists $C_0 > 0$ such that if $\mathbf{T}$ is a sub-logarithmic rate function with $\Omega(\mathbf{T}) < \Omega_0$ and W is $\mathbf{T}$ non-periodic, then there is λ_0 such that for all $\lambda > \lambda_0$*

$$\Big| \int_W \Pi_\lambda(x, x)\, \mathrm{dv}_g(x) - (2\pi)^{-n} \operatorname{vol}_{\mathbb{R}^n}(B) \operatorname{vol}_g(W) \lambda^n \Big| \le C_0\, \lambda^{n-1} \big/ \mathbf{T}\big(\lambda^{-1}\big).$$

In particular, if M is $\mathbf{T}$ non-periodic, then there is λ_0 such that for all $\lambda > \lambda_0$

$$\Big| N(\lambda) - (2\pi)^{-n} \operatorname{vol}_{\mathbb{R}^n}(B) \operatorname{vol}_g(M) \lambda^n \Big| \le C_0\, \lambda^{n-1} \big/ \mathbf{T}\big(\lambda^{-1}\big).$$

We illustrate an application of Theorem 2 in Figure 1. In this example we construct a surface of revolution with both a periodic and a non-periodic set (see Definition 1.2). In particular, Theorem 2 applies with W contained in the non-periodic (green) set. One can obtain little oh improvements for the statement in Theorem 2, but this requires the more general version given in Theorem 6 instead (see Remark 1.6). See Table 1 in §1.3 for some additional examples.

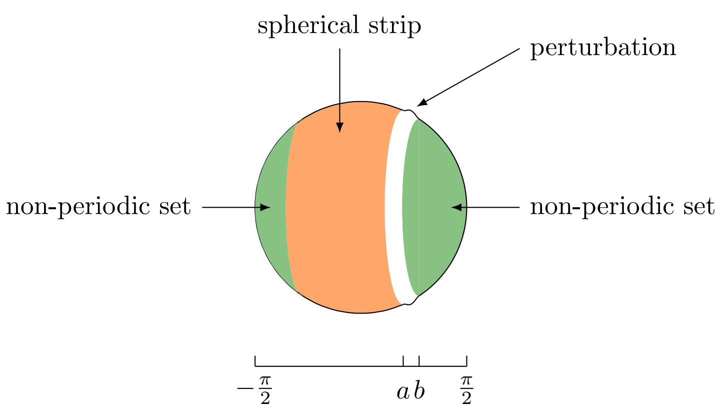

FIGURE 1. An example of a perturbation of the sphere with both a non-periodic (green) and a periodic (orange) physical space set. The perturbed metric coincides with the round metric outside the strip (a, b). Trajectories which remain in the spherical strip are 2π periodic, while those which enter the non-periodic set are mostly non-periodic. See Section B.2.1 for a precise description of this example.

The assumptions of Theorem 2 apply to a wide variety of Riemannian manifolds. Indeed, in addition to the concrete examples in §1.3, the authors [12] use Theorem 2 to give a logarithmic improvement in the remainder for the Weyl law that works for 'typical' metrics on any smooth manifold. This result is the first *quantitative* estimate for the remainder in Weyl laws that holds for most metrics.

We next discuss $E_\lambda(x)$, the remainder in the on diagonal pointwise Weyl law

$$\Pi_\lambda(x,x) = (2\pi)^{-n}\operatorname{vol}_{\mathbb{R}^n}(B)\lambda^n + E_\lambda(x), \qquad x \in M. \tag{1.4}$$

The Weyl remainder in [24] comes from the estimate $E_\lambda(x) = O(\lambda^{n-1})$ for $x \in M$ (again, sharp on the round sphere). The connection between $E_\lambda(x)$ and geodesic loops through x is studied in the works of Safarov, Sogge–Zelditch [37, 40] and often appears in estimates for sup-norms of eigenfunctions. To control the pointwise remainder $E_\lambda(x)$ we impose dynamical conditions on the looping properties of geodesics joining x with itself. For $t_0 > 0$, $T > 0$, $R > 0$, and $x, y \in M$, define

$$\mathcal{L}^R_{x,y}(t_0,T) := \Big\{\rho \in S^*_xM : \bigcup_{t_0\le|t|\le T} \varphi_t(B(\rho,R)) \cap B(S^*_yM, R) \neq \emptyset\Big\}. \tag{1.5}$$

Definition 1.3. Let $\mathbf{T}$ be a resolution function, $t_0 > 0$, $C_{\rm nl} > 0$, and $x, y \in M$. Then, (x,y) *is said to be a* $(t_0,\mathbf{T})$ *non-looping pair with constant* $C_{\rm nl}$ *when*

$$\limsup_{R\to0^+}\Big(\mu_{S^*_xM}\Big(B_{S^*_xM}\big(\mathcal{L}^R_{x,y}(t_0,\mathbf{T}(R)),R\big)\Big)\mu_{S^*_yM}\Big(B_{S^*_yM}\big(\mathcal{L}^R_{y,x}(t_0,\mathbf{T}(R)),R\big)\Big)\mathbf{T}(R)^2\Big) \le C_{\rm nl}.$$

We say x is $(t_0,\mathbf{T})$ *non-looping with constant* $C_{\rm nl}$ if (x,x) is a $(t_0,\mathbf{T})$ non-looping pair with constant $C_{\rm nl}$.

Note that if $t_0 < \operatorname{inj}(M)$, where $\operatorname{inj}(M)$ is the injectivity radius of M, then for x to be $(t_0,\mathbf{T})$ non-looping is the same as being $(\varepsilon,\mathbf{T})$ non-looping for any $0 < \varepsilon \le t_0$. In this case, we write x is $(0,\mathbf{T})$ non-looping.

To state our estimates on the pointwise Weyl remainder, we let $\lambda > 0$, and, for points $x, y \in M$ with $d(x,y) < \operatorname{inj} M$, define

$$E^0_\lambda(x,y) := \Pi_\lambda(x,y) - \frac{1}{(2\pi)^n}\int_{|\xi|_{g_y}<\lambda} e^{i\langle \exp_y^{-1}(x),\xi\rangle}\frac{d\xi}{\sqrt{|g_y|}}. \tag{1.6}$$

Here, the integral is over T^*_yM, $\exp_x : T^*_xM \to M$ is the the exponential map, and $|g_y|$ denotes the determinant of the metric g at y, when g is thought of as matrix in local coordinates.

Theorem 3. *Let* $\alpha, \beta \in \mathbb{N}^n$, $0 < \delta < \frac12$, $C_{\rm nl} > 0$, *and* $\Omega_0 > 0$. *There exists* $C_0 > 0$ *such that the following holds. If* $\mathbf{T}$ *is a sub-logarithmic resolution function with* $\Omega(\mathbf{T}) < \Omega_0$, *there is* $\lambda_0 > 0$ *such that if* $x_0 \in M$ *is* $(0,\mathbf{T})$ *non-looping with constant* $C_{\rm nl}$, *then for all* $\lambda > \lambda_0$

$$\sup_{x,y\in B(x_0,\lambda^{-\delta})}\big|\partial_x^\alpha\partial_y^\beta E^0_\lambda(x,y)\big| \le C_0\,\lambda^{n-1+|\alpha|+|\beta|}\big/\,\mathbf{T}\big(\lambda^{-1}\big).$$

See Table 2 in §1.3 for some examples to which Theorem 3 applies.

Theorems 2 and 3 fit in a long history of work on asymptotics of the kernel of the spectral projector and the eigenvalue counting function. Many authors considered pointwise Weyl sums [35, 20, 1, 34, 38, 24], eventually proving the sharp remainder estimates. The article [24] provided a method which was used in many later works: [17] showed $E(\lambda) = o(\lambda^{n-1})$ under the assumption that the set of periodic trajectories has measure 0, [37, 40] improved estimates on $E_\lambda(x)$ to $o(\lambda^{n-1})$

under the assumption that the set of looping directions through x has measure 0 (see also the book of Safarov–Vassiliev [36]). See [13, 14] for corresponding estimates that are uniform in a small neighborhood of the diagonal and Ivrii [27] for the case of manifolds with boundaries.

While $o(1)$ improvements were available under dynamical assumptions, until now, quantitative improvements in remainders were available in geometries where one has an effective parametrix to $\log\lambda$ times e.g. manifolds without conjugate points [2, 4, 30] or non-Zoll convex analytic rotation surfaces [41, 42]. We point out that the closest results to ours are those of Volovoy [41]. There, quantitative estimates on $E(\lambda)$ are obtained under stronger assumptions than those of Theorem 2. In particular, W is required to be equal to M and the volume in (1.3) is required to be bounded by a positive power of R, rather than $\mathbf{T}(R)^{-1}$.

The estimates in this article are available *without additional* geometric assumptions. This comes from our use of the 'geodesic beam techniques' developed in the authors' work [21, 9, 11] and which in turn draw upon the semiclassical approach of Koch–Tataru–Zworski [32]. Theorems 2 and 3 can be thought of as the quantitative analogs of the main results in [17] and of [37], [40] respectively. In fact, these results can be recovered from Theorems 2 and 3 by allowing $\mathbf{T}(R)$ to grow arbitrarily slowly as $R \to 0^+$ (see [11, Appendix B]). We also note that our estimates include both C^∞ asymptotics for $\Pi_\lambda(x, y)$ and uniformity in certain shrinking neighborhoods of the diagonal without any additional effort and hence include the results from [13, 14].

Remark 1.4. To recover the results of [37, 40, 13, 14] one needs uniformity in $o(1)$ neighborhoods of points of interest. As stated, Theorem 3 does not quite include this since it works in a $\lambda^{-\delta}$ neighborhood of x. However, the full version of our estimates, Theorem 9, allows for the neighborhood of x to shrink arbitrarily slowly and thus recovers these earlier results.

1.2. **Off diagonal Weyl remainders.** The off diagonal behavior of $\Pi_\lambda(x, y)$ plays a crucial role in understanding monochromatic random waves (see e.g. [7]) as well as in estimates for L^p norms of Laplace eigenfunctions (see e.g. [39, Section 5.1]). This problem is more complicated than the on diagonal situation since understanding the far off diagonal (i.e., $d(x, y) > \mathrm{inj}(M)$) regime typically involves parametrices for $e^{it\sqrt{-\Delta_g}}$ for $t > \mathrm{inj}(M)$, which are difficult to control. Notably, our geodesic beam techniques allow us to overcome this difficulty when estimating errors.

To control $\Pi_\lambda(x, y)$ off-diagonal, we introduce a dynamical condition on the non-recurrence properties of the geodesics joining a point x with itself. To our knowledge, this is the first time non-recurrence is used in understanding off-diagonal Weyl remainders. For $x \in M$, $U \subset S^*_xM$, $t_0 > 0$, $T > 0$, and $R > 0$, let

$$\mathcal{R}^R_{U,\pm}(t_0, T) := \bigcup_{t_0 \le \pm t \le T} \varphi_t\big(B(U, R)\big) \cap B_{S^*_xM}(U, R).$$

Definition 1.5. Let $\mathfrak{t}$ and $\mathbf{T}$ be resolution functions and $R_0 > 0$. We say $x \in M$ *is* $(\mathfrak{t},\mathbf{T})$ *non-recurrent at scale* R_0 if for all $\rho \in S^*_xM$ there exists a choice of $\pm$ such that for all $A \subset B_{S^*_xM}(\rho, R_0)$, $\varepsilon > 0$, $r > 0$ with $\mathbf{T}(r) > \mathfrak{t}(\varepsilon)$, and $0 < R < R_0$,

$$\mu_{S^*_xM}\Big(B_{S^*_xM}\Big(\mathcal{R}^{rR}_{A,\pm}(\mathfrak{t}(\varepsilon), \mathbf{T}(r))\,,\, rR\Big)\Big) < \varepsilon\, \mu_{S^*_xM}\Big(B_{S^*_xM}(A, R)\Big).$$

If (x, y) is a $(t_0, \mathbf{T})$ non looping pair for some $t_0 > 0$ we measure the difference between $\Pi_\lambda(x, y)$ and its smoothed version which takes into account propagation up to time t_0. Let $\rho \in \mathcal{S}(\mathbb{R})$ with

$\hat{\rho}(0) \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\hat{\rho} \subset [-2,2]$. For $\sigma > 0$ we define

$$\rho_\sigma(s) := \sigma\,\rho\big(\sigma\, s\big). \tag{1.7}$$

For $x, y \in M$, $t_0 > 0$, and $\lambda > 0$, let

$$E_\lambda^{t_0} := \Pi_\lambda - \rho_{t_0} * \Pi_\lambda, \tag{1.8}$$

where the convolution is taken in the λ variable. Below is our first off diagonal result.

Theorem 4. *Let $\alpha, \beta \in \mathbb{N}^n$, $0 < \delta < \frac{1}{2}$, $C_{\rm nl} > 0$, $R_0 > 0$, $\Omega_0 > 0$, $\varepsilon > 0$, and $\mathfrak{t}$ be a resolution function, there is $C_0 > 0$ such that if $\mathbf{T}_j$ is a sub-logarithmic resolution function with $\Omega(\mathbf{T}_j) < \Omega_0$ for $j = 1, 2$ and $\mathbf{T}_{\max} = \max(\mathbf{T}_1, \mathbf{T}_2)$, then there is $\lambda_0 > 0$ such the following holds. If $x_0, y_0 \in M$ and $t_0 > 0$ are such that x_0 and y_0 are respectively $(\mathfrak{t}, \mathbf{T}_1)$ and $(\mathfrak{t}, \mathbf{T}_2)$ non-recurrent at scale R_0, and (x_0, y_0) is a $(t_0, \mathbf{T}_{\max})$ non-looping pair with constant $C_{\rm nl}$, then for $\lambda > \lambda_0$*

$$\sup_{x \in B(x_0, \lambda^{-\delta})} \sup_{y \in B(y_0, \lambda^{-\delta})} \big|\partial_x^\alpha \partial_y^\beta E_\lambda^{t_0+\varepsilon}(x,y)\big| \le C_0\, \lambda^{n-1+|\alpha|+|\beta|} \Big/ \sqrt{\mathbf{T}_1\big(\lambda^{-1}\big)\mathbf{T}_2(\lambda^{-1})}.$$

See Table 2 in §1.3 for some examples to which Theorem 4 applies.

To compare Theorems 3 and 4, note that for $x, y \in M$ with $d(x,y) < \varepsilon < \operatorname{inj}(M)$,

$$\left|\partial_x^\alpha \partial_y^\beta \left(\rho_{\varepsilon\lambda} * \Pi_\lambda(x,y) - \frac{1}{(2\pi)^n} \int_{|\xi|_{g_y} < \lambda} e^{i\langle \exp_y^{-1}(x), \xi\rangle} q_\lambda(x,y,\xi) \frac{d\xi}{\sqrt{|g_y|}}\right)\right| \le C_0 \lambda^{n-2+|\alpha|+|\beta|}$$

where $q_\lambda(x,y,\xi){=}1 + \lambda^{-1} q_{-1}(x,y,\xi)$ and $q_{-1}(x,y,\xi) = O(d(x,y))$ (see e.g. [13, Proof of Proposition 10]). Then, for points x, y with $d(x,y) < \lambda^{-\delta}$, modulo terms smaller than our remainder, $E_\lambda^0(x,y)$ as defined in (1.6) is the same as $E_\lambda^\varepsilon(x,y)$.

For any $t_0 < \infty$, it is possible to write an oscillatory integral expression for $\rho_{t_0} * \Pi_\lambda(x,y)$. However, its precise behavior in λ depends heavily on the geometry of (M,g); in particular, on the structure of the set of geodesics from x to y. This explains why we state our estimates in terms of $E_\lambda^{t_0}$.

More generally, our results apply to averages of $\Pi_\lambda(x,y)$ with $x \in H_1$ and $y \in H_2$, where H_1, H_2 are any two smooth submanifolds of M. This type of integral is known as a Kuznecov sum [45] and appears in the analytic theory of automorphic forms [5, 33, 28, 22, 23]. All our dynamical assumptions for points $x, y \in M$ above may be defined for the submanifolds $H_1, H_2 \subset M$ instead. In doing so, the only change needed is to use the sets of unit co-normal directions SN^*H_1 and SN^*H_2, instead of S_x^*M and S_y^*M. See Definitions 1.10 and 1.11 for a detailed explanation. In what follows $d\sigma_{H_1}$ and $d\sigma_{H_2}$ denote the volume measures induced by the Riemannian metric on H_1 and H_2 respectively.

Theorem 5. *Let $\alpha, \beta \in \mathbb{N}^n$, $1 \le k_1 \le n$, $1 \le k_2 \le n$, $C_{\rm nl} > 0$, $\Omega_0 > 0$, $\varepsilon > 0$, $R_0 > 0$, and $\mathfrak{t}$ be a resolution function. There is $C_0 > 0$ such that if $\mathbf{T}_j$ is a sub-logarithmic resolution function with $\Omega(\mathbf{T}_j) < \Omega_0$ for $j = 1, 2$ and $\mathbf{T}_{\max} = \max(\mathbf{T}_1, \mathbf{T}_2)$ the following holds. If $t_0 > 0$, and $H_j \subset M$ are submanifolds of codimension k_j such that (H_1, H_2) is a $(t_0, \mathbf{T}_{\max})$ non-looping pair with constant $C_{\rm nl}$, and H_j is $(\mathfrak{t}, \mathbf{T}_j)$ non-recurrent at scale R_0 for $j = 1, 2$, then there is $\lambda_0 > 0$ such that for $\lambda > \lambda_0$*

$$\left|\int_{H_1} \int_{H_2} \partial_x^\alpha \partial_y^\beta E_\lambda^{t_0+\varepsilon}(x,y)\, d\sigma_{H_1}(x) d\sigma_{H_2}(y)\right| \le C_0\, \lambda^{\frac{k_1+k_2}{2}-1+|\alpha|+|\beta|} \Big/ \sqrt{\mathbf{T}_1\big(\lambda^{-1}\big)\mathbf{T}_2(\lambda^{-1})}.$$

See Table 2 in §1.3 for some examples to which Theorem 5 applies.

To our knowledge, Theorem 5 is the first theorem to give improved remainders for Kuznecov sum remainders under dynamical assumptions. Theorems 3, 4, and 5 are consequences of our results for general semiclassical pseudodifferential operators (see Theorems 8 and 9).

Remark 1.6 (Little oh improvements)**.** When the expansion rate $\Lambda_{\max} = 0$ (see (1.11)) and our dynamical assumptions hold for $\mathbf{T}(R) \gg \log R^{-1}$, our theorems can be used to obtain $o(1/\log\lambda)$ improvements over standard remainders. In special situations where the geodesic flow has sub-exponential expansion, we expect similar results with improvements beyond $o(1/\log\lambda)$.

1.3. **Applications.** In this section we present some examples to which our theorems apply. For each of them we give a reference for the detailed proofs that the relevant assumptions are satisfied. Note that Appendix B contains many examples not listed in Tables 1 and 2, and that the results from [8] can be used to find additional examples. With the exception of the final three rows of Table 1 with $W = M$, all the estimates in Tables 1 and 2 are new.

In Table 1, we list examples where the assumptions of Theorem 2 hold. The final two examples are due to Volovoy [42].

M	W	$\lvert E_\lambda\rvert \lesssim$	§
product manifolds	any	$\frac{\lambda^{n-1}}{\log\lambda}$	B.1.1
perturbed spheres	in the non-periodic set	$\frac{\lambda^{n-1}}{\log\lambda}$	B.2.1
manifolds without conjugate points	any	$\frac{\lambda^{n-1}}{\log\lambda}$	B.1
non-Zoll convex analytic surfaces of revolution	any	$\frac{\lambda^{n-1}}{\log\lambda}$	[42]
compact Lie group rank > 1 with bi-invariant metric	any	$\frac{\lambda^{n-1}}{\log\lambda}$	[42]

Table 1. This table lists examples with $\mathbf{T}$ non-periodic subsets with $\mathbf{T}(R) = c\log R^{-1}$. Theorem 2 holds for all these examples. Here, $E_\lambda = \int_W E_\lambda(x)\,\mathrm{dv}_g$ with $E_\lambda(x)$ as in (1.4).

In Table 2 we list some examples for which Theorems 4 and 5 hold. In each case there exists $t_0 > 0$ such that (H_1, H_2) is a $(t_0, \max(\mathbf{T}_1, \mathbf{T}_2))$ non-looping pair. Note that we omit labeling points for which $\mathbf{T}_2 = \mathrm{inj}(M)$ since being $\mathrm{inj}(M)$ non-recurrent is an empty statement. In these cases the gain in the pointwise Weyl law is $\sqrt{\log\lambda}$ instead of $\log\lambda$.

1.4. **Further improvements.** Many experts believe that, for a Baire generic Riemannian metric on a smooth compact manifold, there is $\delta > 0$ such that $E(\lambda) = O(\lambda^{n-1-\delta})$. Presently, this type of improved remainder is only available when the geodesic flow has special structure e.g. the flat torus, non-Zoll convex analytic surfaces of revolution, or compact Lie groups of rank > 1 with bi-invariant metric [42]. Specifically, the geodesic flow must expand only polynomially in time, $\|d\varphi_t\|_{L^\infty(TS^*M)} \leq C\langle t\rangle^N$ for some $N > 0$. Typically, geodesics will instead expand exponentially in some places and, because of this, Egorov's theorem generally only holds to logarithmic times.

	H_1	H_2	$\mathbf{T}_1$	$\mathbf{T}_2$	$\lvert E_\lambda\rvert \lesssim$	§
Manifolds with conjugate points						
product manifolds	x any point (nL)	y any point (nL)	$\log R$	$\log R$	$\frac{\lambda^{n-1}}{\log\lambda}$	B.1.1
spherical pendulum	x not a pole (nL)	x (nL)	$\log R$	$\log R$	$\frac{\lambda^{n-1}}{\log\lambda}$	B.2.2
spherical pendulum	x not a pole (nL)	y a pole	$\log R$	$\operatorname{inj} M$	$\frac{\lambda^{n-1}}{\sqrt{\log\lambda}}$	B.2.2
perturbed spheres	x non-periodic, not a pole, (nL)	y a pole	$\log R$	$\operatorname{inj} M$	$\frac{\lambda^{n-1}}{\sqrt{\log\lambda}}$	B.2.1
perturbed spheres	x non-periodic, not a pole, (nL)	x (nL)	$\log R$	$\log R$	$\frac{\lambda^{n-1}}{\log\lambda}$	B.2.1
Manifolds without conjugate points						
any	$k_1 > 1$ (nL)	$k_2 > 1$, $k_1 + k_2 > n + 1$ (nL)	$\log R$	$\log R$	$\frac{\lambda^{\frac{k_1+k_2}{2}-1}}{\log\lambda}$	B.1
any	geodesic sphere (nL)	geodesic sphere (nL)	$\log R$	$\log R$	$\frac{1}{\log\lambda}$	B.1.2
Anosov	horosphere$^+$ (nRvc)	horosphere$^-$ (nRvc)	$\log R$	$\log R$	$\frac{1}{\log\lambda}$	B.3
Anosov, $K_g \le 0$	totally geodesic (nL)	totally geodesic (nL)	$\log R$	$\log R$	$\frac{\lambda^{\frac{k_1+k_2}{2}-1}}{\log\lambda}$	B.3
Anosov, $K_g \le 0$	totally geodesic (nL)	horosphere (nRvc)	$\log R$	$\log R$	$\frac{\lambda^{\frac{k_1-1}{2}}}{\log\lambda}$	B.3

TABLE 2. The table lists examples where Theorems 4 and 5 hold. We write (nL) when H_i is $\mathbf{T}_i$ non-looping and (nRvc) when H_i is $\mathbf{T}_i$ non-recurrent via coverings. Horosphere$^\pm$ denotes stable/unstable horospheres, and K_g the sectional curvature. A manifold is called Anosov if it has Anosov geodesic flow (see §B.3 for a definition). The label E_λ represents the integrated error term $\int_{H_1}\int_{H_2} E_\lambda(x,y)d\sigma_{H_1}d\sigma_{H_2}$.

In fact, the only ingredient in our proof which restricts us to logarithmic improvements is Egorov's theorem. Under the assumption of polynomial expansion one can prove an Egorov theorem to polynomial times and hence obtain polynomially improved remainders using our methods. We do not pursue this here since the present article is intended to apply on a general manifold and the polynomial times involved in such an Egorov theorem are not explicit. We instead plan to address the integrable case specifically in a future article.

1.5. **Weyl laws for general operators.** Let $P(h) \in \Psi^m(M)$ be a self-adjoint, semiclassical pseudodifferential operator with principal symbol p, that is positive and classically elliptic in the sense that there is $C > 0$ such that

$$p(x,\xi) \geq \tfrac{1}{C}|\xi|^m, \qquad |\xi| \geq C. \tag{1.9}$$

Let $\{E_j(h)\}_j$ be the eigenvalues of P repeated with multiplicity. For $s \in \mathbb{R}$ we work with $\Pi_h(s) := \mathbb{1}_{(-\infty,s]}(P(h))$, which is the orthogonal projection operator

$$\Pi_h(s) : L^2(M) \to \bigoplus_{E_j(h) \leq s} \ker(P(h) - E_j(h)).$$

For $x, y \in M$ we write $\Pi_h(s; x, y)$ for its kernel

$$\Pi_h(s;x,y) := \sum_{E_j(h) \leq s} \phi_{E_j(h)}(x)\overline{\phi_{E_j(h)}}(y), \tag{1.10}$$

where $\{\phi_{E_j(h)}\}_j$ is an orthonormal basis for $L^2(M)$ with $P(h)\phi_{E_j(h)} = E_j(h)\phi_{E_j(h)}$.

Let $\varphi_t : T^*M \to T^*M$ denote the Hamiltonian flow for p at time t. We recall the *maximal expansion rate* for the flow and the *Ehrenfest time* at frequency h^{-1} respectively:

$$\Lambda_{\max} := \limsup_{|t|\to\infty} \frac{1}{|t|} \log \sup_{\{p \in [a-\varepsilon, b+\varepsilon]\}} \|d\varphi_t(x,\xi)\|, \qquad T_e(h) := \frac{\log h^{-1}}{2\Lambda_{\max}}. \tag{1.11}$$

Note that $\Lambda_{\max} \in [0,\infty)$ and if $\Lambda_{\max} = 0$, we may replace it by an arbitrarily small constant.

Definition 1.7. Let $a, b \in \mathbb{R}$ with $a \leq b$. Let $t_0 > 0$ and $\mathbf{T}$ be a resolution function. A set $U \subset T^*M$ is said to be $\mathbf{T}$ *non-periodic for* p *in the window* $[a,b]$ provided that for all $E \in [a,b]$ Definition 1.2 holds with φ_t being the Hamiltonian flow for p, and with S^*M replaced by $p^{-1}(E)$.

The following is our most general version of the Weyl Law. We write $\pi_M : T^*M \to M$ for the natural projection and H_p for the Hamiltonian vector field for p.

Theorem 6. *Let $0 < \delta < \frac{1}{2}$, $\ell \in \mathbb{R}$, and $\mathcal{V} \subset \Psi^\ell(M)$ a bounded subset, $U \subset T^*M$ open, $t_0 > 0$, $C_U > 0$, and $a, b \in \mathbb{R}$ with $a \leq b$. Suppose $d\pi_M H_p \neq 0$ on $p^{-1}([a,b]) \cap \overline{U}$. Then, there is $C_0 > 0$ such that the following holds. Let $K > 0$, $A \in \mathcal{V}$ with $\mathrm{WF_h}(A) \subset U$, $\Lambda > \Lambda_{\max}$, $\mathbf{T}$ be a sub-logarithmic resolution function with $\Lambda\Omega(\mathbf{T}) < 1 - 2\delta$, and suppose U is $\mathbf{T}$ non-periodic in the window $[a,b]$ with*

$$\limsup_{R\to 0} \sup_{t\in[a,b]} \mathbf{T}(R)\mu_{p^{-1}(t)}(B(\partial U, R)) \leq C_U. \tag{1.12}$$

Then, there is $h_0 > 0$ such that for all $0 < h < h_0$, and $E \in [a, b + Kh]$

$$\Big| \sum_{-\infty < E_j(h) \leq E} \langle A\phi_{E_j(h)}, \phi_{E_j(h)}\rangle - \operatorname{tr}\left(A\,\rho_{t_0/h} * \Pi_h(E)\right)\Big| \leq C_0\, h^{1-n} / \,\mathbf{T}(h). \tag{1.13}$$

Since the second term in (1.13) involves only short time propagation for the Schrödinger group $e^{itP/h}$, its asymptotic expansion in powers of h can in principle be obtained. This calculation is routine, but long, so we do not include it here. For the details when $P = -h^2\Delta_g$, we refer the reader to [17, Proposition 2.1]. In addition, if $U \subset T^*M$ has smooth boundary which intersects $p^{-1}(E)$ transversally for $E \in [a, b]$, then (1.12) holds. Although the statement of Theorem 6 is cumbersome when U with rough boundary is allowed, it is natural to consider dynamical assumptions on this type of set. Indeed, many dynamical systems exhibit the so-called 'chaotic sea' with 'integrable islands' behavior where the dynamics are aperiodic in the sea; a set which typically has very rough boundary.

Next, we consider generalized Kuznecov [33] type sums of the form

$$\Pi^{A_1,A_2}_{H_1,H_2}(s) := \int_{H_1}\int_{H_2} A_1\Pi_h(s)A_2^*\,(x,y)\,d\sigma_{H_1}(x)d\sigma_{H_2}(y),$$

where $A_1, A_2 \in \Psi^\infty(M)$ and $H_1, H_2 \subset M$ are two submanifolds of M.

Let $H \subset M$ be a smooth submanifold. For $a, b \in \mathbb{R}$, $a \le b$, define

$$\Sigma^H_{[a,b]} := p^{-1}([a,b]) \cap N^*H. \tag{1.14}$$

Definition 1.8. We say a submanifold $H \subset M$ of codimension k is *conormally transverse for* p *in the window* $[a, b]$ if given $f_1, \dots, f_k \in C^\infty_c(M;\mathbb{R})$ locally defining H, i.e. with $H = \bigcap_{i=1}^k \{f_i = 0\}$ and $\{df_i\}$ linearly independent on H, we have

$$\Sigma^H_{[a,b]} \subset \bigcup_{i=1}^k \{H_p f_i \neq 0\}, \tag{1.15}$$

Here, we interpret f_i as a function on the cotangent bundle by pulling it back through the canonical projection map.

Remark 1.9. If $P(h) = -h^2\Delta_g$, then $p(x,\xi) = |\xi|^2_{g(x)}$. Working with $a = b = 1$, we have $\Sigma^H_{[a,b]} = SN^*H$. In this setup every submanifold $H \subset M$ is conormally transverse for p.

Definition 1.10. Let $H_1, H_2 \subset M$ be two smooth submanifolds. Let $a, b \in \mathbb{R}$ with $a \le b$. Let $t_0 > 0$, $\mathbf{T}$ a resolution function, and $C_{\rm nl} > 0$. We say (H_1, H_2) is a $(t_0, \mathbf{T})$ *non-looping pair in the window* $[a, b]$ *with constant* $C_{\rm nl}$ provided that Definition 1.3 holds for all $E \in [a, b]$ with φ_t being the Hamiltonian flow for p and with $\mathcal{L}^R_{x,y}$ changed to

$$\mathcal{L}^{R,E}_{H_1,H_2}(t_0,T) := \Big\{\rho \in \Sigma^{H_1}_E : \bigcup_{t_0 \le |t| \le T} \varphi_t(B(\rho,R)) \cap B\big(\Sigma^{H_2}_E, R\big) \neq \emptyset\Big\},$$

and with S^*_xM and S^*_yM replaced with $\Sigma^{H_1}_E$ and $\Sigma^{H_2}_E$ respectively. We say H *is* $(t_0, \mathbf{T})$ *non-looping* if (H, H) is a $(t_0, \mathbf{T})$ non-looping pair.

Definition 1.11. Let $H \subset M$ be a smooth submanifold. Let $a, b \in \mathbb{R}$ with $a \le b$. Let $t_0 > 0$, $R_0 > 0$, $0 < C_{\rm nr} < 1$, and let $\mathbf{T}$ be a resolution function. H is said to be $\mathbf{T}$ *non-recurrent in the window* $[a, b]$ *with constants* $(R_0, C_{\rm nr})$ provided Definition 1.5 holds for any $E \in [a, b]$ with S^*_xM replaced by Σ^H_E and where φ_t is the Hamiltonian flow for p.

To state our main estimate for Kuznecov sums, let $\rho \in \mathcal{S}(\mathbb{R})$ with $\hat{\rho}(0) \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\hat{\rho} \subset [-2,2]$. For $T > 0$ we define

$$\rho_{h,T}(t) := \tfrac{T}{h}\, \rho\Big(\tfrac{T}{h}\, t\Big). \tag{1.16}$$

We then introduce the remainder

$$E^{A_1,A_2}_{H_1,H_2}(T,h;s) = \Pi^{A_1,A_2}_{H_1,H_2}(s) - \rho_{h,T} * \Pi^{A_1,A_2}_{H_1,H_2}(s). \tag{1.17}$$

Theorem 7. *Let $P(h) \in \Psi^m(M)$ be a self-adjoint semiclassical pseudodifferential operator with classically elliptic symbol p. Let $\mathfrak{t}$ be a resolution function and $\varepsilon > 0$. For $j = 1,2$, let $H_j \subset M$ be submanifolds with co-dimension k_j. Let $a,b \in \mathbb{R}$ such that H_j is conormally transverse for p in the window $[a,b]$ for $j = 1,2$. Let $R_0 > 0$, $t_0 > 0$, and for $j = 1,2$, let $\mathbf{T}_j$ be sub-logarithmic resolution functions and $\mathbf{T}_{\max} = \max(\mathbf{T}_1, \mathbf{T}_2)$. Suppose H_j is $(\mathfrak{t}, \mathbf{T}_j)$ non-recurrent in the window $[a,b]$ with constant R_0 for each $j = 1,2$, and (H_1,H_2) is a $(t_0, \mathbf{T}_{\max})$ non-looping pair in the window $[a,b]$ with constant $C_{\rm nl}$. Then, for all $A_1, A_2 \in \Psi^\infty(M)$, there exist $h_0 > 0$ and $C_0 > 0$ such that for all $0 < h \le h_0$, $K > 0$, and $s \in [a - Kh, b + Kh]$*

$$\Big|E^{A_1,A_2}_{H_1,H_2}(t_0{+}\varepsilon, h; s)\Big| \le C_0\, h^{1-\frac{k_1+k_2}{2}} \Big/ \sqrt{\mathbf{T}_1(h)\mathbf{T}_2(h)}.$$

Remark 1.12. We omit the precise dependence of the constant C_0 on various parameters in Theorem 7. Instead, we refer the reader to our main theorem on averages, Theorem 8, where we have introduced notation to handle uniformity in families of submanifolds H_1 and H_2.

1.6. **Outline of the paper and ideas from the proof.** In §2 we introduce the notion of good coverings by tubes and various assumptions on these coverings which allow us to adapt the results of [11] to our setup. We also state our main averages theorem in its full generality (Theorem 8). §3 studies how the dynamical assumptions in the introduction relate to the assumptions on coverings by tubes from §2. In §4 we adapt the crucial estimates coming from the geodesic beam techniques [11] so that they can be applied to the study of Weyl remainders. Next, in §5, we estimate the scale (in the energy) at which averages of the spectral projector behave like Lipschitz functions in the spectral parameter. With this in hand, we are able to approximate Π_h using $\rho_{h,T(h)} * \Pi_h$ with $T(h) = \sqrt{\mathbf{T}_1(h)\mathbf{T}_2(h)}$. Finally, §6 shows that the $\rho_{h,T(h)} * \Pi_h$ approximation is close to $\rho_{h,t_0} * \Pi_h$, finishing the proof of our main theorem on averages. §7 contains the proof of our theorems on the Weyl remainder. This section follows the same strategy as that for averages: an estimate for the Lipschitz scale of the trace of the spectral projector, followed by relating $\rho_{h,T(h)} * \Pi_h$ to $\rho_{h,t_0} * \Pi_h$. In Appendix A we present an index of notation and in Appendix B we give examples including those from Table 2 to which our theorems can be applied.

The main idea of this article is to view the kernel of the spectral projector $\mathbb{1}_{[t-s,t]}(P)$ as a quasimode for P. This allows us to use the geodesic beam techniques from [11] to control the energy scale at which the projector behaves like a Lipschitz function and hence to estimate the error when the projector is smoothed at very small scales. This idea is used a second time when controlling $(\rho_{h,T(h)} - \rho_{h,t_0}) * \Pi_h$ to estimate the contribution from small volumes of the possibly looping tubes. A simple argument using Egorov's theorem controls the remaining non-looping tubes. The crucial insight used to handle the Weyl law is to view the kernel of the spectral projector as a distribution on $M \times M$, where it is a quasimode for $\mathbf{P} := P \otimes 1$, and to study the Weyl Law via integration of the kernel over the diagonal. By doing this, we are able to reduce the problem to bounding an average of a quasimode over a submanifold, a setting in which geodesic beam techniques apply.

Note that Theorems 2 and 6 are proved in §7.1.4 and §7.1.3 respectively. Theorem 1 is a corollary of Theorem 2; the necessary dynamical properties are proved in Appendix B.1.1. Theorems 3, 4, 5, and 7 follow from an application of Theorem 9 (See §2.4 for Theorems 3, 4, and 5. Theorem 7 is a direct corollary of Theorem 9.). The fact that Theorem 9 follows from Theorem 8 is proved in § 9 and Theorem 8 is proved in §6.2.

Remark 1.13 (Resolution functions)**.** There are several reasons why we state our theorems in terms of a general resolution function. First, it is necessary to allow $\mathbf{T}(R)$ to grow arbitrarily slowly as $R \to 0$ to recover the $o(1)$ results of [37, 40, 17] (see Remark 1.6). Second, while it may appear from Tables 1 and 2, that $\mathbf{T}(R)$ is always either $c \log R^{-1}$ or the trivial case of $\mathrm{inj}(M)$, this is not always true. In fact, one can check that many integrable examples are non-looping or non-periodic for $\mathbf{T}(R) \gg \log R^{-1}$. At the moment, the authors are not aware of concrete examples with $\mathbf{T}(R) \ll \log R$. However, it is likely that for any sub-logarithmic resolution function $\mathbf{T}$, with $\mathbf{T}(R) \to \infty$ as $R \to 0^+$, a modification of the construction from [6] yields a metric on the sphere for which there is a point x such that x is not $(t_0, \mathbf{T})$ non-looping for any $t_0 > 0$, but there is a resolution function $\mathbf{T}_1$ with $\mathbf{T}_1(R) \longrightarrow \infty$ as $R \to 0^+$ and $t_0 > 0$ such that x is $(t_0, \mathbf{T}_1)$ non-looping. Also, note that our non-periodic, non-looping, and non-recurrent conditions are all monotonic in $\mathbf{T}$ in the sense that if $\mathbf{T}_1(R) \leq \mathbf{T}_2(R)$, and one of these conditions hold with the resolution function $\mathbf{T}_2$, then it also holds with $\mathbf{T}_1$.

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2. Results with dynamical assumptions via coverings by tubes

We divide this section in four parts. In Section 2.1 we introduce the analogues of Definitions 1.10 and 1.11 via the use of coverings by bicharacteristic tubes. Microlocalization to these tubes will eventually be used to generate bicharacteristic beams. In Section 2.2 we introduce the uniformity assumptions that allow us to obtain uniform control of the constants in all our results. In Section 2.3 we state the most general version of our results, using the definitions via coverings by tubes, and the uniformity assumptions.

2.1. Dynamical assumptions via coverings by tubes.

Let $H \subset M$ be a smooth submanifold that is conormally transverse for p in the window $[a, b]$. Let $\mathcal{Z} \subset T^*M$ with

$$\Sigma^H_{[a,b]} \subset \mathcal{Z} \tag{2.1}$$

be a hypersurface that is transverse to the flow, and φ_t continue to denote the Hamiltonian flow for p at time t. Given $A \subset \Sigma^H_{[a,b]}$, $\tau > 0$, and $r > 0$, we define

$$\Lambda^\tau_A(r) := \bigcup_{|t|\le \tau+r} \varphi_t\big(B_{\mathcal{Z}}(A,r)\big). \tag{2.2}$$

Let $\tau_{\mathrm{inj}_H} > 0$ be small enough so that the map

$$(-\tau_{\mathrm{inj}_H}, \tau_{\mathrm{inj}_H}) \times \mathcal{Z} \to T^*M, \qquad (t,q) \mapsto \varphi_t(q), \tag{2.3}$$

is injective. Given $r > 0$, $0 < \tau < \tau_{\mathrm{inj}_H}$, and a collection of points $\{\rho_j\}_{j\in\mathcal{J}(r)}$, we will work with the tubes

$$\mathcal{T}_j = \mathcal{T}_j(r) := \Lambda^\tau_{\rho_j}(r).$$

A (τ, r)-*cover for* $A \subset T^*M$ is a collection of tubes $\{\mathcal{T}_j(r)\}_{j\in\mathcal{J}(r)}$ for which

$$\Lambda^\tau_A(\tfrac{1}{2}r) \subset \bigcup_{j\in\mathcal{J}(r)} \mathcal{T}_j(r), \qquad \text{and} \qquad \mathcal{T}_j(r) \cap \Lambda^\tau_A(\tfrac{1}{2}r) \neq \emptyset, \quad \text{for all } j \in \mathcal{J}(r).$$

Let $\mathfrak{D} > 0$. We say a (τ, r)-cover is a $(\mathfrak{D}, \tau, r)$-*good cover*, if there is a splitting $\mathcal{J}(r) = \sqcup_{i=1}^{\mathfrak{D}} \mathcal{J}_i(r)$ such that for all $1 \le i \le \mathfrak{D}$ and $k \neq \ell \in \mathcal{J}_i(r)$,

$$\mathcal{T}_k(3r) \cap \mathcal{T}_\ell(3r) = \emptyset. \tag{2.4}$$

For $E \in \mathbb{R}$ and $r > 0$, we adopt the notation

$$\mathcal{J}_E(r) := \Big\{ j \in \mathcal{J}(r) : \ \mathcal{T}_j(r) \cap \mathcal{Z} \cap B(\Sigma^H_E, r) \neq \emptyset \Big\}. \tag{2.5}$$

We are now ready to introduce the definitions via coverings of our dynamical assumptions. First, for $0 < t_0 < T_0$, we say $A \subset T^*M$ is $[t_0, T_0]$ *non-self looping* if

$$\bigcup_{t=t_0}^{T_0} \varphi_t(A) \cap A = \emptyset \qquad \text{or} \qquad \bigcup_{t=-T_0}^{-t_0} \varphi_t(A) \cap A = \emptyset. \tag{2.6}$$

Definition 2.1 (non looping pairs via coverings)**.** Let $t_0 > 0$, $\tau_0 > 0$, $\mathfrak{D} > 0$, and $\mathbf{T}$ be a resolution function. Let H_1, H_2 be two submanifolds and $U_1 \subset N^*H_1$, $U_2 \subset N^*H_2$. We say (U_1, U_2) *is a* $(t_0, \mathbf{T})$ *non-looping pair in the window* $[a, b]$ *via* τ_0*-coverings with constant* C_{nl} provided for all $0 < \tau < \tau_0$ there exists $r_0 > 0$ such that for $0 < r < r_0$, any two $(\mathfrak{D}, \tau, r)$-good covers of $U_1 \cap \Sigma^{H_1}_{[a,b]}$ and $U_2 \cap \Sigma^{H_2}_{[a,b]}$, $\{\mathcal{T}^1_j(r)\}_{j\in\mathcal{J}^1(r)}$ and $\{\mathcal{T}^2_j(r)\}_{j\in\mathcal{J}^2(r)}$ respectively, and every $E \in [a, b]$, there is splittings of indices

$$\mathcal{J}^1_E(r) = \mathcal{B}^1_E(r) \cup \mathcal{G}^1_E(r), \qquad \mathcal{J}^2_E(r) = \mathcal{B}^2_E(r) \cup \mathcal{G}^2_E(r),$$

satisfying

(1) for each $i, k \in \{1, 2\}$, $i \neq k$ every $\ell \in \mathcal{G}^i_E(r)$,

$$\Bigg(\bigcup_{t_0+\tau \le |t| \le \mathbf{T}(r)-\tau} \varphi_t\Big(\mathcal{T}^i_\ell(r)\Big) \Bigg) \bigcap \Bigg(\bigcup_{j\in\mathcal{J}^k_E(r)} \mathcal{T}^k_j(r) \Bigg) = \emptyset,$$

(2) $r^{2(n-1)} |\mathcal{B}^1_E(r)| |\mathcal{B}^2_E(r)| \mathbf{T}(r)^2 \le \mathfrak{D}^2 C_{\mathrm{nl}}$.

We will say (H_1, H_2) *is a* $(t_0, \mathbf{T})$ *non-looping pair in the window* $[a,b]$ *via* τ*-coverings* if (N^*H_1, N^*H_2) is. We will also say H *is* $(t_0, \mathbf{T})$ *non-looping in the window* $[a,b]$ *via* τ *coverings* whenever (H,H) is a non-looping pair.

In Section 3, we prove that non looping in the sense of Definition 1.10 is equivalent to non looping by coverings in the sense of Definition 2.1.

Definition 2.2 (non-recurrence via coverings)**.** Let $\tau_0 > 0$, $\mathfrak{D} > 0$, and $\mathbf{T}$ be a resolution function. We say H is $\mathbf{T}$ *non-recurrent in the window* $[a,b]$ *via* τ_0*-coverings with constant* $C_{\rm nr}$ provided for all $0 < \tau < \tau_0$ there exists $r_0 > 0$ such that for $0 < r < r_0$, every $(\mathfrak{D}, \tau, r)$-good cover of $\Sigma^H_{[a,b]}$, $\{\mathcal{T}_j(r)\}_{j \in \mathcal{J}(r)}$, and $E \in [a,b]$, there exists a finite collection of sets of indices $\{\mathcal{G}_{E,\ell}(r)\}_{\ell \in \mathcal{L}_E(r)}$ with $\mathcal{J}_E(r) = \bigcup_{\ell \in \mathcal{L}_E(r)} \mathcal{G}_{E,\ell}(r)$, and so that for every $\ell \in \mathcal{L}_E(r)$ there exist functions $t_\ell(r) > 0$ and $T_\ell(r) > 0$, with $0 \le t_\ell(r) \le T_\ell(r) \le \mathbf{T}(r)$, so that

(1) $\bigcup_{j \in \mathcal{G}_{E,\ell}(r)} \mathcal{T}_j(r)$ is $[t_\ell(r), T_\ell(r)]$ non-self looping,

(2) $r^{\frac{n-1}{2}} \sum_{\ell \in \mathrm{L}_E(r)} \big(|\mathcal{G}_{E,\ell}(r)| t_\ell(r) T_\ell(r)^{-1}\big)^{\frac{1}{2}} \le \mathfrak{D}^{\frac{1}{2}} C_{\rm nr}\, \mathbf{T}(r)^{-\frac{1}{2}}$.

In Lemma 3.5 below we prove that non recurrence in the sense of Definition 1.11 implies non recurrence by coverings in the sense of Definition 2.2. At the moment, we are unable to determine whether these two definitions are equivalent.

2.2. **Uniformity assumptions.** Let $H \subset M$ be a smooth submanifold. In practice, we prove estimates on $\{\tilde{H}_h\}_h$, where $\{\tilde{H}_h\}_h$ is a family of submanifolds such that

$$\sup \Big\{ d\big(\rho, \Sigma^{\tilde{H}_h}_{[a,b]}\big) \;\Big|\; \rho \in \Sigma^H_{[a,b]} \Big\} \le R(h) \qquad h > 0, \tag{2.7}$$

where $R(h) > 0$ and for every multi-index α there is $\mathcal{K}_\alpha > 0$ such that for all $h > 0$

$$|\partial_x^\alpha \mathbf{R}_{\tilde{H}_h}| + |\partial_x^\alpha \mathbf{\Pi}_{\tilde{H}_h}| \le \mathcal{K}_\alpha. \tag{2.8}$$

Here $\mathbf{R}_{\tilde{H}_h}$ and $\mathbf{\Pi}_{\tilde{H}_h}$ denote the sectional curvature and the second fundamental form of $\tilde{H}_h$. Without loss of generality, we will assume $\mathcal{Z}$ is chosen so that there exist $N > 0$, $C = C(p, a, b, \{\mathcal{K}_\alpha\}_{|\alpha| \le N}) > 0$, and $r_0 > 0$ such that for all $E \in [a,b]$, $A \subset \Sigma^H_E$ and $0 < r < r_0$,

$$\mathrm{vol}\Big(B_{\mathcal{Z}}(A, r)\Big) \le C r^n \mu_{\Sigma^H_E}\Big(B_{\Sigma^H_E}(A, r)\Big).$$

We may do this since $\dim \mathcal{Z} = 2n-1$, $\dim \Sigma^H_E = n-1$, and $\Sigma^H_E \subset \mathcal{Z}$.

Note that when $H = \{x_0\}$ is a point, the curvature bounds become trivial, and so in place of (2.7) we work with $d(x_0, \tilde{x}_h) < R(h)$ and may take $\mathcal{K}_\alpha$ to be arbitrarily close to 0. In what follows, let $r_H : T^*M \to \mathbb{R}$ be the geodesic distance to H, i.e., $r_H(x,\xi) = d(x,H)$ for $(x,\xi) \in T^*M$, and write $\pi_M : T^*M \to M$ for the natural projection.

Definition 2.3 (regular families)**.** We will say a family of submanifolds $\{H_h\}_h$ is *regular in the window* $[a,b]$ if it satisfies (2.8) and there is $\varepsilon > 0$ so that for all $h > 0$, the map $(-\varepsilon, \varepsilon) \times \Sigma^H_{[a,b]} \to M$,

$$(t, \rho) \mapsto \pi_M(\varphi_t(\rho)) \quad \text{is a diffeomorphism.} \tag{2.9}$$

Then, define $|H_p r_{_H}| : \Sigma^H_{_{[a,b]}} \to \mathbb{R}$ by

$$|H_p r_{_H}|(\rho) := \lim_{t\to 0} |H_p r_{_H}(\varphi_t(\rho))|. \tag{2.10}$$

Definition 2.4 (uniformly conormally transverse submanifolds)**.** A family of submanifolds $\{\tilde{H}_h\}_h$ is said to be *uniformly conormally transverse for p in the window* $[a,b]$ provided

1. $\tilde{H}_h$ is conormally transverse for p in the window $[a,b]$ for all $h>0$,
2. there exists $\mathfrak{I}_{_0} > 0$ so that for all $h > 0$

$$\inf\Big\{|H_p r_{_{\tilde{H}_h}}|(\rho) \;\big|\; \rho \in \Sigma^H_{_{[a,b]}}\Big\} \geq \mathfrak{I}_{_0}. \tag{2.11}$$

When the constants involved in our estimates depend on $\{\tilde{H}_h\}_h$, they will do so *only* through finitely many of the $\mathcal{K}_{_\alpha}$ constants and the constant $\mathfrak{I}_{_0}$.

Remark 2.5. We note that for $p(x,\xi) = |\xi|^2_{g(x)}$, $a=b=1$, and $\Sigma^H_{_{[a,b]}} = SN^*H$, we have $|H_p r_{_H}|(\rho) = 2$ for all $\rho \in SN^*H$. It follows that every family of submanifolds is uniformly conormally transverse and we may take $\mathfrak{I}_{_0} = 2$.

2.3. **Main results.** We now state the main results from which all of our Kuznecov type asymptotics follow. Throughout the text, the notation $C = C(a_1,\dots,a_k)$ means that the constant C depends *only* on $a_1,\dots,a_k$.

Theorem 8. *For $j=1,2$, let $k_j \in \{1,\dots,n\}$, $\mathfrak{I}^j_{_0} > 0$, $A_j \in \Psi^\infty(M)$. Let $C^1_{_{\rm nr}} > 0$, $C^2_{_{\rm nr}} > 0$ and $C_{_{\rm nl}} > 0$. There is*

$$C_{_0} = C_{_0}(n, k_1, k_2, A_1, A_2, \mathfrak{I}^1_{_0}, \mathfrak{I}^2_{_0}, C^1_{_{\rm nr}}, C^2_{_{\rm nr}}, C_{_{\rm nl}}) > 0$$

such that the following holds.

Let $P(h) \in \Psi^m(M)$ be a self-adjoint semiclassical pseudodifferential operator, with classically elliptic symbol p. Let $0 < \delta < \frac{1}{2}$, $K > 0$, $a,b \in \mathbb{R}$ with $a \leq b$, and for $j=1,2$ let $H_j \subset M$ be a submanifold with co-dimension k_j that is regular and uniformly conormally transverse for p in the window $[a,b]$ (with constant $\mathfrak{I}^j_{_0}$ as in (2.11)). Then, there exists $\tau_0 > 0$ with the following property. Let $\Lambda > \Lambda_{\max}$, and $t_0 > 0$. For $j=1,2$ let $\mathbf{T}_j$ be a sub-logarithmic resolution function with $\Lambda\Omega(\mathbf{T}_j) < 1-2\delta$ and such that the submanifold H_j is $\mathbf{T}_j$ non-recurrent in the window $[a,b]$ via τ_0-coverings with constant $C^j_{_{\rm nr}}$. Suppose (H_1,H_2) is a $(t_0, \mathbf{T}_{\max})$ non-looping pair in the window $[a,b]$ via τ_0-coverings with constant $C_{_{\rm nl}}$ where $\mathbf{T}_{\max} = \max(\mathbf{T}_1, \mathbf{T}_2)$. Let $h^\delta \leq R(h) = o(1)$ and for $j=1,2$ let $\{\tilde{H}_{j,h}\}_h$ be a family of submanifolds of codimension k_j that is regular, uniformly conormally transverse for p in the window $[a,b]$, and satisfies

$$\sup\Big\{d\big(\rho, \Sigma^{\tilde{H}_{j,h}}_{_{[a,b]}}\big) \;\big|\; \rho \in \Sigma^{H_j}_{_{[a,b]}}\Big\} \leq R(h).$$

Then, there is $h_0 > 0$ such that for all $0 < h \leq h_0$ and $s \in [a-Kh, b+Kh]$,

$$\Big|E^{A_1,A_2}_{\tilde{H}_{1,h},\tilde{H}_{2,h}}(t_0,h;s)\Big| \leq C_{_0}\, h^{1-\frac{k_1+k_2}{2}} \Big/ \sqrt{\mathbf{T}_1(R(h))\mathbf{T}_2(R(h))}.$$

We also have the following corollary involving the definitions of non-looping (Definition 1.10) and non-recurrence (Definition 1.11).

Theorem 9. *Let* $\mathfrak{t}$ *be a resolution function,* $\Lambda > \Lambda_{\max}$*,* $K > 0$*,* $\varepsilon > 0$*,* $R_0 > 0$*,* $0 < \delta < \frac{1}{2}$*, and for* $j = 1, 2$ *let* $\mathbf{T}_j$ *be a sub-logarithmic resolution function with* $\Lambda\Omega(\mathbf{T}_j) < 1 - 2\delta$ *and let* $\mathbf{T}_{\max} = \max(\mathbf{T}_1, \mathbf{T}_2)$*. Suppose the same assumptions as Theorem 8, but assume instead that for* $j = 1, 2$ *the submanifold* H_j *is* $(\mathfrak{t}, \mathbf{T}_j)$ *non-recurrent in the window* $[a, b]$ *at scale* R_0*, and* (H_1, H_2) *is a* $(t_0, \mathbf{T}_{\max})$ *non-looping pair in the window* $[a, b]$ *with constant* $C_{\rm nl}$*. Then, there exist* $C_0 = C_0(n, k_1, k_2, A_1, A_2, \mathfrak{I}_0^1, \mathfrak{I}_0^2, \mathfrak{t}, C_{\rm nl})$ *and* $h_0 > 0$ *such that for all* $0 < h \leq h_0$ *and* $s \in [a - Kh, b + Kh]$

$$\left| E^{A_1, A_2}_{\tilde{H}_{1,h}, \tilde{H}_{2,h}}(t_0 + \varepsilon, h; s) \right| \leq C_0\, h^{1 - \frac{k_1 + k_2}{2}} \Big/ \sqrt{\mathbf{T}_1(R(h)) \mathbf{T}_2(R(h))}.$$

For the proof of Theorem 8, see §6.2 and for the proof of Theorem 9 see § 9.

2.4. **Application to the Laplacian.** In this section we show how to obtain Theorems 3, 4, and 5 from Theorem 9. To do this, we work with an operator Q such that $\sigma(Q)(x, \xi) = |\xi|_{g(x)}$ near $\{(x, \xi) : |\xi|_{g(x)} = 1\}$, Theorem 9 applies with $P = Q$, and for $\lambda = h^{-1}$ and all $N > 0$

$$\mathbb{1}_{(-\infty,1]}(Q) = \Pi_\lambda, \qquad (\rho_{h,t_0} * \mathbb{1}_{(-\infty,s]}(Q))(1) = \rho_{t_0} * \Pi_\lambda + O(h^\infty)_{H^{-N}_{\rm scl} \to H^N_{\rm scl}}. \tag{2.12}$$

To build Q, let $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}; [0, 1])$ with $\operatorname{supp} \psi_1 \subset (-1/4, 1/4)$, $\operatorname{supp} \psi_2 \subset [-16, 16]$, $\psi_1 \equiv 1$ on $[-1/16, 1/16]$ and $\psi_2 \equiv 1$ on $[-4, 4]$. We claim

$$Q = (1 - \psi_1(-h^2\Delta_g))\psi_2(-h^2\Delta_g)\sqrt{-h^2\Delta_g} - h^2\Delta_g(1 - \psi_2(-h^2\Delta_g)) \tag{2.13}$$

satisfies the desired properties. Observe that the second term in (2.14) is added to make Q classically elliptic, and that we use $-h^2\Delta_g$ rather than $\sqrt{-h^2\Delta_g}$ in order to apply [46, Theorem 14.9] to obtain $Q \in \Psi^2(M)$. Note also that Q is self-adjoint and $\sigma(Q) = |\xi|_g$ on $\{\frac{1}{2} \leq |\xi|_g \leq 2\}$,

$$\rho_{t_0} * \Pi_\lambda = \Big(\rho_{t_0,h} * \mathbb{1}_{(-\infty,s]}\Big(\sqrt{-h^2\Delta_g}\Big)\Big)(1), \qquad \Pi_\lambda = \mathbb{1}_{(-\infty,1]}\Big(\sqrt{-h^2\Delta_g}\Big) \tag{2.14}$$

$$\mathbb{1}_{(-\infty,s]}(Q) = \mathbb{1}_{(-\infty,s]}\Big(\sqrt{-h^2\Delta_g}\Big), \qquad s \in [\tfrac{1}{2}, 2] \tag{2.15}$$

and $\mathbb{1}_{(-\infty,s]}(Q) = \mathbb{1}_{(-\infty,s]}(\sqrt{-h^2\Delta_g}) = 0$ for $s < 0$. Finally, we use the ellipticity of both Q and $-h^2\Delta_g$ to obtain that for $N \geq 0$

$$\mathbb{1}_{(-\infty,s]}(Q) = O_N(\langle s \rangle^N)_{H^{-N}_{\rm scl} \to H^N_{\rm scl}}, \qquad \mathbb{1}_{(-\infty,s]}\Big(\sqrt{-h^2\Delta_g}\Big) = O_N(\langle s \rangle^{2N})_{H^{-N}_{\rm scl} \to H^N_{\rm scl}}.$$

Now, for all $N > 0$ and $L > 1$ there is $C_{N,L} > 0$ so that $|\rho\Big(\frac{t_0}{h}(1 - s)\Big)| \leq C_{N,L} h^{2N+L} \langle s \rangle^{-2N-L}$ on $|s - 1| > \frac{1}{2}$. Therefore

$$\begin{aligned} &\Big[\rho_{t_0,h} * \Big(\mathbb{1}_{(-\infty,s]}(Q) - \mathbb{1}_{(-\infty,s]}(\sqrt{-h^2\Delta_g})\Big)\Big](1) \\ &= \int_{\substack{s \notin [1/2,2] \\ s \geq 0}} \frac{t_0}{h}\rho\Big(\frac{t_0}{h}(1 - s)\Big)\Big(\mathbb{1}_{(-\infty,s]}(Q) - \mathbb{1}_{(-\infty,s]}\Big(\sqrt{-h^2\Delta_g}\Big)\Big) ds = O_N(h^{2N+L-1})_{H^{-N}_{\rm scl} \to H^N_{\rm scl}}. \end{aligned} \tag{2.16}$$

Combining (2.14) with (2.15) and (2.16), we obtain (2.12).

Now, every submanifold is conormally transverse for $p(x, \xi) = |\xi|_{g(x)}$ at $p^{-1}(1)$ with constant $\mathfrak{I}_0 = 1$. Therefore, Theorems 3, 4, and 5 follow from Theorem 9. To see this, we set $P = Q$, $a = b = 1$, and observe that the Hamiltonian flow for $\sigma(Q)$ near $S^*_x M$ is equal to the geodesic flow. In particular, the dynamical definitions 1.10 and 1.11 applied to Q at $E = 1$ are exactly Definitions 1.3 and 1.5 with S^*_xM replaced by SN^*H. This is true because Definitions 1.3 and 1.5 are stated with φ_t being the *homogeneous* geodesic flow, i.e., the flow generated by $|\xi|_{g(x)}$. Next, we apply Theorem 5 with $\Lambda = 2\Lambda_{\max}+1$, $h = \lambda^{-1}$, and work with the resolution functions $\widetilde{\mathbf{T}}_j = (\Lambda\Omega_0)^{-1}(1-2\delta)\mathbf{T}_j$ for $j = 1, 2$.

3. Dynamical assumptions and coverings

In this section we relate the non-looping and non-recurrence concepts introduced in Definitions 1.10 , 1.11, to their analogues via coverings given in Definitions 2.1, 2.2.

Proposition 3.1. *Let $H_1, H_2 \subset M$ be smooth submanifolds. Let $a, b \in \mathbb{R}$ be such that H_1, H_2 are conormally transverse for p in the window $[a, b]$, and $\tau_0 > 0$. Let $t_0 > 0$, $\mathbf{T}$ a resolution function, and suppose (H_1, H_2) is a $(t_0, \mathbf{T})$ non-looping pair in the window $[a, b]$ with constant $C_{\rm nl}$. Then, there is $\widetilde{C_{\rm nl}} = \widetilde{C_{\rm nl}}(p, a, b, n, C_{\rm nl}, H_1, H_2) > 0$ such that (H_1, H_2) is a $(t_0{+}3\tau_0, \widetilde{\mathbf{T}})$ non-looping pair in the window $[a, b]$ via τ_0-coverings with constant $\widetilde{C_{\rm nl}}$ and with $\widetilde{\mathbf{T}}(R) = \mathbf{T}(4R){-}3\tau_0$.*

Before proving the proposition, we record some facts about sub-logarithmic resolution functions.

Lemma 3.2. *Suppose $\mathbf{T}$ is a sub-logarithmic resolution function.*

(1) For $0 < a < b < 1$,
$$\mathbf{T}(b) \le \mathbf{T}(a) \le \frac{\log a}{\log b}\,\mathbf{T}(b).$$
In particular, $\mathbf{T}(R) \le \frac{\log R}{\log \mu + \log R}\mathbf{T}(\mu R)$ for $0 < \mu < R^{-1}$.

(2) Let $f(s) := -\log(\mathbf{T}^{-1}(s))$. Then, f extends to a differentiable function on $[0, \infty)$, $f(0) = 0$, and $f(a) \le \frac{a}{b} f(b)$ for $0 < a < b$.

(3) Let $0 < \delta < \frac{1}{2}$, and $R(h) \ge h^\delta$ with $R(h) = o(1)$. Then for all $\Lambda > \Lambda_{\max}$, $\varepsilon > 0$, there is $h_0 > 0$ such that for $0 < h < h_0$
$$\mathbf{T}(R(h)) \le (\Omega(\mathbf{T})\Lambda + \varepsilon) T_e(h).$$

Proof. Note that
$$0 \le \log\frac{\mathbf{T}(a)}{\mathbf{T}(b)} = -\int_a^b \frac{\mathbf{T}'(s)}{\mathbf{T}(s)} ds \le \int_a^b \frac{1}{s\log s^{-1}} ds = \log\Big(\frac{\log a^{-1}}{\log b^{-1}}\Big),$$
and hence the first claim holds. For the second claim, observe that since $\mathbf{T}$ is sub-logarithmic, $f'(s) \ge -\frac{\log(\mathbf{T}^{-1}(s))}{s} = \frac{f(s)}{s}$.

To prove the last claim, observe that since $R(h) = o(1)$, for all $\Lambda > \Lambda_{\max}$ and $\varepsilon > 0$, there is $h_0 > 0$ such that for $0 < h < h_0$,
$$\mathbf{T}(R(h)) \le (\Omega(\mathbf{T}) + \varepsilon\Lambda^{-1}) \log R(h)^{-1} \le (\Omega(\mathbf{T})\Lambda + \varepsilon) T_e(h).$$
The second inequality follows from definitions (1.2), (1.11), and $R(h) \ge h^\delta$ with $0 < \delta < \frac{1}{2}$. □

In the following lemma we explain how to partition a $(\mathfrak{D},\tau,r)$-good cover for $\Sigma_E^{H_1}$ into tubes that do not loop through $\Sigma_E^{H_2}$ for times in (t_0,T), and tubes that are 'bad' in the sense that they do loop through $\Sigma_E^{H_2}$. We do this while controlling the number of 'bad' tubes in terms of the size of the set $\mathcal{L}^{S,E}_{H_1,H_2}(t_0,T)$ for $S>4r$.

Lemma 3.3. *Let $a,b\in\mathbb{R}$, $H_1,H_2\subset M$ be smooth submanifolds such that H_1,H_2 are conormally transverse for p in the window $[a,b]$. Then there is $C_0=C_0(p,a,b,n,H_1,H_2)$ such that the following holds. Let $\tau_0>0$, $r>0$, and $0<\tau<\tau_0$. For $i=1,2$ let $\{\mathcal{T}_j^i(r)\}_{j\in\mathcal{J}^i(r)}$ be a $(\mathfrak{D},\tau,r)$-good cover of $\Sigma^{H_i}_{[a,b]}$. Let $t_0>0$, $T>0$. Then, for all $E\in[a,b]$ and $S\geq 4r$ there is a splitting $\mathcal{J}_E^1(r)=\mathcal{B}_E^1(r)\cup\mathcal{G}_E^1(r)$ such that*

(1) for $j\in\mathcal{G}_E^1(r)$ and $k\in\mathcal{J}_E^2(r)$

$$\bigcup_{t_0+2(\tau+r)\leq|t|\leq T-2(\tau+r)}\varphi_t(\mathcal{T}_j^1(r))\cap\mathcal{T}_k^2(r)=\emptyset,$$

(2) $|\mathcal{B}_E^1(r)|\leq\mathfrak{D}C_0r^{1-n}\mu_{\Sigma_E^{H_1}}\Big(B_{\Sigma_E^{H_1}}\Big(\mathcal{L}^{S,E}_{H_1,H_2}(t_0,T),S\Big)\Big)$.

Proof. For $j=1,2$ let $\mathcal{Z}_j\subset T^*M$ be the hypersurface transverse to the flow, with $\Sigma^{H_j}_{[a,b]}\subset\mathcal{Z}_j$, used to build the tubes of the cover, as explained in (2.1). Let $E\in[a,b]$ and for $S>0$ set

$$\mathcal{B}_E^1(r):=\big\{j\in\mathcal{J}_E^1(r)\,:\,\mathcal{T}_j^1(r)\cap B_{\mathcal{Z}_1}(\mathcal{L}^{S,E}_{H_1,H_2}(t_0,T),2r)\neq\emptyset\big\}.$$

Then, for $j\in\mathcal{B}_E^1(r)$,

$$\mathcal{T}_j^1(r)\cap\mathcal{Z}_1\;\subset B_{\mathcal{Z}_1}(\mathcal{L}^{S,E}_{H_1,H_2}(t_0,T),4r).$$

In particular, there exists $C_0=C_0(p,a,b,n,H_1,H_2)>0$ such that for all $S\geq4r$

$$|\mathcal{B}_E^1(r)|\leq\mathfrak{D}r^{1-2n}\operatorname{vol}\Big(B_{\mathcal{Z}_1}(\mathcal{L}^{S,E}_{H_1,H_2}(t_0,T),4r)\Big)\leq C_0\mathfrak{D}r^{1-n}\mu_{\Sigma_E^{H_1}}\Big(B_{\Sigma_E^{H_1}}\big(\mathcal{L}^{S,E}_{H_1,H_2}(t_0,T),S\big)\Big).$$

This proves the claim in (2).

To see the claim in (1), let $j\in\mathcal{G}_E^1(r):=\mathcal{J}_E^1(r)\setminus\mathcal{B}_E^1(r)$. Then, $\mathcal{T}_j^1(r)=\Lambda^\tau_{\rho_j}(r)$ for some $\rho_j\in\mathcal{Z}_1$ with $d(\rho_j,\Sigma_E^{H_1})<2r$ and $d(\rho_j,\mathcal{L}^{S,E}_{H_1,H_2}(t_0,T))>3r$. This yields that there is $\rho_0\in\Sigma_E^{H_1}\setminus\mathcal{L}^{S,E}_{H_1,H_2}(t_0,T)$ such that $d(\rho_0,\rho_j)<2r$. In particular, since $\bigcup_{t_0\leq|t|\leq T}\varphi_t(B(\rho_0,S))\cap B(\Sigma_E^{H_2},S)=\emptyset$ and $\mathcal{T}_j^1(r)\subset\bigcup_{|t|\leq\tau+r}\varphi_t(B(\rho_0,3r))$, this yields

$$\bigcup_{t_0+\tau+r\leq|t|\leq T-(\tau+r)}\varphi_t(\mathcal{T}_j^1(r))\cap B(\Sigma_E^{H_2},S)=\emptyset \tag{3.1}$$

for $S\geq4r$. On the other hand, since for all $k\in\mathcal{J}_E^2(r)$, we have $\mathcal{T}_k^2(r)\cap\mathcal{Z}_2\subset B(\Sigma_E^{H_2},3r)$,

$$\mathcal{T}_k^2(r)\subset\bigcup_{|t|\leq\tau+r}\varphi_t(B(\Sigma_E^{H_2},3r)) \tag{3.2}$$

In particular, combining (3.1) and (3.2) we have

$$\bigcup_{t_0+2(\tau+r)\leq|t|\leq T-2(\tau+r)} \varphi_t(\mathcal{T}_j^1(r))\cap B(\Sigma_E^{H_2},S)=\emptyset.$$

Thus, the claim (1) holds, provided $S\geq 4r$. □

With Lemmas 3.2 and 3.3 in place, we are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let $C_0=C_0(p,a,b,n,H_1,H_2)$ be as in Lemma 3.3. We apply Lemma 3.3 with $r=R$, $T=\mathbf{T}(S)$, $S=4R$, $0<R<\frac{1}{2}\tau_0$. This shows that (H_1,H_2) is a $[t_0+3\tau_0,\widetilde{\mathbf{T}}]$ non-looping pair in the window $[a,b]$ via τ-coverings with constant $\widetilde{C_{\rm nl}}=C_0^2C_{\rm nl}$. □

Lemma 3.4. *There is a constant $C_n>0$, depending only on n, such that the following holds. Let $\tau_0>0$, $t_0>0$, $H_1,H_2\subset M$ be smooth submanifolds such that H_1 and H_2 are conormally transverse for p in the window $[a,b]$. Let $\mathbf{T}$ be a resolution function. If (H_1,H_2) is a $(t_0,\mathbf{T})$ non-looping pair in the window $[a,b]$ via τ_0-coverings with constant $C_{\rm nl}$, then (H_1,H_2) is a $(t_0,\widetilde{\mathbf{T}})$ non-looping pair in the window $[a,b]$ with constant $C_{\rm nl}C_n$ and $\widetilde{\mathbf{T}}(R)=\mathbf{T}(2R)$.*

Proof. Let $E\in[a,b]$ and fix $i,j\in\{1,2\}$, $i\neq j$. For each $R>0$ consider the non-looping partition $\mathcal{J}_E^i(R)=\mathcal{G}_E^i(R)\sqcup\mathcal{B}_E^i(R)$ given by Definition (2.1). Let $\rho\in\mathcal{L}_{H_i,H_j}^{R/2,E}(t_0,\mathbf{T}(R))$. Then, there are $\rho_1\in B(\rho,R/2)$ and $t_0\leq|t|\leq\mathbf{T}(R)$ such that $\varphi_t(\rho_1)\in B(\Sigma_E^{H_j},R/2)$. Hence, there is $\ell\in\mathcal{B}_E^i(R)$ such that $\rho_1\in\mathcal{T}_\ell^i(R)$ and hence $\rho\in\mathcal{T}_\ell^i(2R)$. This implies $B_{\Sigma_E^{H_i}}(\rho,R/2)\subset\mathcal{T}_\ell^i(3R)$. Thus,

$$B_{\Sigma_E^{H_i}}(\mathcal{L}_{H_i,H_j}^{R/2,E}(t_0,\mathbf{T}(R)),R/2)\subset\bigcup_{\ell\in\mathcal{B}_E^i(R)}\mathcal{T}_\ell^i(3R).$$

In particular, there exists $C_n>0$ such that

$$\mu_{\Sigma_E^{H_i}}\Big(B_{\Sigma_E^{H_i}}(\mathcal{L}_{H_i,H_j}^{R/2,E}(t_0,\mathbf{T}(R)),R/2)\Big)\leq C_nR^{n-1}|\mathcal{B}_E^i(R)|.$$

Therefore,

$$\begin{aligned}\mu_{\Sigma_E^{H_1}}\Big(B_{\Sigma_E^{H_1}}(\mathcal{L}_{H_1,H_2}^{R/2,E}(t_0,\mathbf{T}(R)),R/2)\Big)\mu_{\Sigma_E^{H_2}}\Big(B_{\Sigma_E^{H_2}}(\mathcal{L}_{H_2,H_1}^{R/2,E}(t_0,\mathbf{T}(R)),R/2)\Big)\mathbf{T}(R)^2\\ \leq C_n^2R^{2n-2}|\mathcal{B}_E^1(R)||\mathcal{B}_E^2(R)|\mathbf{T}(R)^2\leq C_n^2\mathfrak{D}^2C_{\rm nl}.\end{aligned}$$

The lemma follows from Definition 1.10 after taking the limit $R\to0^+$ and redefining C_n. □

Proposition 3.5. *Let $\mathfrak{t}$, $\mathbf{T}$ be resolution functions and $H\subset M$ be a smooth submanifold. Let $a,b\in\mathbb{R}$ be such that H is conormally transverse for p in the window $[a,b]$. Suppose H is $(\mathfrak{t},\mathbf{T})$ non-recurrent in the window $[a,b]$ at scale R_0.*

Then, there exists $C_{\rm nr}=C_{\rm nr}(M,p,\mathfrak{t},R_0)>0$ such that for all $\tau_0>0$, there is a resolution function $\widetilde{\mathbf{T}}$ such that the submanifold H is $\widetilde{\mathbf{T}}$ non-recurrent in the window $[a,b]$ via τ_0-coverings with constant $C_{\rm nr}$. Moreover, there is $c>0$ such that if $\mathbf{T}$ is sub-logarithmic, then $\widetilde{\mathbf{T}}(R)\geq c\mathbf{T}(R)$ for all R.

The proof of this result hinges on two lemmas. To state the first one, we introduce a slight adaptation of [8, Definition 3]. Let $\varepsilon_0 > 0$, $\digamma > 0$, $\mathfrak{t}_0 : [\varepsilon_0, +\infty) \to [1, +\infty)$, and $f : [0, \infty) \to [0, \infty)$. We say a set A_0 is $(\varepsilon_0, \mathfrak{t}_0, \digamma, f)$ *controlled up to time* T provided it is $(\varepsilon_0, \mathfrak{t}_0, \digamma)$ controlled up to time T in the sense of [8, Definition 3] except that we replace the condition on r by

$$0 < r < \tfrac{1}{\digamma} e^{-\digamma \Lambda T - f(T)} r_0 \tag{3.3}$$

and replace point (3) by

$$\inf_k R_{1,k} \geq \tfrac{1}{4} e^{-f(T)} \inf_i R_{0,i}. \tag{3.4}$$

Next, fix $E \in [a, b]$. Since H is $(\mathfrak{t}, \mathbf{T})$ non-recurrent in the window $[a, b]$ at scale R_0, for all $\rho \in \Sigma^H_E$ there exists a choice of $\pm$ such that for all $A \subset B_{\Sigma^H_E}(\rho, R_0)$, $0 < R < R_0$, $\varepsilon > 0$, and $T > \mathfrak{t}(\varepsilon)$

$$\mu_{\Sigma^H_E}\Big(B_{\Sigma^H_E}\Big(\mathcal{R}^{e^{-f(T)}R}_{A,\pm}(\mathfrak{t}(\varepsilon), T)\,,\, e^{-f(T)}R\Big) \leq \varepsilon\, \mu_{\Sigma^H_E}(B_{\Sigma^H_E}(A, R)\Big), \tag{3.5}$$

with f as in Lemma 3.2. Then, extract a finite cover of Σ^H_E by balls $\tilde{B}_\rho = B(\rho, R_0/2)$ and set

$$\tilde{\mathcal{A}}_E := \{\tilde{B}_{\rho_i}\}_{i=1}^K, \qquad \text{and} \qquad \mathcal{A}_E := \{B_{\rho_i}\}_{i=1}^K, \tag{3.6}$$

where $B_\rho = B(\rho, R_0)$. Note that, again using that H is non-recurrent with at scale R_0, we may assume $K \leq C_n R_0^{1-n}$ where C_n is a constant depending only on n.

Lemma 3.6. *Let H, $\mathfrak{t}$ and $\mathbf{T}$ be as in Proposition 3.5 and $f(T) := -\log(\mathbf{T}^{-1}(T))$. Then, there exist $c_n > 0$ depending only on n and $\digamma > 0$ such that for all $E \in [a, b]$ and $T > 1$ every ball in $\mathcal{A}_E$ is $(0, \mathfrak{t}_0, \digamma, f)$ controlled up to time T with $\mathfrak{t}_0(\varepsilon) = \mathfrak{t}(c_n \varepsilon)$.*

Proof. Let $E \in [a, b]$. Let $A_0 := B_{\rho_0}$ for some $B_{\rho_0} \in \mathcal{A}_E$, $\varepsilon_1 > 0$, $\Lambda > \Lambda_{\max}$, and $0 < \tau < \frac{1}{2}\tau_{\text{inj}_H}$. Let $T > 1$ and $0 \leq \tilde{R}_0 \leq \frac{1}{\digamma} e^{-\digamma \Lambda T}$ for $\digamma > 2R_0^{-1}$ to be determined later. Let $0 < r_0 < \tilde{R}_0$. Suppose $A_1 \subset A_0$ and $\{B_{0,i}\}_{i=1}^N$ are balls centered in A_0 with radii $R_{0,i} \in [r_0, \tilde{R}_0]$ such that $A_1 \subset \cup_{i=1}^N B_{0,i} \subset A_0$.

Let $R := \frac{1}{2} \inf_i R_{0,i}$. There exist $C_n > 0$, depending only on n, and a collection of balls $\{\tilde{B}_{0,i}\}_{i=1}^{N_0}$ of radius R, such that

$$A_1 \subset \bigcup_{i=1}^{N_0} \tilde{B}_{0,i}, \qquad N_0 R^{n-1} \leq C_n \sum_{i=1}^{N} R_{0,i}^{n-1}. \tag{3.7}$$

Fix $0 \leq r \leq \frac{1}{\digamma} e^{-\digamma \Lambda T - f(T)} r_0$. Next, let $\{B(q_j, r)\}_{j \in \mathcal{J}} \subset \Sigma^H_E$ be a cover of Σ^H_E by balls of radius r such that there are at most $\mathfrak{D}_n$ balls over each point in Σ^H_E, where $\mathfrak{D}_n > 0$ depends only on n. Assume, without loss of generality, that (3.5) holds for ρ_0 with the choice $\pm = +$. Next, set $\mathcal{J}_{A_1} := \{j \in \mathcal{J} :\ B(q_j, \frac{1}{2} e^{-f(T)} R) \cap \mathcal{R}^{e^{-f(T)}R}_{A_1,+}(\mathfrak{t}(\varepsilon_1), T) \neq \emptyset\}$. Defining the collection

$$\{B_{1,i}\}_{i=1}^{N_1} := \Big\{B_{\Sigma^H_E}\big(q_j, \tfrac{1}{2} e^{-f(T)} R\big) :\ j \in \mathcal{J}_{A_1}\Big\},$$

we have $\bigcup_{i=1}^{N_1} B_{1,i} \subset B_{\Sigma_E^H}\Big(\mathcal{R}^{e^{-f(T)}R}_{A_1,+}(\mathfrak{t}(\varepsilon_1),T)\,,\, e^{-f(T)}R\Big)$. Then, letting $R_{1,i} := \frac{1}{2}e^{-f(T)}R$, we have $R_{1,i} \in [0, \frac{1}{4}\tilde{R}_0]$, and using that $R < R_0/2$ the bound in (3.5) applied to A_1 yields

$$\sum_{i=1}^{N_1} R_{1,i}^{n-1} \le \varepsilon_1\, \mathfrak{D}_n\, \mu_{\Sigma_E^H}\Big(B_{\Sigma_E^H}(A_1, R)\Big). \tag{3.8}$$

Next, by (3.7) note that $B_{\Sigma_E^H}(A_1, R) \subset \bigcup_{i=1}^{N_0} 2\tilde{B}_{0,i}$, where $2\tilde{B}_{0,i}$ denotes the ball with the same center as $\tilde{B}_{0,i}$ but with radius $2R$. Using (3.7) again there is $C_n > 0$ such that

$$\mu_{\Sigma_E^H}(B_{\Sigma_E^H}(A_1, R)) \le \mu_{\Sigma_E^H}\Big(\bigcup_{i=1}^{N_0} 2\tilde{B}_{0,i}\Big) \le C_n \sum_{i=1}^{N} R_{0,i}^{n-1}. \tag{3.9}$$

Let $\varepsilon := \varepsilon_1 C_n \mathfrak{D}_n$. Combining (3.8) and (3.9) yields point (2) of [8, Definition 3] with $\mathfrak{t}_0(\varepsilon) = \mathfrak{t}(\varepsilon/(C_n\mathfrak{D}_n))$. By the definition of R, we also note that point (3), which was replaced by (3.4), also holds.

It remains to check point (1) i.e. there is $\digamma > 0$ such that $\Lambda^\tau_{A_1\setminus\cup_k B_{1,k}}(r)$ is $[\mathfrak{t}_0(\varepsilon), T]$ non-self looping for $0 < r < \frac{1}{\digamma}e^{-\digamma\Lambda T - f(T)}R$. For this, suppose $\rho_1, \rho_2 \in \Lambda^\tau_{A_1\setminus\cup_k B_{1,k}}(r)$ and $t \in [\mathfrak{t}_0(\varepsilon), T]$ such that $\varphi_t(\rho_1) = \rho_2$. Then, there are $s_1, s_2 \in [-\tau - r, \tau + r]$, $q_1, q_2 \in A_1 \setminus \cup_k B_{1,k}$ such that $d(\rho_i, \varphi_{s_i}(q_i)) < r$. In particular, there is $C_0 > 0$ depending only on (M, p, a, b, Λ) such that

$$d(\varphi_{s_2 - t - s_1}(q_2), A_1) < (1 + C_0 e^{\Lambda(|t| + 2\tau + 2r)})r.$$

Finally, let $\digamma > 0$ be large enough so that $\frac{1}{\digamma}e^{-\digamma\Lambda T} < \min((1 + C_0 e^{\Lambda(|T|+2\tau+2r)})^{-1}, R_0/2)$. Note that the choice of $\digamma$ does not need to depend on T. Then, since $r < (1+C_0 e^{\Lambda(|T|+2\tau+2r)})^{-1}e^{-f(T)}R$, we have $q_2 \in \mathcal{R}^{e^{-f(T)}R}_{A_1,+}(\mathfrak{t}_0(\varepsilon), T)$, which is a contradiction since $\mathcal{R}^{e^{-f(T)}R}_{A_1,+}(\mathfrak{t}_0(\varepsilon), T) \subset \cup_i B_{1,i}$. $\square$

In what follows we fix $1 < \beta_0 < \varepsilon_0^{-1}$ and define

$$\mathbf{F}(T) := \sum_{k=0}^{\log_{\beta_0} T} f\big(\beta_0^{-k}\, T\big).$$

Lemma 3.7. *Let $B \subset \Sigma_E^H$ be a ball of radius $\delta > 0$. Let $0 < \varepsilon_0 < 1$, $\mathfrak{t}_0 : [\varepsilon_0, +\infty) \to [1, +\infty)$, $f : [0, \infty) \to [0, \infty)$ increasing with $f(e^{-x}) \in L^1([0, \infty))$, $T_0 > 0$, and $\digamma > 0$, such that B can be $(\varepsilon_0, \mathfrak{t}_0, \digamma, f)$-controlled up to time T_0. Let $0 < m < \frac{\log T_0 - \log \mathfrak{t}_0(\varepsilon_0)}{\log \beta_0}$ be a positive integer, $\Lambda > \Lambda_{\max}$,*

$$0 < \tilde{R}_0 \le \min\Big\{\tfrac{1}{\digamma}e^{-\digamma\Lambda T_0}, \tfrac{\delta}{10}\Big\}, \qquad 0 < r_1 < \tfrac{1}{5\digamma}e^{-(\digamma\Lambda T_0 + \mathbf{F}(T_0) + f(T_0))}\tilde{R}_0,$$

and $B_0 \subset B$ with $d(B_0, B^c) > \tilde{R}_0$. Let $0 < \tau < \tau_0$ and suppose $\{\Lambda^\tau_{\rho_j}(r_1)\}_{j=1}^{N_{r_1}}$ is a $(\mathfrak{D}, \tau, r_1)$ good cover of $\Sigma_{H,p}$ and set $\mathcal{E} := \{j \in \{1, \dots, N_{r_1}\} : \Lambda^\tau_{\rho_j}(r_1) \cap \Lambda^\tau_{B_0}(\frac{r_1}{5}) \neq \emptyset\}$.

Then, there exist $C_{M,p} > 0$ depending only on (M,p) and sets $\{\mathcal{G}_{E,\ell}\}_{\ell=0}^m \subset \{1,\dots N_{r_1}\}$, $\mathcal{B}_E \subset \{1,\dots N_{r_1}\}$ so that $\mathcal{E} \subset \mathcal{B}_E \cup \cup_{\ell=0}^m \mathcal{G}_{E,\ell}$ and

- $\bigcup_{i\in\mathcal{G}_{E,\ell}} \Lambda^\tau_{\rho_i}(r_1)$ *is* $[\mathfrak{t}_0(\varepsilon_0), \beta_0^{-\ell}T_0]$ *non-self looping for* $\ell \in \{0,\dots,m\}$, (3.10)
- $|\mathcal{G}_{E,\ell}| \le C_{M,p}\mathfrak{D}\varepsilon_0^\ell \delta^{n-1} r_1^{1-n}$ *for every* $\ell \in \{0,\dots,m\}$, (3.11)
- $|\mathcal{B}_E| \le C_{M,p}\mathfrak{D}\varepsilon_0^{m+1}\delta^{n-1}r_1^{1-n}$. (3.12)

Proof. The proof is the same as that of [8, Lemma 3.2], with a very minor modification. Namely, we replace R_0 by $\tilde R_0$ and put $r_0 = e^{-\mathbf{F}(T_0)}\tilde R_0$ instead of $r_0 = e^{2\mathbf{D}\Lambda T_0}\tilde R_0$. We then obtain the following instead of the leftmost equation in [8, (3.21)]

$$\inf_k R_{2,k} \ge \tfrac14 e^{-f(T_0)} \inf_i R_{1,i}.$$

Which in turn changes the leftmost equation in [8, (3.22)] to

$$\inf_k R_{\ell,k} \ge e^{-\mathbf{F}(T_0)}\tilde R_0 = r_0.$$

This follows from the argument below [8, Remark 8], that yields, since $\ell \le m$,

$$\inf_k R_{\ell,k} \ge \frac{1}{4^\ell}\prod_{j=0}^{\ell} e^{-f(\beta_0^{-j}T_0)}R_0 = \frac{1}{4^\ell} e^{-\sum_{j=0}^\ell f(\beta_0^{-j}T_0)}\tilde R_0 \ge e^{-\mathbf{F}(T_0)}\tilde R_0.$$

□

With Lemmas 3.6 and 3.7 in place, we are now ready to prove Proposition 3.5.

Proof of Proposition 3.5. Let $\{\mathcal{T}_j(R)\}_{j\in\mathcal{J}(h)} = \{\Lambda^\tau_{\rho_j}(R)\}_{j\in\mathcal{J}(h)}$ be a $(\mathfrak{D},\tau,R)$ good covering of $\Sigma^H_{[a,b]}$. Let $E\in[a,b]$ and $\mathcal{A}_E := \{B_{\rho_i}\}_{i=1}^K$ be the covering of Σ^H_E as described in (3.6). Let $\mathfrak{t}_0$ be as in Lemma 3.6 and fix $0<\varepsilon_0<\frac12$. There exists $\digamma > 0$ such that each ball in $\mathcal{A}_E$ can be $(\varepsilon_0, \mathfrak{t}_0, \digamma, f)$ controlled for time $T>1$.

We then apply Lemma 3.7 to each ball in $\mathcal{A}_E$. Let $\delta_0 := R_0/2$ be the radius of the balls in $\mathcal{A}_E$, and $\mathbf{T}_0 = \mathbf{T}_0(R)$ such that $\mathbf{T}_0 > \mathfrak{t}_0(\varepsilon_0)$ and

$$R \le \frac{1}{10\digamma^2} e^{-\left(2\digamma\Lambda \mathbf{T}_0(R) + \mathbf{F}(\mathbf{T}_0(R)) + f(\mathbf{T}_0(R))\right)}. \tag{3.13}$$

Without loss of generality, we may assume $\digamma$ is large enough so that $\frac{1}{\digamma}e^{-\digamma\Lambda\mathfrak{t}_0(\varepsilon_0)} \le \frac{\delta_0}{10}$. Then, putting $\tilde R_0 = \frac{1}{\digamma}e^{-\digamma\Lambda T_0}$ in Lemma 3.7, and using condition (3.13) allows us to set $r_1 = R$ in Lemma 3.7 and apply it to each ball B_{ρ_0} in $\mathcal{A}_E$. Let $\tilde B_{\rho_0}$ be the ball with the same center as B_{ρ_0} but with a radius $R_0/2$ so that $d(\tilde B_{\rho_0}, B^c_{\rho_0}) = R_0/2 > \tilde R_0$. Let $\tau_0 > 0$, $0<\tau<\tau_0$, and set $\mathcal{J}_E^{\rho_0}(R) = \{j \in \mathcal{J}_E(R):\ \Lambda^\tau_{\rho_j}(R) \cap \Lambda^\tau_{\tilde B_{\rho_0}}(\frac15 R) \ne \emptyset\}$, there is $C_{M,p}>0$ and sets $\{\mathcal{G}_{E,\ell}\}_{\ell=0}^m \subset \mathcal{J}_E(R)$, $\mathcal{B}_E \subset \mathcal{J}_E(R)$ so that $\mathcal{J}_E^{\rho_0}(R) \subset \mathcal{B}_E \cup \cup_{\ell=0}^m \mathcal{G}_{E,\ell}$, and (3.10), (3.11), (3.12) hold.

Therefore, letting $T_\ell = \beta_0^{-\ell}\mathbf{T}_0$ and $t_\ell = \mathfrak{t}_0(\varepsilon_0)$ for $1 \leq \ell \leq m$, and setting $\mathcal{G}_{m+1} := \mathcal{B}_{E}$, $T_{m+1} = t_{m+1} = 1$, yields that there exists $C_{\rm nr} = C_{\rm nr}(M, p, \mathfrak{t}) > 0$ such that

$$R^{\frac{n-1}{2}} \sum_{\ell=0}^{m+1} \left(\frac{|\mathcal{G}_\ell| t_\ell}{T_\ell}\right)^{1/2} \leq \left(\frac{C_{M,p}\mathfrak{D}\delta_0^{n-1}}{\mathbf{T}_0(R)} \sum_{\ell=0}^{m+1} (\beta_0\varepsilon_0)^\ell\right)^{\frac{1}{2}} \leq \frac{C_{\rm nr}\mathfrak{D}^{\frac{1}{2}}}{\sqrt{\mathbf{T}_0(R)}}.$$

The existence of $C_{\rm nr} > 0$ is justified since $\beta_0\varepsilon_0 < 1$. Repeating for each ball $B_{\rho_i} \in \mathcal{A}_E$ and using $K \leq C_n R_0^{1-n}$, proves that H is $\mathbf{T}_0$ non-recurrent in the window $[a, b]$ via τ_0-coverings with constant $C_{\rm nr} C_n R_0^{1-n}$.

By Lemma 3.2, when $\mathbf{T}$ is sub-logarithmic and $0 < a < b$ we have $f(b) \geq \frac{b}{a} f(a)$. In particular,

$$\mathbf{F}(T_0) = \sum_j f(2^{-j}T_0) \leq \sum_j 2^{-j} f(T_0) \leq 2f(T_0).$$

Therefore, using $f(T) = -\log(\mathbf{T}^{-1}(T))$, there exists $c > 0$ such that we may define

$$\mathbf{T}_0(R) = cf^{-1}(\log R) \geq c\mathbf{T}(R).$$

□

Remark 3.8. We note that our definition of recurrence (Definition 1.11) is equivalent to the following. There is $\digamma > 0$ such that for all $\rho \in \Sigma_E^H$ there is $R_0 > 0$ such that $B(\rho, R_0)$ is $(\varepsilon_0, \mathfrak{t}_0, \digamma, f)$ controlled with an additional small modification of the definition of $(\varepsilon_0, \mathfrak{t}_0, \digamma, f)$ controlled (see (3.3) and (3.4)): One needs to replace (1) by

$$\bigcup_{t_0 \leq \pm t \leq T} \Lambda^\tau_{A_1 \setminus \cup \tilde{B}_{1,k}}(r) \cap \Lambda^\tau_{A_1}(r) = \emptyset.$$

To see these are equivalent, we identify $B(\rho, R_0)$ with A_0 and A with A_1.

One can check that all of the proofs of being $(\varepsilon_0, \mathfrak{t}_0, \digamma, f)$ controlled in [8] actually prove this slightly stronger condition with $f(T) = CT$ for some $C > 0$.

4. Basic Estimates for averages over submanifolds

Let $P(h) \in \Psi^m(M)$ be a self-adjoint semiclassical pseudodifferential operator, with classically elliptic symbol p. Throughout this section we assume $H \subset M$ is a smooth submanifold of co-dimension k, and $a, b \in \mathbb{R}$ are such that H is conormally transverse for p in the window $[a, b]$.

As explained in §1.6, we crucially view the kernel of the spectral projector $\mathbb{1}_{[t-s,t]}(P)$ as a quasimode for P. We are then able to use estimates from [11] to estimate the error when the projector is smoothed at very small scales. This section is dedicated to adapting the estimates from [11] to the current setup.

All our estimates are made in terms of $(\mathfrak{D}, \tau, R(h))$-good covers and δ-partitions associated to them. For the definition of a good cover see (2.4). Note, in addition, that there is a constant $\mathfrak{D}_n$ depending only on n such that we may work with a $(\mathfrak{D}_n, \tau, R(h))$ good cover [10, Lemma 2.2] [11, Proposition 3.3].

We now define the concept of δ-partitions. Let $\tau > 0$, $0 < \delta < \frac{1}{2}$, and $R(h) \geq h^\delta$. Let $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$ be a $(\tau, R(h))$-cover for $\Sigma^H_{[a,b]}$ with $\mathcal{T}_j = \Lambda^\tau_{\rho_j}(R(h))$, and for $E \in [a,b]$ let $\mathcal{J}_E(h) := \mathcal{J}_E(R(h))$ as defined in (2.5). We say

$$\{\chi_{\mathcal{T}_j}\}_{j\in\mathcal{J}_E(h)} \subset S_\delta(T^*M;[0,1]) \tag{4.1}$$

is a *δ-partition for* Σ^H_E associated to $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$ provided the families $\{\chi_j\}_{j\in\mathcal{J}_E(h)}$ and $\{h^{-1}[P,\chi_j]\}_{j\in\mathcal{J}_E(h)}$ are bounded in $S_\delta(T^*M;[0,1])$ and

$$(1)\,\operatorname{supp}\chi_j \subset \Lambda^\tau_{\rho_j}(R(h)),\ \text{ for all } j\in\mathcal{J}_E(h), \qquad (2)\sum_{j\in\mathcal{J}_E(h)}\chi_j \geq 1 \text{ on } \Lambda^{\tau/2}_{\Sigma^H_E}(\tfrac{1}{2}R(h)).$$

For the construction of such a partition we refer the reader to [11, Proposition 3.4].

The next lemma controls the average of Au over a submanifold H in terms of the L^2 masses of the bicharacteristic beams intersecting the microsupport of A. Here, u is a quasimode for P and A is a pseudodifferential operator. When we apply this lemma, u will be the kernel of the spectral projector onto a small window, and A will either represent a localizer to a family of tubes or differentiation in one of the coordinates.

To ease notation, for $E \in \mathbb{R}$ we write $P_E = P_E(h)$

$$P_E := P - E. \tag{4.2}$$

In addition, given $A \in \Psi^\infty_\delta(M)$, $\psi \in C^\infty_0(\mathbb{R};[0,1])$, $E\in\mathbb{R}$, $h>0$, $C>0$, $C_N>0$, and $u\in\mathcal{D}'(M)$ we set $\alpha := \frac{k-2m+1}{2}$ and

$$Q^{A,\psi}_{E,h}(C,C_N,u) := Ch^{-\frac{1}{2}-\delta}\big\|\big(1-\psi\big(\tfrac{P_E}{h^\delta}\big)\big)P_E Au\big\|_{H^\alpha_{\rm scl}} + C_N h^N\Big(\|u\|_{L^2(M)} + \|P_E u\|_{H^\alpha_{\rm scl}}\Big). \tag{4.3}$$

We fix $\varepsilon_0 > 0$ and a continuous family $[a-\varepsilon_0, b+\varepsilon_0] \ni E \mapsto B_E \in \Psi^0_\delta(M)$ such that

$$\mathrm{MS_h}(B_E) \subset \Lambda^{\tau_0+\varepsilon_0}_{\Sigma^H_E}(3R(h)) \qquad\text{and}\qquad \mathrm{MS_h}(I - B_E)\cap\Lambda^{\tau_0+\varepsilon_0}_{\Sigma^H_E}(2R(h)) = \emptyset. \tag{4.4}$$

This will serve as a microlocalizer to the region of interest. We recall the constants $\mathcal{K}_0$, $\tau_{\rm inj}$, $\mathfrak{I}_0$ defined in (2.8), (2.3), and (2.11) respectively.

Lemma 4.1. *There exist $\tau_0 = \tau_0(M,p,\tau_{\rm inj},\mathfrak{I}_0) > 0)$ and$R_0 = R_0(M,p,k,\mathcal{K}_0,\tau_{\rm inj},\mathfrak{I}_0) > 0$, such that the following holds.*

Let $0<\tau<\tau_0$, $0<\delta<\frac{1}{2}$ and $h^\delta \leq R(h) \leq R_0$. For $h>0$ let $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$ be a $(\mathfrak{D}_n,\tau,R(h))$ good cover of $\Sigma^H_{[a,b]}$. Let $\mathcal{V}\subset S_\delta(T^*M;[0,1])$ be bounded. Let $\psi\in C^\infty_0(\mathbb{R};[0,1])$ with $\psi(t)=1$ for $|t|\leq\frac{1}{4}$ and $\psi(t)=0$ for $|t|\geq 1$. Let $\ell\in\mathbb{R}$, $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ be bounded subsets of $\Psi_\delta(M)$ and $\Psi^\ell_\delta(M)$ respectively, and B_E be as in (4.4).

Then, there exist $C_0 = C_0(n,k,\mathfrak{I}_0,\mathcal{V},\mathcal{W},\widetilde{\mathcal{W}})$, $C>0$, and for all $K>0$ there is $h_0>0$, such that for all $N>0$ there exists $C_N>0$, with the following properties. For all $u\in\mathcal{D}'(M)$, $0<h<h_0$, $E\in[a-Kh,b+Kh]$, every δ-partition $\{\chi_{\mathcal{T}_j}\}_{j\in\mathcal{J}_E(h)}\subset\mathcal{V}$ associated to $\{\mathcal{T}_j\}_{j\in\mathcal{J}_E(h)}$,

and every $A \in \widetilde{\mathcal{W}}$ *such that* $B_E \frac{1}{h}[P, A] \in \mathcal{W}$,

$$h^{\frac{k-1}{2}} \Big| \int_H Au \, d\sigma_H \Big| \le C_0 R(h)^{\frac{n-1}{2}} \sum_{j \in \mathcal{I}_E(h)} \Big(\frac{\|Op_h(\tilde{\chi}_{\mathcal{T}_j}) u\|_{L^2(M)}}{\tau^{\frac{1}{2}}} + \frac{C}{h} \|Op_h(\tilde{\chi}_{\mathcal{T}_j}) P_E u\|_{L^2(M)} \Big) + Q^{A,\psi}_{E,h}(C, C_N, u). \tag{4.5}$$

Here, $\mathcal{I}_E(h) := \{j \in \mathcal{J}_E(h) : \mathcal{T}_j \cap \mathrm{MS_h}(A) \cap \Lambda^{\tau}_{\Sigma^H_E}(R(h)/2) \neq \emptyset\}$, $\psi \in S_\delta \cap C^\infty_c(T^*M; [0,1])$ *is any symbol with* $\operatorname{supp} \psi \subset \big(\Lambda^{\tau}_{\Sigma^H_E}(2h^\delta)\big)^c$, *and for each* $j \in \mathcal{J}_E(h)$ *we let* $\tilde{\chi}_{\mathcal{T}_j}$ *be any symbol in* $S_\delta(T^*M; [0,1]) \cap C^\infty_c(T^*M; [0,1])$ *such that* $\tilde{\chi}_{\mathcal{T}_j} \equiv 1$ *on* $\operatorname{supp} \chi_{\mathcal{T}_j}$ *and* $\operatorname{supp} \tilde{\chi}_{\mathcal{T}_j} \subset \mathcal{T}_j$. *In addition, if* $\widetilde{\mathcal{W}} \subset \Psi^\ell_0(M)$, *then* $C_0 = C_0(n, k, \mathfrak{I}_0, \mathcal{V}, \widetilde{\mathcal{W}})$.

Proof. First, we prove the statement for the case $A = I$. Note that in this case the sets $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ play no role. The result for $A = I$ is a direct combination of the estimate in [11, (3.16)] and [11, Proposition 3.2]. Indeed, [11, Proposition 3.2] yields the existence of $\tau_0, R_0, h_0 > 0$ as claimed, and the estimate [11, (3.16)] yields the same bound as above, but with three modifications.

First, the constant $C_0 = C_0(n, k, \mathfrak{I}_0) > 0$ is the constant $C_{n,k}$ divided by $\mathfrak{I}_0$, because we absorb the $|H_p r_H(\rho_j)|$ factors in [11, (3.16)]. Second, the estimate in [11, (3.16)] is given for for $\Big| \int_H Op_h(\beta_\delta) u \, d\sigma_H \Big|$, where $\beta_\delta : T^*H \to \mathbb{R}$ is a localizer to near conormal directions defined by $\beta_\delta(x', \xi') = \chi\big(h^{-\delta} |\xi'|_H\big)$ where $\chi \in C^\infty_0(\mathbb{R}; [0,1])$ is a smooth cut-off with $\chi(t) = 1$ for $t \le \frac{1}{2}$ and $\chi(t) = 0$ for $t \ge 1$. It turns out that this estimate is all we need since [11, Proposition 3.2] yields that for every $N > 0$ there exists $c_N > 0$ such that for all $u \in \mathcal{D}'(M)$

$$\Big| \int_H (1 - Op_h(\beta_\delta)) u \, d\sigma_H \Big| \le c_N h^N \Big(\|u\|_{L^2(H)} + \|P_E u\|_{H_{\rm scl}^{\frac{k-2m+1}{2}}(M)} \Big). \tag{4.6}$$

The third modification is that in [11, (3.16)] the first error term is $Ch^{-\frac{1}{2}-\delta} \big\| P_E u \big\|_{H_{\rm scl}^{\frac{k-2m+1}{2}}(M)}$ instead of $Ch^{-\frac{1}{2}-\delta} \big\| \big(1 - \psi\big(\frac{P_E}{h^\delta}\big)\big) P_E u \big\|_{H_{\rm scl}^{\frac{k-2m+1}{2}}(M)}$. The operator $\big(1 - \psi\big(\frac{P_E}{h^\delta}\big)\big)$ can be added since the error term is a consequence of the bound in [11, (3.10)], and that bound is for $Op_h(\chi) u$ where χ is supported in $\{(x, \xi) : \; |p_E(x, \xi)| \ge \frac{1}{3} h^\delta\}$. One then uses $\operatorname{supp} \chi \subset \operatorname{supp} \big(1 - \psi\big(\frac{p_E}{h^\delta}\big)\big)$.

We note that the desired bound holds for every δ-partition $\{\chi_{\mathcal{T}_j}\}_{j \in \mathcal{J}_E(h)} \subset \mathcal{V}$ associated to $\{\mathcal{T}_j\}_{j \in \mathcal{J}_E(h)}$, since the constants C, C_N, h_0 provided by [11, Proposition 3.5] are uniform for $\chi_{\mathcal{T}_j}$ in bounded subsets of S_δ.

Given $\varepsilon_0 > 0$ we note that the statement holds for every $E \in [a - \varepsilon_0, b + \varepsilon_0]$ since the constants C, C_N, h_0 provided by [11, Proposition 3.5] depend on P_E only through P. Therefore, given $K > 0$, the statement for $A = I$ holds for $E \in [a - Kh, b + Kh]$ provided h_0 depends on K.

We now treat the case $A \neq I$. Let $\mathcal{V}, \mathcal{W}, \widetilde{\mathcal{W}}$, and $\{B_E\}_{E \in [a-\varepsilon_0, b+\varepsilon_0]}$ be as in the assumptions. Let $E \in [a - \varepsilon_0, b + \varepsilon_0]$. Let $X \in \Psi_\delta(M)$ with $\mathrm{MS_h}(I - X) \cap \Lambda^{\tau}_{\Sigma^H_E}(\frac{1}{3} R(h)) = \emptyset$, $\mathrm{MS_h}(X) \subset \Lambda^{\tau_0 + \varepsilon_0}_{\Sigma^H_E}(\frac{1}{2} R(h))$

and $B_E[P,X]\in\Psi_\delta(M)$. Then, for all $N>0$ there is $C_N>0$ depending on $\mathcal{V}$

$$\Big|\int_H (I-X)Aud\sigma_H\Big|\leq C_N h^N,$$

so we may replace A by XA and assume $\mathrm{MS_h}(A)\subset\Lambda^{\tau_0+\varepsilon_0}_{\Sigma_E}(R(h)/2)$ from now on. Since the estimate holds when $A=I$, there exist $C_0=C_0(n,k,\mathfrak{I}_0)$, $C>0$, and for all $K>0$ there is $h_0>0$ such that for all $N>0$ there exists $C_N>0$ with the following properties. For all $u\in\mathcal{D}'(M)$, $0<h<h_0$, $E\in[a-Kh,b+Kh]$, and every δ-partition $\{\chi_{\mathcal{T}_j}\}_{j\in\mathcal{J}_E(h)}\subset\mathcal{V}$ associated to $\{\mathcal{T}_j\}_{j\in\mathcal{J}_E(h)}$, the bound in (4.5) holds with I in place of A, and with Au in place of u:

$$h^{\frac{k-1}{2}}\Big|\int_H Au\,d\sigma_H\Big|\leq C_0R(h)^{\frac{n-1}{2}}\sum_{j\in\mathcal{I}_E(h)}\Big(\frac{\|Op_h(\tilde\chi_{\mathcal{T}_j})Au\|}{\tau^{\frac12}}+Ch^{-1}\|Op_h(\tilde\chi_{\mathcal{T}_j})P_EAu\|\Big)$$
$$+Q^{I,\psi}_{E,h}(C,C_N,Au).$$

We may sum over $j\in\mathcal{I}_E(h)$ instead of $j\in\mathcal{J}_E(h)$ since $\mathrm{MS_h}(A)\cap\Lambda^\tau_{\Sigma^H_E}(\frac12R(h))\subset\cup_{j\in\mathcal{I}_E(h)}\mathcal{T}_j$.

Next, we explain how to write u in place of Au in each of the terms of the sum over $j\in\mathcal{I}_E(h)$ in (4.5). To replace the term $\|Op_h(\chi_{\mathcal{T}_j})Au\|_{L^2(M)}$ with $\|Op_h(\tilde\chi_{\mathcal{T}_j})u\|_{L^2(M)}$, we use $\mathrm{MS_h}(Op_h(\chi_{\mathcal{T}_j})A)\subset\mathrm{Ell}(Op_h(\tilde\chi_{\mathcal{T}_j}))$ and apply the elliptic parametrix construction to find $F_1\in\Psi_\delta(M)$ with

$$Op_h(\chi_{\mathcal{T}_j})A=F_1Op_h(\tilde\chi_{\mathcal{T}_j}). \tag{4.7}$$

Next, to replace the term $\|Op_h(\chi_{\mathcal{T}_j})P_EAu\|_{L^2(M)}$ with $\|Op_h(\tilde\chi_{\mathcal{T}_j})P_Eu\|_{L^2(M)}$, we decompose

$$Op_h(\chi_{\mathcal{T}_j})P_EA=Op_h(\chi_{\mathcal{T}_j})[P_E,A]+Op_h(\chi_{\mathcal{T}_j})AP_E$$

for each $j\in\mathcal{I}_E(h)$, and apply the elliptic parametrix construction and find $F_2\in\Psi_\delta(M)$ with

$$h^{-1}Op_h(\chi_{\mathcal{T}_j})[P_E,A]=F_2Op_h(\tilde\chi_{\mathcal{T}_j}). \tag{4.8}$$

To do this we used the assumptions: B_E is microlocally the identity on $\Lambda^{\tau_0+\varepsilon_0}_{\Sigma^H_E}(2R(h))$, $\mathrm{MS_h}(A)\subset\Lambda^{\tau_0+\varepsilon_0}_{\Sigma^H_E}(\frac12R(h))$, and A is such that $B_E\frac1h[P,A]\in\mathcal{W}\subset\Psi_\delta(M)$. This allows us to apply the parametrix construction to $Op_h(\chi_{\mathcal{T}_j})B_E\frac1h[P_E,A]$.

Using (4.7) and (4.8), we may modify C_0, and having it now also depend on A, $\mathcal{V}$ and $\mathcal{W}$, to obtain the claim. Note that if $A\in\Psi_0^\infty(M)$, then $\frac1h[P_E,A]\in\Psi^\infty_\delta(M)$ and so we may apply the elliptic parametrix construction to obtain (4.8) without the need of introducing the operator B_E or the set $\mathcal{W}$. In this case, we have $C_0=C_0(n,k,\mathfrak{I}_0,\mathcal{V},\widetilde{\mathcal{W}})$ as claimed. □

Definition 4.2 (low density tubes)**.** Let $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$ be a cover by tubes of $\Sigma^H_{[a,b]}$ and $0<\delta<\frac12$. Let $\mathcal{G}(h)\subset\mathcal{J}(h)$ and for each $j\in\mathcal{G}(h)$ let $1<t_j(E,h)\leq T_j(E,h)$, where $h>0$ and $E\in\mathbb{R}$.

We say $\{\mathcal{T}_j\}_{j\in\mathcal{G}(h)}$ *has* $\{(t_j,T_j)\}_{j\in\mathcal{G}(h)}$ *density on* $[a,b]$ if the following holds. For all $\mathcal{V}\subset S_\delta$ bounded, $K>0$ there is $h_0>0$ such that for all $0<h<h_0$, $E\in[a-Kh,b+Kh]$, every

δ-partition $\{\chi_j\}_{j\in\mathcal{G}_E(h)}\subset\mathcal{V}$ associated to $\{\mathcal{T}_j\}_{j\in\mathcal{G}_E(h)}$, and all $u\in\mathcal{D}'(M)$,

$$\sum_{j\in\mathcal{G}_E(h)}\|Op_h(\chi_j)u\|^2_{L^2(M)}\frac{T_j(E,h)}{t_j(E,h)}\leq 4\|u\|^2_{L^2(M)}+4\max_{j\in\mathcal{G}_E(h)}\frac{T_j(E,h)^2}{h^2}\|P_Eu\|^2_{L^2(M)},$$

where $\mathcal{G}_E(h)=\mathcal{G}(h)\cap\mathcal{J}_E(h)$.

As a consequence of [11, Lemma 4.1] one has: if a collection of families of tubes is non self-looping for different times, then the tubes have a low density dictated by those times.

Lemma 4.3. *Let R_0, τ_0, δ, $R(h)$, τ, and $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$ be as in Lemma 4.1. Let $0<\alpha<1-\limsup_{h\to0^+}2\frac{\log R(h)}{\log h}$ and $K>0$. There exists $h_0>0$ such that the following holds. Let $0<h<h_0$, $E\in[a-Kh,b+Kh]$, and $\mathcal{G}_E(h)\subset\mathcal{J}_E(h)$ with $\mathcal{G}_E(h)=\sqcup_{\ell\in\mathcal{L}_E(h)}\mathcal{G}_{E,\ell}(h)$. For every $\ell\in\mathcal{L}_E(h)$ suppose $t_\ell(E,h)>0$, $0<T_\ell(E,h)\leq 2\alpha T_e(h)$, and*

$$\bigcup_{j\in\mathcal{G}_{E,\ell}(h)}\mathcal{T}_j\qquad\textit{is }[t_\ell,T_\ell]\quad\textit{non-self looping for every }\ \ell\in\mathcal{L}_E(h).$$

Then, $\{\mathcal{T}_j\}_{j\in\mathcal{G}(h)}$ has $\{(t_j,T_j)\}_{j\in\mathcal{G}(h)}$ density on $[a,b]$, where for $0<h<h_0$, $j\in\mathcal{J}(h)$, and $E\in[a-Kh,b+Kh]$, we set $(t_j(E,h),T_j(E,h)):=(t_\ell(E,h),T_\ell(E,h))$ whenever $j\in\mathcal{G}_{E,\ell}(h)$.

We note that the statement of [11, Lemma 4.1] does not provide the requisite uniformity for $E\in[a-Kh,b+Kh]$; however, this follows from the same argument.

Our next estimate shows that if a family of tubes has low density, then averages of a quasimode over H can be controlled in terms of the density times.

Lemma 4.4. *Let R_0, τ_0, δ, $R(h)$, τ, $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$, $\mathcal{W}$, $\widetilde{\mathcal{W}}$, and ψ be as in Lemma 4.1. Then, there exist $C_0=C_0(n,k,p,\mathfrak{I}_0,\mathcal{W})$ and $C>0$, and for all $N>0$, $K>0$ there are $h_0>0$ and $C_N>0$, such that the following holds.*

Suppose that for all $0<h<h_0$ and $E\in[a-Kh,b+Kh]$ there exists $\mathcal{G}_E(h)\subset\mathcal{J}_E(h)$ with $\mathcal{G}_E(h)=\sqcup_{\ell\in\mathcal{L}_E(h)}\mathcal{G}_{E,\ell}(h)$, such that for every $\ell\in\mathcal{L}_E(h)$ there exist $t_\ell=t_\ell(E,h)>0$ and $T_\ell=T_\ell(E,h)>0$ so that, with $(t_j,T_j):=(t_\ell,T_\ell)$ for every $j\in\mathcal{G}_{E,\ell}(h)$, then

$$(1)\{\mathcal{T}_j\}_{j\in\mathcal{G}(h)}\textit{ has }\{(t_j,T_j)\}_{j\in\mathcal{G}(h)}\textit{ density on }[a,b],\qquad (2)\,\mathrm{MS_h}(A)\cap\Lambda^\tau_{\Sigma^H_E}(\tfrac12R(h))\subset\bigcup_{j\in\mathcal{G}_E(h)}\mathcal{T}_j.$$

Then, for all $u\in\mathcal{D}'(M)$, $0<h<h_0$, $E\in[a-Kh,b+Kh]$, and every $A\in\widetilde{\mathcal{W}}$ with $B_E\frac1h[P,A]\in\mathcal{W}$,

$$h^{\frac{k-1}{2}}\Big|\int_H Au\,d\sigma_H\Big|\leq C_0R(h)^{\frac{n-1}{2}}\sum_{\ell\in\mathcal{L}_E(h)}\Big(\frac{(|\mathcal{G}_{E,\ell}|t_\ell)^{\frac12}}{\tau^{\frac12}T_\ell^{\frac12}}\|u\|_{L^2(M)}+\frac{(|\mathcal{G}_{E,\ell}|t_\ell T_\ell)^{\frac12}}{h}\|P_Eu\|_{L^2(M)}\Big)$$
$$+\,Q^{A,\psi}_{E,h}(C,C_N,u).$$

In addition, if $\widetilde{\mathcal{W}}\subset\Psi^\infty_0(M)$, the estimate holds with $C_0=C_0(n,k,p,\mathfrak{I}_0,\widetilde{\mathcal{W}})$.

Proof. Let $\mathcal{V}$ a bounded subset of $S_\delta(T^*M;[0,1])$. By Lemma 4.1 there exist $C_0=C_0(n,k,\mathfrak{I}_0,\mathcal{V},\mathcal{W})$, $C>0$, and $h_0>0$, such that for all $N>0$ there exist $C_N>0$, with the following properties. For all $u \in \mathcal{D}'(M)$, $K > 0$, $0 < h < h_0$, $E \in [a - Kh, b + Kh]$, and every δ-partition $\{\chi_{\mathcal{T}_j}\}_{j \in \mathcal{J}_E(h)} \subset \mathcal{V}$ associated to $\{\mathcal{T}_j\}_{j \in \mathcal{J}_E(h)}$,

$$h^{\frac{k-1}{2}} \Big| \int_H A u ds H \Big| \leq C_0 R(h)^{\frac{n-1}{2}} \sum_{j \in \mathcal{I}_E(h)} \Big(\frac{\|Op_h(\tilde{\chi}_{\mathcal{T}_j}) u\|_{L^2}}{\tau^{\frac{1}{2}}} + \frac{C}{h} \|Op_h(\tilde{\chi}_{\mathcal{T}_j}) P_E u\|_{L^2} \Big) + Q^{A,\psi}_{E,h}(C, C_N, u),$$

where $\mathcal{I}_E(h) := \bigcup_{\ell \in \mathcal{L}_{h,E}} \mathcal{G}_{E,\ell}$. Note that if $A \in \Psi^\infty_0(M)$, then the estimate holds with $C_0 = C_0(n, k, p, \mathfrak{I}_0, \mathcal{V}, \widetilde{\mathcal{W}})$. Next, note that

$$\sum_{j \in \mathcal{I}_E(h)} \|Op_h(\tilde{\chi}_{\mathcal{T}_j}) P_E u\| \leq |\mathcal{J}_E(h)|^{\frac{1}{2}} \Big(\sum_{j \in \mathcal{J}_E(h)} \|Op_h(\tilde{\chi}_{\mathcal{T}_j}) P_E u\|^2 \Big)^{\frac{1}{2}},$$

and so, since $|\mathcal{J}_E(h)| \leq C_n \operatorname{vol}(\Sigma^H_E) R(h)^{1-n}$ for some $C_n > 0$, we have, after adjusting $C > 0$, that for all $0 < h < h_0$

$$h^{\frac{k-1}{2}} \Big| \int_H A u \, d\sigma_H \Big| \leq C_0 \frac{R(h)^{\frac{n-1}{2}}}{\tau^{\frac{1}{2}}} \sum_{j \in \mathcal{I}_E(h)} \|Op_h(\tilde{\chi}_{\mathcal{T}_j}) u\|_{L^2(M)} + \frac{C}{h} \|P_E u\|_{L^2(M)} + Q^{A,\psi}_{E,h}(C, C_N, u). \quad (4.9)$$

Since we are working with a $(\mathfrak{D}_n, \tau, R(h))$-good cover, we split each $\mathcal{G}_{E,\ell}$ into $\mathfrak{D}_n$ families $\{\mathcal{G}_{E,\ell,i}\}_{i=1}^{\mathfrak{D}_n}$ of disjoint tubes. Note that

$$\sum_{j \in \mathcal{I}_E(h)} \|Op_h(\tilde{\chi}_j) u\|_{L^2(M)} \leq \sum_{\ell \in \mathcal{L}} \sum_{i=1}^{\mathfrak{D}_n} \sum_{j \in \mathcal{G}_{E,\ell,i}} \|Op_h(\tilde{\chi}_j) u\|_{L^2(M)}. \quad (4.10)$$

Next, since $\{\mathcal{T}_j\}_{j \in \mathcal{G}(h)}$ has $\{(t_j, T_j)\}_{j \in \mathcal{G}(h)}$ density on $[a, b]$, after possibly shrinking h_0 (depending on the S_δ bounds for $\tilde{\chi}_j$ and $K > 0$), Cauchy-Schwarz yields that for all $0 < h < h_0$

$$\sum_{j \in \mathcal{G}_{E,\ell,i}} \|Op_h(\tilde{\chi}_j) u\|_{L^2(M)} \leq 2 \Big(\frac{t_\ell |\mathcal{G}_{E,\ell}|}{T_\ell} \Big)^{\frac{1}{2}} \Big(\|u\|^2_{L^2(M)} + \frac{T_\ell^2}{h^2} \|P_E u\|^2_{L^2(M)} \Big)^{\frac{1}{2}}. \quad (4.11)$$

The result follows from combining (4.11) and (4.10), and feeding this to (4.9). Note that C_0 needs to be modified, but only in a way that depends on n via $\mathfrak{D}_n$. □

We also need the following basic estimate for averages over submanifolds to control averages of $u = \mathbf{1}_{(-\infty, s]}(P)$ when s is large.

Lemma 4.5. *Suppose $H \subset M$ is a submanifold of codimension k and $P \in \Psi^m(M)$, with $m > 0$, is such that there exists $C > 0$ for which*

$$|\sigma(P)(x, \xi)| \geq |\xi|^m / C, \qquad (x, \xi) \in N^*H, \qquad |\xi| \geq C.$$

Let $\psi \in S^0(T^*M; [0, 1])$ with $\psi \equiv 1$ on N^*H, and let $\ell \in \mathbb{R}$. Let $A \in \Psi^\ell_\delta(M)$ and $r > \frac{k+2\ell}{2m}$. Then, there are $C_0 > 0$ and $h_0 > 0$ such that for all $N > 0$ there is $C_N > 0$ satisfying

$$h^{\frac{k}{2}} \Big| \int_H A u d\sigma_H \Big| \leq C_0 \Big(\|Op_h(\psi) u\|_{L^2(M)} + \|Op_h(\psi) P_E^r u\|_{L^2(M)} \Big) + C_N h^N \|u\|_{H^{-N}_{\rm scl}(M)}, \qquad 0 < h < h_0.$$

Proof. Let $\tilde\psi \in S^0(T^*M;[0,1])$ with $\tilde\psi \equiv 1$ on N^*H, $\operatorname{supp}\tilde\psi \subset \{\psi \equiv 1\}$, and such that

$$|\sigma(P_E)(x,\xi)| \geq \tfrac{1}{C}|\xi|^m, \qquad (x,\xi) \in \operatorname{supp}\tilde\psi, \qquad |\xi| \geq C.$$

Then, since $\mathrm{WF_h}(\delta_H) = N^*H$, for any $N > 0$ there is $C_N > 0$ such that

$$\Big|\int_H AOp_h(1-\tilde\psi)u d\sigma_H\Big| \leq C_N h^N \|u\|_{H^{-N}_{\mathrm{scl}}(M)}. \tag{4.12}$$

Next, by the Sobolev embedding theorem, for any $\varepsilon > 0$ there exists $C_0 > 0$ such that

$$\Big|\int_H AOp_h(\tilde\psi)u d\sigma_H\Big| \leq C_0 h^{-\frac{k}{2}} \|Op_h(\tilde\psi)u\|_{H^{\frac{k}{2}+\varepsilon+\ell}_{\mathrm{scl}}(M)}.$$

Taking r with $rm > \frac{k}{2} + \ell$ and using an elliptic parametrix, for any $N > 0$ there is $C_N > 0$ with

$$\begin{aligned} h^{\frac{k}{2}}\Big|\int_H AOp_h(\psi)u d\sigma_H\Big| \leq C_0\|Op_h(\tilde\psi)u\|_{H^{rm}_{\mathrm{scl}}(M)} &\leq C_0\big(\|Op_h(\psi)u\|_{L^2(M)} + \|Op_h(\psi)P_E^r u\|_{L^2(M)}\big) \\ &\quad + C_N h^N \|u\|_{H^{-N}_{\mathrm{scl}}(M)}. \end{aligned} \tag{4.13}$$

Indeed, this follows from letting $\chi \in S^0(T^*M;[0,1])$ so that $|\sigma(P_E)(x,\xi)| \geq \frac{1}{C}|\xi|^m$ in the support of $\tilde\psi(1-\chi)$, and then using the elliptic parametrix construction to find $F_1, F_2 \in \Psi^0(M)$ such that

$$\langle hD\rangle^{rm} Op_h(\tilde\psi)(1 - Op_h(\chi)) = F_1 Op_h(\psi) P_E^r + O(h^\infty)_{\Psi^{-\infty}},$$
$$\langle hD\rangle^{rm} Op_h(\tilde\psi) Op_h(\chi) = F_2 Op_h(\psi) + O(h^\infty)_{\Psi^{-\infty}}.$$

Combining with (4.12) and (4.13) completes the proof. □

5. Lipschitz Scale for Spectral Projectors

In this section we estimate the scale at which averages of the spectral projector behave like Lipschitz functions of the spectral parameter, and use this to approximate Π_h using $\rho_{h,T(h)} * \Pi_h$.

Throughout this section we assume $H_1, H_2 \subset M$ are two smooth submanifolds of co-dimension k_1 and k_2 respectively. The goal for this section is to prove the following proposition.

Proposition 5.1. *Suppose $a, b \in \mathbb{R}$ such that H_1, H_2 are uniformly conormally transverse for p in the window $[a,b]$. Let τ_0, R_0 be as in Lemma 4.1. Let $0 < \tau < \tau_0$ and $0 < \delta < \frac{1}{2}$. For $i = 1,2$, let $\mathbf{T}_i$ be sub-logarithmic resolution functions with $\Omega(\mathbf{T}_i)\Lambda < 1 - 2\delta$ and suppose H_i is $\mathbf{T}_i$ non-recurrent in the window $[a,b]$ via τ-coverings with constant C^i_{nr}.*

Let $A_1, A_2 \in \Psi^\infty(M)$, $K > 0$, $R(h) \geq h^\delta$, and $\mathbf{T} := \sqrt{\mathbf{T}_1\mathbf{T}_2}$. Then, there exist $h_0 > 0$ and

$$C_0 = C_0(n, k_1, k_2, \mathfrak{I}^1_0, \mathfrak{I}^2_0, A_1, A_2, C^1_{\mathrm{nr}}, C^2_{\mathrm{nr}}) > 0,$$

such that for all $0 < h \leq h_0$ and $E \in [a - Kh, b + Kh]$,

$$\Big|\Pi^{A_1,A_2}_{H_1,H_2}(E) - \rho_{h,T_{\max}(h)} * \Pi^{A_1,A_2}_{H_1,H_2}(E)\Big| \leq C_0 h^{\frac{2-k_1-k_2}{2}} \Big/ \mathbf{T}(R(h)).$$

Remark 5.2. To ease notation, throughout this section we write $T_i(h) := \mathbf{T}_i(R(h))$, $T(h) := \mathbf{T}(R(h))$, and $T_{\max}(h) := \max(\mathbf{T}_1(R(h), \mathbf{T}_2(R(h))))$.

Proof. We split the proof into Lemmas 5.3, 5.4, and 5.5 below. Lemmas 5.4 and 5.5 show that there exist $C_0 = C_0(n, k_1, k_2, \mathfrak{I}_0^1, \mathfrak{I}_0^2, A_1, A_2, C_{\mathrm{nr}}^1, C_{\mathrm{nr}}^2) > 0$, $C_1 > 0$, and $h_0 > 0$ such that $w_h(E) := \Pi_{H_1,H_2}^{A_1,A_2}(E)$ satisfies the hypotheses of Lemma 5.3 with $I_h := [a - Kh, b + Kh]$, $\rho_h := \rho_{h,T_{\max}(h)}$, $\sigma_h := T_{\max}(h)/h$,

$$L_h := C_0 h^{\frac{2-k_1-k_2}{2}} \Big/ T(h) \qquad \text{and} \qquad B_h := C_1 h^{-\frac{k_1+k_2}{2}},$$

and $0 < h < h_0$. Next, let $\{K_j\}_{j=1}^{\infty} \subset \mathbb{R}_+$ be given by the choice of ρ in (1.16). Since $\left\langle \frac{T_1(h)s}{h} \right\rangle^{\frac{1}{2}} \left\langle \frac{T_2(h)s}{h} \right\rangle^{\frac{1}{2}} \leq \langle \sigma_h s \rangle$ for all $s \in \mathbb{R}$, Lemma 5.3 yields that there exists $C_\rho > 0$ and for all $N > 0$ there exists $C_N > 0$ such that

$$\left| \Pi_{H_1,H_2}^{A_1,A_2}(E) - \rho_{h,T(h)} * \Pi_{H_1,H_2}^{A_1,A_2}(E) \right| \leq C_\rho C_0 \frac{h^{\frac{2-k_1-k_2}{2}}}{T(h)} + C_N C_1 h^{-\frac{k_1+k_2}{2}} \Big(\frac{h}{T_{\max}(h)} \Big)^N,$$

for all $0 < h < h_0$. This completes the proof after choosing h_0 small enough. □

We now present the lemmas used in the proof of Proposition 5.1. The first shows that if a family of functions $\{w_h\}_h$ is Lipstchitz at scale σ_h^{-1} with (at most) polynomial growth at infinity, then the family can be well approximated by its convolution $\rho_h * w_h$ where $\{\rho_h\}_h$ is a family of Schwartz functions

Lemma 5.3. *Let $\{K_j\}_{j=0}^{\infty} \subset \mathbb{R}_+$. Then, there exists $C > 0$ and for all $N_0 \in \mathbb{R}$, $N > 0$ there exists $C_N > 0$, such that the following holds. Let $\{\rho_h\}_{h>0} \subset \mathcal{S}(\mathbb{R})$ be a family of functions and $\{\sigma_h\}_{h>0} \subset \mathbb{R}_+$ such that for all $j \geq 1$ and $h > 0$,*

$$|\rho_h(s)| \leq \sigma_h K_j \langle \sigma_h s \rangle^{-j} \qquad \text{for all } s \in \mathbb{R}.$$

Let $\{L_h\}_{h>0} \subset \mathbb{R}_+$, $\{B_h\}_{h>0} \subset \mathbb{R}_+$, $\{w_h : \mathbb{R} \to \mathbb{R}\}_{h>0}$, $I_h \subset [-K_0, K_0]$, $h_0 > 0$ and $\varepsilon_0 > 0$, be so that for all $0 < h < h_0$

- *$|w_h(t - s) - w_h(t)| \leq L_h \langle \sigma_h s \rangle$ for all $t \in I_h$ and $|s| \leq \varepsilon_0$,*
- *$|w_h(s)| \leq B_h \langle s \rangle^{N_0}$ for all $s \in \mathbb{R}$.*

Then, for all $0 < h < h_0$ and $t \in I_h$

$$\left| (\rho_h * w_h)(t) - w_h(t) \int_{\mathbb{R}} \rho_h(s) ds \right| \leq C L_h + C_N B_h \sigma_h^{-N} \varepsilon_0^{-N}.$$

Proof. For all $0 < h < h_0$ and $t \in I_h$

$$\begin{aligned}
&\left| (\rho_h * w_h)(t) - w_h(t) \int_{\mathbb{R}} \rho_h(s) ds \right| = \left| \int_{\mathbb{R}} \rho_h(s) \big(w_h(t-s) - w_h(t) \big) ds \right| \\
&\leq L_h \int_{|s| \leq \varepsilon_0} |\rho_h(s)| \langle \sigma_h s \rangle ds + B_h \int_{|s| \geq \varepsilon_0} |\rho_h(s)| \Big(\langle t - s \rangle^{N_0} + \langle t \rangle^{N_0} \Big) ds \\
&\leq L_h \int_{|s| \leq \varepsilon_0} \sigma_h K_3 \langle \sigma_h s \rangle^{-2} ds + B_h \int_{|s| \geq \varepsilon_0} K_{N_0+2+N} \sigma_h \langle \sigma_h s \rangle^{-(N_0+2+N)} \Big(\langle t - s \rangle^{N_0} + \langle t \rangle^{N_0} \Big) ds.
\end{aligned}$$

The existence of C and C_N follows from integrability of each term and the boundedness of I_h. □

The next lemma shows that the family of functions $w_h(t) = \Pi^{A_1,A_2}_{H_1,H_2}(t)$ is Lipschitz at scales dictated by the non-recurrence times for H_1 and H_2.

Lemma 5.4. *Suppose $a, b \in \mathbb{R}$, $\varepsilon_0 > 0$ are such that H_1, H_2 are conormally transverse for p in the window $[a-\varepsilon_0, b+\varepsilon_0]$. Let A_1, A_2, τ_0, R_0, τ, δ, $R(h)$, and α be as in Proposition 5.1. Let $C_{\rm nr} >$ and $K > 0$. Then, there exist $h_0 > 0$ and*

$$C_0 = C_0(n, k_1, k_2, \mathfrak{I}^1_0, \mathfrak{I}^2_0, A_1, A_2, C_{\rm nr}) > 0$$

such that the following holds.

For $i = 1, 2$, let $\mathbf{T}_i$ be a sub-logarithmic resolution function with $\Omega(\mathbf{T}_i)\Lambda < 1-2\delta$. Suppose H_i is $\mathbf{T}_i$ non-recurrent in the window $[a, b]$ via τ-coverings with constant $C^i_{\rm nr} \le C_{\rm nr}$. Then for all $0 < h \le h_0$, $|s| \le \varepsilon_0$, and $t \in [a-Kh, b+Kh]$,

$$\left|\Pi^{A_1,A_2}_{H_1,H_2}(t) - \Pi^{A_1,A_2}_{H_1,H_2}(t-s)\right| \le C_0 \frac{h^{\frac{2-k_1-k_2}{2}}}{\sqrt{T_1(h)T_2(h)}} \Big\langle \frac{T_1(h)s}{h} \Big\rangle^{\frac12} \Big\langle \frac{T_2(h)s}{h} \Big\rangle^{\frac12}.$$

Proof. We first assume the statement for $|s| \le 2h$. Suppose $s \ge 2h$. The case of $s \le -2h$ being similar. Define $k_0 := \lfloor \frac{s}{h} \rfloor$ and $t_k := t - s + kh$ for $0 \le k \le k_0 - 1$, and $t_k := t$ for $k = k_0$. Then

$$\Pi^{A_1,A_2}_{H_1,H_2}(t) - \Pi^{A_1,A_2}_{H_1,H_2}(t-s) = \sum_{k=0}^{k_0-1} \Pi^{A_1,A_2}_{H_1,H_2}(t_{k+1}) - \Pi^{A_1,A_2}_{H_1,H_2}(t_k).$$

Using $|t_{k+1} - t_k| \le 2h$, and putting $t = t_{k+1}$, $s = t_{k+1} - t_k$, we apply the case $|s| \le 2h$ with $T_1 = T_2 = 1$ for each term to obtain

$$\left|\Pi^{A_1,A_2}_{H_1,H_2}(t) - \Pi^{A_1,A_2}_{H_1,H_2}(t-s)\right| \le C_0 k_0 h^{\frac{2-k_1-k_2}{2}} \le C_0 h^{\frac{2-k_1-k_2}{2}} |s/h|,$$

and this proves the claim provided the statement holds for $|s| \le 2h$.

We proceed to prove the statement for $|s| \le 2h$. First, note that by (1.10) and Cauchy-Schwarz

$$\left|\Pi^{A_1,A_2}_{H_1,H_2}(t) - \Pi^{A_1,A_2}_{H_1,H_2}(t-s)\right|^2 \le \sum_{t-s \le E_k \le t} \left|\int_{H_1} A_1 \phi_{E_k} d\sigma_{H_1}\right|^2 \cdot \sum_{t-s \le E_j \le t} \left|\int_{H_2} A_2 \phi_{E_j} d\sigma_{H_2}\right|^2. \quad (5.1)$$

Now, for each $i = 1, 2$,

$$\sum_{t-s \le E_j \le t} \left|\int_{H_i} A_i \phi_{E_j} d\sigma_{H_i}\right|^2 = \|\mathbb{1}_{[t-s,t]}(P)\, A_i^*\, \delta_{H_i}\|^2_{L^2(M)} = \sup_{\|w\|_{L^2(M)}=1} \left|\int_{H_i} A_i \mathbb{1}_{[t-s,t]}(P) w \, d\sigma_{H_i}\right|^2, \quad (5.2)$$

where δ_{H_i} is the delta distribution at H_i and the last equality follows by duality.

We now use the non-recurrence assumption on H_1 and H_2. Since for each $i = 1, 2$, the submanifold H_i is $\mathbf{T}_i$ non-recurrent in the window $[a, b]$ via τ_0-coverings, there is $h_0 > 0$ small enough depending on $R(h), K$ so that for all $0 < h < h_0$ and $t \in [E - Kh, E + Kh]$ there is a partition of indices $\mathcal{J}^i_t(h) = \cup_{\ell \in \mathbf{L}^i_t(h)} \mathcal{G}^i_{t,\ell}(h)$, and times $\{T^i_\ell(h)\}_{\ell \in \mathbf{L}^i_t(h)}$, and $\{t^i_\ell(h)\}_{\ell \in \mathbf{L}^i_t(h)}$ as in Definition 2.2.

Note that we have chosen h_0 small enough so that $\mathcal{J}^i_E(h)$ is a $(\tau, R(h))$ good covering of $\Sigma_t^{H_i}$ for $t \in [E-Kh, E+Kh]$. In particular, for $i=1,2$ and $t \in [E-Kh,\, E+Kh]$

$$R(h)^{\frac{n-1}{2}} \sum_{\ell \in \mathcal{L}^i_E(h)} \frac{(|\mathcal{G}^i_{t,\ell}| t^i_\ell)^{\frac12}}{(T^i_\ell)^{\frac12}} \le \frac{C^i_{\rm nr}}{T_i^{\frac12}}, \qquad R(h)^{\frac{n-1}{2}} \sum_{\ell \in \mathcal{L}^i_E(h)} (|\mathcal{G}^i_{t,\ell}| t^i_\ell)^{\frac12} (T^i_\ell)^{\frac12} \le C^i_{\rm nr} T_i^{\frac12}. \tag{5.3}$$

The first bound is condition (2) in Definition 2.2, and the second bound follows from the first one together with the $T^i_\ell \le T_i$ for all $\ell \in \mathcal{L}^i_{h,E}$. Next, for $\ell \in \mathcal{L}^i_E$ let

$$\tilde{T}^i_\ell(h) := \begin{cases} T^i_\ell(h) \big\langle \frac{T_i(h)s}{h} \big\rangle^{-1} & t^i_\ell \le T^i_\ell \big\langle \frac{T_i(h)s}{h}\big\rangle^{-1} \\ 1 & \text{else} \end{cases}, \quad \tilde{t}^i_\ell(h) := \begin{cases} t^i_\ell(h) & t^i_\ell \le T^i_\ell \big\langle \frac{T_i(h)s}{h}\big\rangle^{-1} \\ 1 & \text{else} \end{cases} \tag{5.4}$$

and note that $\sum_{\tilde{t}^i_\ell = \tilde{T}^i_\ell = 1} |\mathcal{G}^i_{t,\ell}|^{\frac12} \le C^i_{\rm nr} \sqrt{\frac{1}{T_i}\Big\langle \frac{T_i s}{h}\Big\rangle}$. In particular,

$$\sum_{\ell \in \mathcal{L}^i_E(h)} \frac{(|\mathcal{G}^i_{t,\ell}| \tilde{t}^i_\ell)^{\frac12}}{(\tilde{T}^i_\ell)^{\frac12}} \le 2C^i_{\rm nr} \sqrt{\frac{1}{T_i}\Big\langle \frac{T_i s}{h}\Big\rangle}, \qquad \sum_{\ell \in \mathcal{L}^i_E(h)} \sqrt{|\mathcal{G}^i_{t,\ell}| \tilde{t}^i_\ell \tilde{T}^i_\ell} \le 2C^i_{\rm nr} \Big(\frac{1}{T_i}\Big\langle \frac{T_i s}{h}\Big\rangle\Big)^{-\frac12}. \tag{5.5}$$

Then, since for each $\ell \in \mathcal{L}^i_E(h)$ the union of tubes with indices in $\mathcal{G}^i_{E,\ell}$ is also $[\tilde{t}^i_\ell(h), \tilde{T}^i_\ell(h)]$ non-self looping, we may apply Lemma 4.3 with the sets $\{\mathcal{G}^i_{t,\ell}(h)\}_{\ell \in \mathcal{L}^i_E(h)}$, $\{\tilde{T}^i_\ell(h)\}_{\ell \in \mathcal{L}^i_E(h)}$, $\{t^i_\ell(h)\}_{\ell \in \mathcal{L}^i_E(h)}$ to see that $\{\mathcal{T}_j\}_{j \in \mathcal{G}^i_{t,\ell}(h)}$ has $\{(t_j, T_j)\}$ density on $[a,b]$ where $t_j = \tilde{t}^i_j(h)$, $T_j = \tilde{T}^i_j(h)$. Then, using Lemma 4.4 with operators $A_i \in \Psi^\infty(M)$, $\psi \in C_0^\infty(\mathbb{R};[0,1])$ with $\psi(t) = 1$ for $|t| \le \frac14$ and $\psi(t) = 0$ for $|t| \ge 1$, and for $s \in \mathbb{R}$ let $u = \mathbb{1}_{[t-s,t]}(P) w$, where w is any function in $L^2(M)$ with $\|w\|_{L^2(M)} = 1$. Next, by Lemma 4.4, for $i = 1,2$, there exist $C^i_0 = C_0(n, k_i, \mathfrak{I}^i_0, A_i)$, $C > 0$, and for all N there is $C_N > 0$ such that for all $0 < h < h_0$, $s \in \mathbb{R}$, and $t \in [E-Kh, E+Kh]$

$$\begin{aligned} h^{\frac{k_i-1}{2}} \Big| \int_{H_i} A_i \mathbb{1}_{[t-s,t]}(P) w \, d\sigma_{H_i} \Big| &\le C^i_0 R(h)^{\frac{n-1}{2}} \sum_{\ell \in \mathcal{L}^i_E(h)} \frac{(|\mathcal{G}^i_{t,\ell}| \tilde{t}^i_\ell)^{\frac12}}{(\tau \tilde{T}^i_\ell)^{\frac12}} \|\mathbb{1}_{[t-s,t]}(P) w\|_{L^2(M)} \\ &+ C^i_0 R(h)^{\frac{n-1}{2}} \sum_{\ell \in \mathcal{L}^i_E(h)} \frac{(|\mathcal{G}^i_{t,\ell}| \tilde{t}^i_\ell \tilde{T}^i_\ell)^{\frac12}}{h} \|P_t \mathbb{1}_{[t-s,t]}(P) w\|_{L^2(M)} + Q^{A,\psi}_{t,h}(C, C_N, \mathbb{1}_{[t-s,t]}(P) w). \end{aligned} \tag{5.6}$$

Note that for all N there is $C_N > 0$ such that for all $t \in [a-Kh, b+Kh]$, $|s| \le 10$ and $0 < h < 1$

$$\|P_t \mathbb{1}_{[t-s,t]}(P)\|_{L^2 \to H^N_{\rm scl}} \le C_N |s|, \qquad \|\mathbb{1}_{[t-s,t]}(P)\|_{L^2 \to L^2} \le 1. \tag{5.7}$$

In addition, we use the elliptic parametrix construction, together with $|s| \le 2h$ to obtain

$$\|\big(1 - \psi\big(\tfrac{P_t}{h^\delta}\big)\big) P_t A_i \mathbb{1}_{[t-s,t]}(P)\|_{L^2 \to H^N_{\rm scl}} \le C_N h^N. \tag{5.8}$$

We combine these estimates with (5.3) and the definition of $\tilde{T}^i_\ell$ into (5.2) to obtain that for all $0 < h < h_0$, $|s| \le 2h$, $K > 0$, and $t \in [E-Kh, E+Kh]$,

$$h^{\frac{k_i-1}{2}} \|\mathbb{1}_{[t-s,t]}(P)\, A_i^*\, \delta_{H_i}\|_{L^2(M)} \le C^i_0 C^i_{\rm nr} \bigg(\frac{1}{\tau^{\frac12}} \Big(\frac{1}{T_i}\Big\langle \frac{T_i s}{h}\Big\rangle\Big)^{\frac12} + \frac{|s|}{h} \Big(\frac{1}{T_i}\Big\langle \frac{T_i s}{h}\Big\rangle\Big)^{-\frac12} \bigg) + C_N h^N.$$

In particular, since $\tau < 1$, using this estimate in (5.2) we conclude that for all $0 < h < h_0$, $|s| \leq 2h$, $K > 0$, and $t \in [E - Kh, E + Kh]$

$$h^{\frac{k_i-1}{2}}\Big(\sum_{t-s\leq E_j\leq t}\Big|\int_{H_i} A_i\phi_{E_j}\,d\sigma_{H_i}\Big|^2\Big)^{\frac{1}{2}} \leq \frac{C_0^i C_{\rm nr}^i}{\sqrt{\tau T_i(h)}}\Big\langle\frac{T_i(h)s}{h}\Big\rangle^{\frac{1}{2}} + C_N h^N.$$

Combining estimates for H_1 and H_2 using (5.1), and $C_{\rm nr}^i \leq C_{\rm nr}$ completes the proof. □

The last lemma shows that $w_h(s) = \Pi_{H_1,H_2}^{A_1,A_2}(s)$ has at most polynomial growth at infinity.

Lemma 5.5. *Let $\ell_1, \ell_2 \in \mathbb{R}$. Then, there is $N_0 > 0$ such that for all $A_1 \in \Psi_\delta^{\ell_1}(M)$, $A_2 \in \Psi_\delta^{\ell_2}(M)$, there are $C_1 > 0$, $h_0 > 0$, such that for all $0 < h < h_0$ and $s \in \mathbb{R}$,*

$$|\Pi_{H_1,H_2}^{A_1,A_2}(s)| \leq C_1 h^{-\frac{k_1+k_2}{2}}\langle s\rangle^{N_0}.$$

Proof. Arguing as in (5.1), and (5.2), it is enough to prove that there is $C_1 > 0$ such that for each $i = 1, 2$ there is $N_i > 0$ for which

$$\sup_{\|w\|_{L^2(M)}=1}\Big|\int_{H_i} A_i \mathbb{1}_{(-\infty,s]}(P)w\,d\sigma_{H_i}\Big| \leq C_1 h^{-\frac{k_i}{2}}\langle s\rangle^{N_i}.$$

Applying Lemma 4.5 with $u = \mathbb{1}_{(-\infty,s]}(P)w$ yields that for any $\psi \in S^0(T^*M; [0,1])$ with $\psi \equiv 1$ on N^*H and $r_i > \frac{k_i+2\ell_i}{2m}$ there exist $C_1 > 0$ and $h_0 > 0$ such that for all $N > 0$ there is $C_N > 0$ satisfying for $0 < h < h_1$ and $s \in \mathbb{R}$,

$$\begin{aligned} h^{\frac{k_i}{2}}\Big|\int_{H_i} A_i\mathbb{1}_{(-\infty,s]}(P)w\,d\sigma_{H_i}\Big| &\leq C_N h^N\|\mathbb{1}_{(-\infty,s]}(P)w\|_{H_{\rm scl}^{-N}(M)} \\ &\quad + C_1\big(\|Op_h(\psi)\mathbb{1}_{(-\infty,s]}(P)w\|_{L^2(M)} + \|Op_h(\psi)P_s^{r_i}\mathbb{1}_{(-\infty,s]}(P)w\|_{L^2(M)}\big). \end{aligned} \tag{5.9}$$

Finally, the last term is bounded by $C_1(1 + |s|^{r_i})$ since $\|f(P)\|_{L^2\to L^2} \leq \|f\|_{L^\infty}$. □

6. Smoothed projector with non-looping condition

This section is dedicated to the proof of Theorems 8 and 9. The crucial step, completed in §6.1, is to bound $(\rho_{h,\widetilde{T}(h)} - \rho_{h,t_0}) * \Pi_{H_1,H_2}^{A_1,A_2}$ when the pair (H_1, H_2) is $(t_0, \mathbf{T})$ non-looping and $\widetilde{T}(h) = \frac{1}{2}\mathbf{T}(R(h))$. In §6.2 we prove Theorem 8 by combining the estimates from §6.1 with Proposition 5.1. In §6.3 we derive Theorem 9 from Theorem 8.

6.1. **Comparing against a short fixed time.** Throughout this section we continue to assume $H_1 \subset M$ and $H_2 \subset M$ are two submanifolds of co-dimension k_1 and k_2 respectively. The goal is to show that, under the assumption (H_1, H_2) is a $(t_0, \mathbf{T})$ non-looping pair in the window $[a, b]$, we can control $\rho_{\sigma_{h,\widetilde{T}(h)}} * \Pi_h$ by comparing it to $\rho_{h,t_0} * \Pi_h$. For the rest of the section we write

$$\widetilde{T}(h) := \tfrac{1}{2}\mathbf{T}(R(h)), \qquad T(h) := \mathbf{T}(R(h)).$$

Proposition 6.1. *Suppose $a, b \in \mathbb{R}$ are such that H_1, H_2 are conormally transverse for p in the window $[a, b]$. Let τ_0, R_0 be as in Lemma 4.1. Let $0 < \tau < \tau_0$, $0 < \delta < \frac{1}{2}$, and $\mathbf{T}$ a sub-logarithmic resolution function with $\Omega(\mathbf{T})\Lambda < 1 - 2\delta$.*

Suppose (H_1, H_2) is a $(t_0, \mathbf{T})$ non-looping pair in the window $[a, b]$ via τ-coverings with constant $C_{\rm nl}$. Let $A_1, A_2 \in \Psi^\infty(M)$, $h^\delta \leq R(h) \leq R_0$, and $K > 0$. There exist

$$C_0 = C_0(n, k_1, k_2, \mathfrak{I}^1_0, \mathfrak{I}^2_0, A_1, A_2, C_{\rm nl}) > 0$$

and $h_0 > 0$ such that for all $0 < h < h_0$ and all $E \in [a - Kh, b + Kh]$,

$$\Big|(\rho_{h,\widetilde{T}(h)} - \rho_{h,t_0}) * \Pi^{A_1,A_2}_{H_1,H_2}(E)\Big| \leq C_0 h^{\frac{2-k_1-k_2}{2}} \Big/ \mathbf{T}(R(h)). \tag{6.1}$$

We prove the proposition at the end of the section. The proof hinges on four lemmas. The first one, Lemma 6.3, rewrites the left hand side in (6.1) in terms of the function

$$f_{S,T,h}(\lambda) := f_{S,T}(h^{-1}\lambda), \qquad f_{S,T}(\lambda) := \frac{1}{i}\int_{\mathbb{R}} \frac{1}{\tau}\,\hat{\rho}\big(\tfrac{\tau}{T}\big)\big(1 - \hat{\rho}\big(\tfrac{\tau}{S}\big)\big)e^{-i\tau\lambda}d\tau, \tag{6.2}$$

where S, T are two positive constants with $S < T$, and ρ is as in (1.16)

Remark 6.2. We note that for all $N > 0$

$$|f_{S,T}(\lambda)| \leq C_N \langle \lambda S\rangle^{-N}, \qquad \operatorname{supp} \hat{\rho}\big(\tfrac{\tau}{T}\big)\big(1 - \hat{\rho}\big(\tfrac{\tau}{S}\big)\big) \subset \{\tau \in \mathbb{R} :\ |\tau| \in [S, 2T]\}. \tag{6.3}$$

Lemma 6.3. *Suppose $k > 0$ and $P \in \Psi^k(M)$ is self-adjoint with symbol satisfying* (1.9)*. Then, for all $N > 0$,*

$$(\rho_{h,\widetilde{T}} - \rho_{h,t_0}) * \Pi_h(E) = f_{t_0,\widetilde{T},h}(P_E) + O(h^N)_{H^{-N}_{\rm scl}\to H^N_{\rm scl}}.$$

Proof. First, we prove that if P is self-adjoint $E_1, E_2 \in \mathbb{R}$, then

$$\int_{E_1}^{E_2} (\rho_{h,\widetilde{T}(h)} - \rho_{h,t_0}) * \partial_s\Pi_h(s)ds = f_{t_0,\widetilde{T}(h),h}(P_{E_2}) - f_{t_0,\widetilde{T}(h),h}(P_{E_1}). \tag{6.4}$$

To ease notation write $\widetilde{T}$ for $\widetilde{T}(h)$. To prove (6.4) we write

$$\int_{E_1}^{E_2} (\rho_{h,\widetilde{T}} - \rho_{h,t_0}) * \partial_s\Pi_h(s)ds = \int_{E_1}^{E_2}\int_{\mathbb{R}} \hat{\rho}\big(\tfrac{w}{\sigma_{h,\widetilde{T}}}\big)\big[1 - \hat{\rho}\big(\tfrac{w}{\sigma_{h,t_0}}\big)\big]e^{-iw(P-s)}dwds,$$

where we use $\hat{\rho}\big(\frac{w}{\sigma_{h,t_0}}\big) = \hat{\rho}\big(\frac{w}{\sigma_{h,\widetilde{T}}}\big)\hat{\rho}\big(\frac{w}{\sigma_{h,t_0}}\big)$. Putting $\tau := hw$, (6.4) follows.

Next, let $N > 0$. By (6.4) it suffices to find $E_1 \in \mathbb{R}$ such that for all $t > c > 0$

$$\|f_{t_0,\widetilde{T},h}(P_{E_1})\|_{H^{-N}_{\rm scl}\to H^N_{\rm scl}} \leq C_N h^{2N}, \qquad \|\rho_{h,t} * \Pi_h(E_1)\|_{H^{-N}_{\rm scl}\to H^N_{\rm scl}} = O(h^N). \tag{6.5}$$

To prove the first claim in (6.5), note that by (6.3) for all $N > 0$ there is $C_N > 0$ such that

$$\|P^N_{E_1} f_{t_0,\widetilde{T},h}(P_{E_1})P^N_{E_1}\|_{L^2\to L^2} \leq C_N h^{2N}.$$

Next, since P satisfies (1.9), there is $a > 0$ such that $p(x, \xi) > -a$ for all $(x, \xi) \in T^*M$. In particular, for $E_1 \leq -2a$, P_{E_1} is elliptic and we have $P^{-1}_{E_1} : H^s_{\rm scl}(M) \to H^{s+k}_{\rm scl}(M) = O_s(1)$ for all $s \in \mathbb{R}$. Then, for $E_1 \leq -2a$ the first claim in (6.5) follows.

Next, by the sharp Gårding inequality, there is $C > 0$ such that $\Pi_h(s) \equiv 0$ for $s \leq -a - Ch$. Thus, for $E_1 \leq -3a$ and all $N, M \geq 0$ there is $C_{M,N} > 0$ such that

$$\|(\rho_{h,t} * \Pi_h)(E_1)\|_{H^{-N}_{\rm scl} \to H^N_{\rm scl}} \leq \int_{\mathbb{R}} \tfrac{t}{h}\rho\big(\tfrac{t}{h}s\big)\|\Pi_h(E_1 - s)\|_{H^{-N}_{\rm scl}\to H^N_{\rm scl}} ds \leq C_{M,N} \int_{s\leq -a} \tfrac{t}{h}\big\langle \tfrac{t}{h}s\big\rangle^{-M} \langle s\rangle^{2N/k}.$$

The claim follows after choosing M large enough. □

Let $H_1, H_2,\ t_0, T(h),\ \tau$, and $R(h)$ be as in Proposition 6.1. Since (H_1, H_2) is a $(t_0, \mathbf{T})$ non-looping pair in the window $[a, b]$ via τ_0-coverings, for $i = 1, 2$ and $h > 0$ we let

$$\{\mathcal{T}^i_j\}_{j\in\mathcal{J}^i(h)} \quad \text{a } (\mathfrak{D}_n, \tau, R(h))\text{-good cover of } \Sigma^{H_i}_{[a,b]} \text{ satisfying (1) and (2) in Definition 2.1.} \tag{6.6}$$

We study $A_1 f_{t_0,\widetilde{T},h}\left(P_E\right) A_2^*$ by understanding the behavior of

$$F^{A_1,A_2}_{j,\ell}(E, h) := Op_h(\chi_{\mathcal{T}^1_j})A_1 f_{t_0,\widetilde{T},h}\left(P_E\right) A^*_2 Op_h(\chi_{\mathcal{T}^2_\ell}) \tag{6.7}$$

for $j \in \mathcal{J}^1(h)$ and $k \in \mathcal{J}^2(h)$. Next, we study the case when $\mathcal{T}^1_j$ does not loop through $\mathcal{T}^2_k$.

Lemma 6.4. *Assume H_1 and H_2 are conormally transverse for p in the window $[a, b]$. For $i = 1, 2$ let $\{\mathcal{T}^i_j\}_{j\in\mathcal{J}^i(h)}$ as in* (6.6) *and $j \in \mathcal{J}^1(h)$, $\ell \in \mathcal{J}^2(h)$ be such that*

$$\varphi_t(\mathcal{T}^1_j) \cap \mathcal{T}^2_\ell = \emptyset, \qquad |t| \in [t_0{+}\tau, T(h){-}\tau].$$

Let $K > 0$ and $\mathcal{V}$ be a bounded subset of $S_\delta(T^*M; [0, 1])$. Then, there exists $h_0 > 0$ and for all $N > 0$ there exists $C_N > 0$ such that for all $0 < h < h_0$, $E \in [a - Kh, b + Kh]$, and every δ-partition $\{\chi_{\mathcal{T}^i_j}\}_{j\in\mathcal{J}^i_E(h)} \subset \mathcal{V}$ associated to $\{\mathcal{T}_j\}_{j\in\mathcal{J}^i_E(h)}$, $i = 1, 2$,

$$\|F^{A_1,A_2}_{j,\ell}(E, h)\|_{H^{-N}_{\rm scl}(M)\to H^N_{\rm scl}(M)} \leq C_N h^N.$$

Proof. By Egorov's theorem, for all $N > 0$ there exist $h_0 > 0$ and $C_N > 0$ such that for all $0 < h < h_0$, and $E \in [a - Kh, b + Kh]$,

$$\big\|Op_h(\chi_{\mathcal{T}^1_j})A_1\, e^{-it\frac{P_E}{h}} A^*_2\, Op_h(\chi_{\mathcal{T}^2_\ell})\big\|_{H^{-N}_{\rm scl}(M)\to H^N_{\rm scl}(M)} \leq C_N h^N, \qquad |t| \in [t_0{+}\tau, T(h){-}\tau]$$

(see e.g. [18, Proposition 3.9]). The claim follows from the definition (6.2) together with the facts that by (6.3) the support of its integrand has $\tau \in [t_0, 2\widetilde{T}(h)]$, and $\widetilde{T}(h) = \frac{1}{2}T(h)$. □

The next lemma provides an estimate for $F^{A_1,A_2}_{j,\ell}(E, h)$ based on volumes of tubes.

Lemma 6.5. *Assume H_1 and H_2 are conormally transverse for p in the window $[a, b]$. Let A_1, A_2, τ_0, R_0, τ, δ, and $R(h)$ be as in Proposition 6.1. For $i = 1, 2$ let $\{\mathcal{T}^i_j\}_{j\in\mathcal{J}^i(h)}$ be a $(\mathfrak{D}_n, \tau, R(h))$-good covering of $\Sigma^{H_i}_{[a,b]}$. Let $K > 0$ and $\mathcal{V}$ a bounded subset of $S_\delta(T^*M; [0, 1])$. Then, there are $C_0 = C_0(n, k_1, k_2, \mathfrak{I}^1_0, \mathfrak{I}^2_0, A_1, A_2, \mathcal{V})$ and $h_0 > 0$, and for all $N > 0$ there exists $C_N > 0$ such that the following holds. For all $0 < h < h_0$, $E \in [a - Kh, b + Kh]$, all δ-partitions $\{\chi_{\mathcal{T}^i_j}\}_{j\in\mathcal{J}^i_E(h)} \subset \mathcal{V}$ and $\mathcal{I}_i \subset \mathcal{J}^i_E(h)$ for $i = 1, 2$, and all $t_0, \tilde{T}$ with $0 < t_0 < \widetilde{T}$,*

$$\Big|\int_{H_1}\int_{H_2} \sum_{\ell\in\mathcal{I}_1, j\in\mathcal{I}_2} F^{A_1,A_2}_{j,\ell}(E, h)(x, y)d\sigma_{H_2}(y)d\sigma_{H_1}(x)\Big| \leq C_0\tau^{-1}h^{\frac{2-k_1-k_2}{2}} R(h)^{n-1}|\mathcal{I}_1|^{\frac{1}{2}}|\mathcal{I}_2|^{\frac{1}{2}} + C_N h^N.$$

Proof. The first step in our proof is to define for $0 < t_0 < \widetilde{T}$ the functions
$$g^2_{t_0,\widetilde{T}}(\lambda) g^1_{t_0,\widetilde{T}}(\lambda) := f_{t_0,\widetilde{T}}(\lambda), \qquad g^2_{t_0,\widetilde{T}}(\lambda) := \langle t_0 \lambda \rangle^{-N_0},$$
where $N_0 \geq 1$ will be chosen later. Note that by (6.3) for all $L > 0$ there is $C_L > 0$ such that
$$|g^1_{t_0,\widetilde{T}}(\lambda)| \leq C_L \langle t_0 \lambda \rangle^{-L+1}. \tag{6.8}$$
Since $f_{t_0,\widetilde{T},h}(P_E) = g^1_{t_0,\widetilde{T},h}(P_E) g^2_{t_0,\widetilde{T},h}(P_E)$, we may use Cauchy-Schwarz to bound
$$\begin{aligned}\Big|\int_{H_1}\int_{H_2} \sum_{\ell\in\mathcal{I}_1, j\in\mathcal{I}_2} &\Big[F^{A_1,A_2}_{j,\ell}(E,h)\Big](x,y) d\sigma_{H_2}(y) d\sigma_{H_1}(x)\Big| \\ &\leq \Big\| \sum_{\ell\in\mathcal{I}_1} g^1_{t_0,\widetilde{T}}(P_E) A_1^* Op_h(\chi_{\mathcal{T}^1_\ell})\delta_{H_1}\Big\|_{L^2(M)} \Big\| \sum_{\ell\in\mathcal{I}_2} g^2_{t_0,\widetilde{T},h}(P_E) A_2^* Op_h(\chi_{\mathcal{T}^2_\ell})\delta_{H_2}\Big\|_{L^2(M)}.\end{aligned}$$

Next, we use that for $i = 1, 2$,
$$\Big\| \sum_{\ell\in\mathcal{I}_i} g^i_{t_0,\widetilde{T},h}(P_E) A_i^* Op_h(\chi_{\mathcal{T}^i_\ell})\delta_{H_i}\Big\|_{L^2(M)} \leq \sup_{\|w\|=1} \Big| \int_{H_i} \sum_{\ell\in\mathcal{I}_i} Op_h(\chi_{\mathcal{T}^i_\ell}) A_i\, g^i_{t_0,\widetilde{T},h}(P_E) w\, d\sigma_{H_i}\Big|.$$
Thus, let $w \in L^2(M)$ and fix $i \in \{1, 2\}$. We next apply Lemma 4.4 to the function $u = g^i_{t_0,\widetilde{T},h}(P_E)w$ and operator $A = \sum_{j\in\mathcal{I}_i} Op_h(\chi_{\mathcal{T}^i_j})A_i \in \Psi^\infty_\delta(M)$. Here, we use that $\mathrm{MS_h}(A) \subset \cup_{j\in\mathcal{I}_i}\mathcal{T}^i_j$ and that $\frac{1}{h}[P_E, A] \in \Psi^\infty_\delta(M)$ (see the definition of a δ-partition (4.1)). In particular, we may fix $\mathcal{W} \subset \Psi^\infty_\delta(M)$ such that $\frac{1}{h}[P_E, A] \in \mathcal{W}$ regardless of the choice of cover and δ-partition contained in $\mathcal{V}$. Then, the constant C^i_0 provided by the Lemma depends on A_i instead of $\mathcal{W}$.

Fix $\psi \in C^\infty_0(\mathbb{R}; [0, 1])$ with $\psi(t) = 1$ for $|t| \leq \frac{1}{4}$ and $\psi(t) = 0$ for $|t| \geq 1$. By Lemma 4.4 with $t_1 = t_0$, $T_1 = t_0$, and $\mathcal{G}_\ell = \emptyset$ for all $\ell > 1$, we obtain that there are $C^i_0 = C^i_0(n, k_i, \mathfrak{I}^i_0, A_i) > 0$, $C > 0$, there exist $h_0 > 0$ and for all $N > 0$ there is $C_N > 0$ such that for all $0 < h < h_0$
$$\begin{aligned}h^{\frac{k_i-1}{2}}\Big| \int_{H_i} \sum_{j\in\mathcal{I}_i} Op_h(\chi_{\mathcal{T}^i_j}) A_i\, g^i_{t_0,\widetilde{T},h}(P_E) w\, d\sigma_{H_i}\Big| &= Q^{A,\psi}_{E,h}(C, C_N, g^i_{t_0,\widetilde{T},h}(P_E)w) \\ &+ C^i_0 R(h)^{\frac{n-1}{2}} |\mathcal{I}_i|^{\frac{1}{2}} \Big(\frac{1}{\tau^{\frac{1}{2}}} \big\| g^i_{t_0,\widetilde{T},h}(P_E) w\big\|_{L^2(M)} + \tfrac{t_0}{h} \big\| P_E g^i_{t_0,\widetilde{T},h}(P_E) w\big\|_{L^2(M)}\Big).\end{aligned}$$
By the definitions $g^i_{t_0,\widetilde{T}}$, $i = 1, 2$ and (6.8) there exists $C > 0$ such that for all $t_0, \widetilde{T}$ with $t_0 < \widetilde{T}$,
$$\big\| g^i_{t_0,\widetilde{T},h}(P_E)\big\|_{L^2\to L^2} \leq C, \qquad \big\| P_E g^i_{t_0,\widetilde{T},h}(P_E)\big\|_{L^2\to L^2} \leq C\frac{h}{t_0}, \quad i = 1, 2.$$
In addition, note that for $i = 1, 2$ there exists $C_{N_0} > 0$ such that
$$\big\| \big(1 - \psi\big(\tfrac{P_E}{h^\delta}\big)\big) P_E A\, g^i_{t_0,\widetilde{T},h}(P_E)\big\|_{L^2\to L^2} \leq C_{N_0} h^{N_0(1-\delta)+\delta}.$$
The claim follows from choosing N_0 large enough that $N_0(1 - \delta) + \delta \geq N$. □

Lemma 6.6. *Assume the same assumptions as in Proposition 6.1. For $i = 1, 2$ let $\{\mathcal{T}^i_j\}_{j\in\mathcal{J}^i(h)}$ be as in (6.6), $\mathcal{V}$ be a bounded subset of $S_\delta(T^*M; [0, 1])$ and $K > 0$. There exists $h_0 > 0$, and for* *all $N>0$ there exists $C_N>0$ such that for all $0<h<h_0$, $E\in[a-Kh,b+Kh]$, and every δ-partition $\{\chi_{\mathcal{T}_j^i}\}_{j\in\mathcal{J}_E^i(h)}\subset\mathcal{V}$ associated to $\{\mathcal{T}_j^i\}_{j\in\mathcal{J}_E^i(h)}$,*

$$\Big\|\gamma_{H_1}A_1f_{t_0,\tilde{T},h}(P_E)A_2^*\delta_{H_2}-\sum_{j\in\mathcal{J}_E^1(h),\,\ell\in\mathcal{J}_E^2(h)}\gamma_{H_1}F_{j,\ell}^{A_1,A_2}(E,h)\delta_{H_2}\Big\|_{H_{\rm scl}^{-N}(H_2)\to H_{\rm scl}^{N}(H_1)}\leq C_Nh^N.$$

Proof. Let $K>0$ and $\psi\in C_c^\infty((-1,1);[0,1])$ with $\psi(t)=1$ for $|t|\leq\frac14$. We claim there exists $h_0>0$ such that for all $N>0$ there is $C_N>0$ so that for $0<h<h_0$,

$$\|\big(1-\psi\big(\tfrac{P_E}{h^\delta}\big)\big)f_{t_0,\tilde{T},h}(P_E)\|_{H_{\rm scl}^{-N}(M)\to H_{\rm scl}^{N}(M)}\leq C_Nh^N,\qquad E\in[a-Kh,b+Kh].\tag{6.9}$$

To see this, first note that for $\tilde\psi\in C_c^\infty$ with $\operatorname{supp}\tilde\psi\subset\{\psi\equiv1\}$ and $L>0$,

$$\big(1-\psi\big(\tfrac{P_E}{h^\delta}\big)\big)f_{t_0,\tilde{T},h}(P_E)=P_E^{-L}\big(1-\psi\big(\tfrac{P_E}{h^\delta}\big)\big)P_E^Lf_{t_0,\tilde{T},h}(P_E)P_E^LP_E^{-L}\big(1-\tilde\psi\big(\tfrac{P_E}{h^\delta}\big)\big).$$

Now, since P_E is classically elliptic in $\Psi^m(M)$, for all $s\in\mathbb{R}$,

$$P_E^{-L}\big(1-\psi\big(\tfrac{P_E}{h^\delta}\big)\big)=O_{L,s}(h^{-\delta L})_{H_{\rm scl}^{s}(M)\to H_{\rm scl}^{s+mL}(M)}.\tag{6.10}$$

Note that (6.10) also holds with $\tilde\psi$ in place of ψ. In addition, by (6.3)

$$P_E^Lf_{t_0,\tilde{T},h}(P_E)P_E^L=O_L(h^{2L})_{L^2(M)\to L^2(M)}.\tag{6.11}$$

Taking $L>\max(N/m,N/(2(1-\delta)))$ and combining (6.10) and (6.11) we obtain (6.9).

Next, for $i=1,2$ we define $G_i:=\mathrm{Id}-\sum_{j\in\mathcal{J}_E^i(h)}Op_h(\chi_{\mathcal{T}_j^i})$, and note that $\mathrm{MS_h}(G_i)\cap\Lambda^\tau_{\Sigma_E^{H_i}}(R(h)/2)=\emptyset$. Therefore, combining Lemma 4.1 together with (6.9), there exists $h_0>0$ such that for all $N>0$ there is $C_N>0$ so that for all $0<h<h_0$,

$$\|\gamma_{H_1}A_1G_1f_{t_0,\tilde{T},h}(P_E)A_2^*\delta_{H_2}\|_{H_{\rm scl}^{-N}(H_2)\to H_{\rm scl}^{N}(H_1)}\leq C_Nh^N,\qquad E\in[a-Kh,b+Kh].\tag{6.12}$$

In particular, the lemma follows from applying (6.12) and its analogs since

$$\begin{aligned}&\gamma_{H_1}A_1f_{t_0,\tilde{T},h}(P_E)A_2^*\delta_{H_2}-\sum_{j\in\mathcal{J}_E^1(h),\ell\in\mathcal{J}_E^2(h)}\gamma_{H_1}F_{j,\ell}^{A_1,A_2}(E,h)\delta_{H_2}\\&=\gamma_{H_1}A_1G_1f_{t_0,\tilde{T},h}(P_E)A_2^*\delta_{H_2}+\gamma_{H_1}A_1f_{t_0,\tilde{T},h}(P_E)G_2A_2^*\delta_{H_2}+\gamma_{H_1}A_1G_1f_{t_0,\tilde{T},h}(P_E)G_2A_2^*\delta_{H_2}.\ \square\end{aligned}$$

Proof of Proposition 6.1. Since (H_1,H_2) is a $(t_0,\mathbf{T})$ non-looping pair in the window $[a,b]$ via τ_0-coverings, for $i=1,2$ and $h>0$ we may work with $\{\mathcal{T}_j^i\}_{j\in\mathcal{J}^i(h)}$, as in (6.6) and $\{\chi_{\mathcal{T}_j^i}\}_{j\in\mathcal{J}^i(h)}$ a δ-partition associated $\{\mathcal{T}_j^i\}$ For each $E\in[a,b]$ and $i=1,2$, let $\mathcal{J}_{E,h}^i=\mathcal{B}_E^i(h)\cup\mathcal{G}_E^i(h)$ be a partition of indices such that property (1) of Definition 2.1 with $r=R(h)$. Then, by Lemma 6.4, for $K>0$ there exists $h_0>0$ such that the following holds: For all $N>0$ there is $C_N>0$ so that for all $0<h<h_0$, $E\in[a-Kh,b+Kh]$, and $i,k=1,2$ with $i\neq k$,

$$\Big|\int_{H_1}\int_{H_2}\sum_{j\in\mathcal{J}_E^k(h)}\sum_{\ell\in\mathcal{G}_E^i(h)}[F_{j,\ell}^{A_1,A_2}(E,h)](x,y)d\sigma_{H_2}(y)d\sigma_{H_1}(x)\Big|\leq C_Nh^N.\tag{6.13}$$

Therefore, considering the remaining term, and applying Lemma 6.5 we obtain the following. There is $C_0 = C_0(n, k_1, k_2, \mathfrak{I}_0^1, \mathfrak{I}_0^2, A_1, A_2) > 0$ and for $K > 0$ there exists $h_0 > 0$ such that the following holds: For all $N > 0$ there is $C_N > 0$ so that for all $0 < h < h_0$, $E \in [a - Kh, b + Kh]$,

$$\Big|\int_{H_1}\int_{H_2}\sum_{j\in\mathcal{B}^1_E(h)}\sum_{\ell\in\mathcal{B}^2_E(h)}[F_{j,\ell}^{A_1,A_2}(E,h)](x,y)d\sigma_{H_2}(y)d\sigma_{H_1}(x)\Big|$$
$$\leq C_0 h^{\frac{2-k_1-k_2}{2}} R(h)^{n-1}|\mathcal{B}^1_E(h)|^{\frac12}|\mathcal{B}^2_E(h)|^{\frac12} + C_N h^N \leq C_0 C_{\rm nl} h^{\frac{2-k_1-k_2}{2}}/T(h). \tag{6.14}$$

To get the last line we used that our covering satisfies property (2) of Definition 2.1. Combining Lemma 6.6 with (6.6), (6.13), and (6.14), we obtain the claim.

6.2. **Proof of Theorem 8.** Since for $i = 1, 2$ the submanifold H_i is $T_i(h)$ non-recurrent in the window $[a, b]$ via τ_0-coverings with constant $C^i_{\rm nr}$, we may apply Proposition 5.1 to obtain the existence of $C_0 = C_0(n, k_1, k_2, \mathfrak{I}_0^1, \mathfrak{I}_0^2, A_1, A_2, C^1_{\rm nr}, C^2_{\rm nr})$ and for all $K > 0$ obtain $h_0 > 0$ such that for all $0 < h \leq h_0$ and $s \in [a - Kh, b + Kh]$,

$$\Big|\Pi^{A_1,A_2}_{H_1,H_2}(s) - \rho_{h,\tilde{T}_{\max}(h)} * \Pi^{A_1,A_2}_{H_1,H_2}(s)\Big| \leq C_0\, h^{\frac{2-k_1-k_2}{2}}\big/\,T(h), \tag{6.15}$$

where $T(h) = (T_1(h)T_2(h))^{\frac12}$ and $T_{\max}(h) = \max(T_1(h), T_2(h))$. Note that we are actually applying the proposition only using that H_i is $\frac12 T_i(h)$ non-recurrent.

On the other hand, since (H_1, H_2) is a $(t_0, \mathbf{T}_{\max})$ non-looping pair in the window $[a, b]$ via τ_0 coverings, we may apply Proposition 6.1 to obtain that there exist $C_1 = C_1(n, k_1, k_2, \mathfrak{I}_0^1, \mathfrak{I}_0^2, A_1, A_2, C_{\rm nl}) > 0$ and for all $K > 0$ there is $h_0 > 0$ such that for all $0 < h < h_0$ and all $s \in [a - Kh, b + Kh]$

$$\Big|(\rho_{h,\tilde{T}_{\max}(h)} - \rho_{h,t_0}) * \Pi^{A_1,A_2}_{H_1,H_2}(s)\Big| \leq C_1\, h^{\frac{2-k_1-k_2}{2}}\big/\,T(h). \tag{6.16}$$

The result follows from combining (6.15) with (6.16). We note that H_1 and H_2 may be replaced by $\tilde{H}_{1,h}$ and $\tilde{H}_{2,h}$ since $C_{\rm nl}$, $C^1_{\rm nr}$, and $C^2_{\rm nr}$ are uniform for $\{\tilde{H}_{1,h}\}_h$ and $\{\tilde{H}_{2,h}\}_h$.

6.3. **Proof of Theorem 9.** Let $0 < \tau < \min(\tau_0, \varepsilon/3)$. By Proposition 3.5 there exists $c_0 > 0$, $C_{\rm nr} = C_{\rm nr}(M, p, \mathfrak{t}, R_0) > 0$ such that for $j = 1, 2$, the submanifold H_j is $c\mathbf{T}_i(R)$ non-recurrent in the window $[a, b]$ via τ coverings with constant $C_{\rm nr}$.

Now, since (H_1, H_2) is a $(t_0, \mathbf{T}_{\max})$ non-looping pair in the window $[a, b]$ with constant $C_{\rm nl}$. Proposition 3.1 implies there is $\widetilde{C_{\rm nl}} = \widetilde{C_{\rm nl}}(p, a, b, n, C_{\rm nl}, H_1, H_2)$ such that (H_1, H_2) is a $(t_0 + 3\tau_0, \tilde{\mathbf{T}})$ non-looping pair in the window $[a, b]$ via τ_0-coverings with constant $\widetilde{C_{\rm nl}}$ where $\tilde{\mathbf{T}}(R) = \mathbf{T}_{\max}(4R) - 3\tau_0$. Since $\mathbf{T}_j$ are sub-logarithmic, there is $c_1 > 0$ such that $\tilde{\mathbf{T}}(R) \geq c_1\mathbf{T}_{\max}(R)$. The proof now follows from a direct application of Theorem 8 with $\mathbf{T}_j$ replaced by $\min(c_0, c_1)\mathbf{T}_j$ and t_0 by $t_0 + \varepsilon$.

7. The Weyl law

In order to improve remainders in the Weyl law itself, we let $\Delta \subset M \times M$ be the diagonal, and for $A_1, A_2 \in \Psi^\infty(M)$ consider the integral

$$\int_M [A_1 \mathbb{1}_{(-\infty,s]}(P) A_2](x,x)\, \mathrm{dv}_g(x) = \int_\Delta \Big((A_1 \otimes A_2^*)\mathbb{1}_{(-\infty,s]}(P)\Big)(x,y) d\sigma_\Delta(x,y),$$

where $d\sigma_\Delta$ is the Riemannian volume form induced on Δ by the product metric on $M \times M$. To ease notation, we write $\mathbf{P}_t = (P - t) \otimes 1 = P \otimes 1 - t\,\mathrm{Id}$. We will view Δ as a hypersurface of codimension n in $M \times M$, and the kernel of $\mathbb{1}_{[t-s,t]}(P)$ as a quasimode for $\mathbf{P}_t$. In particular, observe that for any operator $B : L^2(M) \to L^2(M)$

$$\|\mathbf{P}_t \mathbb{1}_{[t-s,t]}(P)B\|_{L^2(M\times M)} \leq |s| \|\mathbb{1}_{[t-s,t]}(P)B\|_{L^2(M\times M)}. \tag{7.1}$$

In addition, note that for $(x, \xi, y, \eta) \in T^*M \times T^*M$

$$\sigma(\mathbf{P}_t)(x, \xi, y, \eta) = p(x, \xi) - t =: \mathbf{p}(x, \xi, y, \eta) - t =: \mathbf{p}_t(x, \xi, y, \eta).$$

Therefore, for all $c > 0$, there is $C > 0$ such that if $c|\eta| \leq |\xi|$ and $|\xi| \geq C$, then

$$|\sigma(\mathbf{P}_t)(x, \xi, y, \eta)| \geq \tfrac{1}{C}|(\xi, \eta)|^m.$$

In particular, since we work near the $\mathbf{p}$ flow-out of $N^*\Delta \cap \{\mathbf{p} = t\}$ where $t \in [a, b]$, and

$$N^*\Delta = \{(x, \xi, x, -\xi) : \ (x, \xi) \in T^*M\},$$

we may work as though $\mathbf{P}_t$ were elliptic in $\Psi^m(M \times M)$, and apply the results of the previous sections by accepting $O(h^\infty)$ errors. We will do this without further comment.

We next describe the tubes relevant in this section. We will work microlocally near a point $\rho_0 \in N^*\Delta \cap \mathbf{p}^{-1}([a, b])$. Let $\pi_R, \pi_L : T^*(M \times M) \to T^*M$ denote the projections to the right and left factor, and let $\mathcal{Z}_{\pi_L(\rho_0)} \subset T^*M$ be a transversal to the flow for p containing $\pi_L(\rho_0)$. (Such a hypersurface exists since $dp(\rho) \neq 0$ on $p^{-1}([a, b])$.) Define a transversal to the flow for $\mathbf{p}$ by

$$\mathcal{Z}_{\rho_0} := \mathcal{Z}_{\pi_L(\rho_0)} \times T^*M,$$

and let U be a neighborhood of ρ_0 in $N^*\Delta$ such that $U \cap \mathbf{p}^{-1}([a, b]) \subset \mathcal{Z}_{\rho_0}$. We will use the metric $\tilde{d}$ on $T^*M \times M$ defined by $\tilde{d}\Big((\rho_L, \rho_R), (q_L, q_R)\Big) := \max\Big(d(\rho_L, q_L), d(\rho_R, q_R)\Big)$, for $(\rho_L, \rho_R), (q_L, q_R) \in T^*M \times M$. With this definition, for $\rho = (\rho_L, \rho_R) \in N^*\Delta \cap \{\mathbf{p}_t = 0\}$,

$$\mathcal{T}_\rho := \Lambda^\tau_\rho(r) = \tilde{\Lambda}^\tau_{\rho_L}(r) \times B(\rho_R, r)$$

where $\Lambda^\tau_A(r)$ is defined by (2.2) with φ_t the Hamiltonian flow for $\mathbf{p}$ and $\tilde{\mathcal{T}} = \tilde{\Lambda}^\tau_{\rho_L}(r)$ denotes a tube with respect to p and the hypersurface $\mathcal{Z}_{\pi_L(\rho_0)}$. In particular, when we use cutoffs with respect to a tube $\mathcal{T}$, we will always work with cutoffs of the form

$$\chi_{\mathcal{T}}(x, \xi, y, \eta) = \chi_{\tilde{\mathcal{T}}}(x, \xi)\chi_{\rho_R}(y, \eta), \qquad \operatorname{supp}\chi_{\rho_R} \subset B(\rho_R, r).$$

We will refer only to this tube in T^*M, leaving the other implicit and will think of the kernel of $A_1\mathbb{1}_{[a,b]}(P)A_2$ as that of $\mathbb{1}_{[a,b]}(P)$ acted on by $A_1 \otimes A_2^t$. Before we start our proof of the improved Weyl remainder, we need a dynamical lemma.

Lemma 7.1. *Let $C_{\mathrm{np}} > 0$, $a \leq b$, and $U \subset T^*M$ satisfying $d\pi_M H_p \neq 0$ on $p^{-1}([a, b]) \cap \overline{U}$. Then there are $\tau_0 > 0$ and $\widetilde{C_{\mathrm{np}}} = \widetilde{C_{\mathrm{np}}}(p, U, C_{\mathrm{np}})$ such that the following holds. If U is $(t_0, \mathbf{T})$ non-periodic for p in the window $[a, b]$ with constant C_{np}, then $N^*\Delta \cap (U \times T^*M)$ is $(t_0 + 3\tau_0, \mathbf{T}(16R) - 3\tau_0)$ non-looping for $\mathbf{p}$ via τ_0-coverings in the window $[a, b]$ with constant $\widetilde{C_{\mathrm{np}}}$.*

Proof. Let $E\in[a,b]$. We work with $\mathcal{L}^{R,E}_{\Delta,\Delta}(t_0,T)$ as defined in Definition 1.10 but with p replaced by $\mathbf{p}$, $\varphi_t^{\mathbf{p}}:=\exp(tH_{\mathbf{p}})$, and $\Sigma^\Delta_E=N^*\Delta\cap\{\mathbf{p}=E\}$. First, we claim

$$\pi_L\Big(B_{\Sigma^\Delta_E}\big(\mathcal{L}^{R,E}_{\Delta_U,\Delta_U}(t_0,T),R\big)\Big)\subset B_{\Sigma^\Delta_E}\big(\mathcal{P}^R_U(t_0,T),2R\big). \tag{7.2}$$

Here, through a slight abuse of notation, we write $\mathcal{L}^{R,E}_{\Delta_U,\Delta_U}$ for (1.5) with S^*_xM and S^*_yM replaced by $\Delta_U:=N^*\Delta\cap(U\times T^*M)$ and $\varphi_t=\exp(tH_{\mathbf{p}})$. To prove (7.2) suppose $\rho_0\in B_{\Sigma^\Delta_E}\big(\mathcal{L}^{R,E}_{\Delta_U,\Delta_U}(t_0,T),R\big)$. Then, there are $\rho_1\in\Sigma^\Delta_E\cap\Delta_U$ and $\rho_1'\in T^*(M\times M)$ such that

$$\tilde d(\rho_0,\rho_1)<R,\qquad \tilde d(\rho_1,\rho_1')<R,\qquad\text{and}\qquad \bigcup_{t_0\le|t|\le T}\varphi_t^{\mathbf{p}}(\rho_1')\cap B(\Sigma^\Delta_E,R)\neq\emptyset.$$

Therefore, there is $\rho_2\in\Sigma^\Delta_E$ such that $\tilde d\big(\varphi_t^{\mathbf{p}}(\rho_1'),\rho_2\big)<R$ for some $t_0\le|t|\le T$. Let $\rho_1'=(x',\xi',y',-\eta')$ with $(x',\xi'),(y',\eta')\in T^*M$. Then, since $\rho_1=(x,\xi,x,-\xi)$ and $\rho_2=(y,\eta,y,-\eta)$ for some $(x,\xi)\in T^*M$ and $(y,\eta)\in T^*M$, we have $d(\varphi_t(x',\xi'),(x',\xi'))<4R$ and $\pi_L(\rho_1')=(x',\xi')\in\mathcal{P}^{4R}_U(t_0,T)$. On the other hand, since $d(\pi_L(\rho_0),\pi_L(\rho_1'))<2R$ we obtain $\pi_L(\rho_0)\in B_{S^*M}\big(\mathcal{P}^{4R}_U(t_0,T),2R\big)$. This proves claim (7.2).

Next, note that since $\pi_L:\Delta_U\cap\Sigma^\Delta_E\to\{p=E\}\cap U$ is a diffeomorphism for $E\in[a,b]$, it follows that there exists $C=C(p)>0$ such that for all $E\in[a,b]$

$$\mu_E\Big(B_{\Sigma^\Delta_E}\big(\mathcal{L}^{R,E}_{\Delta_U,\Delta_U}(t_0,T),R\big)\Big)\le C\mu_{S^*M}\Big(B_{S^*M}\big(\mathcal{P}^{4R}_U(t_0,T),2R\big)\Big).$$

Hence, if U is $(t_0,\mathbf{T})$ non-periodic for p at energy E, we have

$$\mu_E\Big(B_{\Sigma^\Delta_E}\big(\mathcal{L}^{R,E}_{\Delta_U,\Delta_U}(t_0,\mathbf{T}(4R)),R\big)\Big)\mathbf{T}(4R)\le C\mu_{S^*M}\Big(B_{S^*M}\big(\mathcal{P}^{4R}_U(t_0,\mathbf{T}(4R)),4R\big)\Big)\mathbf{T}(4R)\le CC_{\rm np},$$

and so Δ_U is $(t_0,\mathbf{T}(4R))$ non-looping for $\mathbf{p}$ at energy E. The result follows from Corollary 3.1. ☐

In what follows, we write $\|\cdot\|_{HS}$ for the Hilbert-Schmidt norm.

Lemma 7.2. *Let $\mathcal{V}\subset S_\delta(T^*M;[0,1])$ be a bounded subset. Then, there are $C>0$ and $h_0>0$, and for all $N>0$ there exists $C_N>0$, such that for all $t\in[a,b]$, $\chi\in\mathcal{V}$, $0<h<h_0$, and $|s|\le 2h$,*

$$\|\mathbb{1}_{[t-s,t]}(P)Op_h(\chi)\|^2_{HS}\le Ch^{1-n}\mu_{p^{-1}(t)}(\operatorname{supp}\chi\cap p^{-1}(t))+C_Nh^N, \tag{7.3}$$

$$h^{-2}\|P_t\mathbb{1}_{[t-s,t]}(P)Op_h(\chi)\|^2_{HS}\le Ch^{1-n}\mu_{p^{-1}(t)}(\operatorname{supp}\chi\cap p^{-1}(t))+C_Nh^N. \tag{7.4}$$

Proof. We follow the proof of [18, Lemma 3.11]. Let $\psi\in\mathcal{S}(\mathbb{R};\mathbb{R})$ with $\psi(0)=1$ and $\operatorname{supp}\hat\psi\subset[-1,1]$. Define $\psi_\varepsilon(s):=\psi(\varepsilon s)$. Then, there is $\varepsilon_0>0$ small enough so that $\psi_{\varepsilon_0}(s)>\frac12$ on $[-2,2]$. Abusing notation slightly, put $\psi=\psi_{\varepsilon_0}$. Then, there exists an operator Z_s such that $\mathbb{1}_{[t-s,t]}(P)=Z_s\psi\big(\frac{P_t}{h}\big)$, $[Z_s,P]=0$, and $\|Z_s\|_{L^2\to L^2}\le 3$ for $|s|\le 2h$. Therefore, $\|\mathbb{1}_{[t-s,t]}(P)Op_h(\chi)\|_{HS}\le 3\big\|\psi\big(\frac{P_t}{h}\big)Op_h(\chi)\big\|_{HS}$ and the Hilbert–Schmidt norm is the L^2 norm of the kernel. Next, we recall that after application of a microlocal partition of unity, we may write

$$\psi\Big(\frac{P_t}{h}\Big)(x,y)=h^{-n}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\hat\psi(\tau)e^{\frac{i}{h}(\varphi(\tau,x,\eta)-\langle y,\eta\rangle-t\tau)}a(\tau,x,y,\eta)d\eta d\tau+O(h^\infty)_{HS}$$

for a symbol $a \sim \sum_j h^j a_j$ and phase φ solving $\partial_t\varphi = p(x, \partial_x\varphi)$ and $\varphi(0, x, \eta) = \langle x, \eta\rangle$. At this point the proof of (7.3) follows exactly as in [18, Lemma 3.11].

To obtain (7.4), we write $P_t\mathbb{1}_{[t-s,t]}(P) = Z_sP_t\psi\big(\frac{P_t}{h}\big)$ and note that $\frac{P_t}{h}\psi\big(\frac{P_t}{h}\big) = (t\psi)\big(\frac{P_t}{h}\big)$. Hence the same argument applies with $\widehat{t\psi}(\tau) = -i\partial_\tau\hat{\psi}(\tau)$ replacing $\hat{\psi}(\tau)$. □

We will also need the following trace bound for $\mathbb{1}_{[t-s,t]}$.

Lemma 7.3. *Suppose $a, b \in \mathbb{R}$, $\varepsilon_0 > 0$, $\ell_1, \ell_2 \in \mathbb{R}$, $\mathcal{V}_1 \subset \Psi^{\ell_1}(M)$, and $\mathcal{V}_2 \subset \Psi^{\ell_2}_\delta(M)$ bounded subsets, $U \subset T^*M$ open such that $|d\pi_M H_p| > c > 0$ on $p^{-1}([a-\varepsilon_0, b+\varepsilon_0]) \cap U$. Let τ_0,R_0, δ, $R(h)$, and τ be as in Lemma 4.1. Let $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$ be a $(\mathfrak{D}, \tau, R(h))$ good covering of $\mathbf{p}^{-1}([a, b]) \cap N^*\Delta\cap(U \times T^*M)$ and $\mathcal{V} \subset S_\delta(T^*M \times T^*M; [0, 1])$ bounded. Then, there is $C_0 > 0$ such that for all $\{\chi_{\mathcal{T}_j}\}_{j\in\mathcal{J}(h)} \subset \mathcal{V}$ partitions for $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$, $j \in \mathcal{J}(h)$, $A_1 \in \mathcal{V}_1$, $A_2 \in \mathcal{V}_2$, and $|s| \le \varepsilon_0$*

$$\Big|\int_\Delta Op_h(\chi_{\mathcal{T}_j})A_1\mathbb{1}_{[t-s,t]}(P)A_2 d\sigma_\Delta\Big| \le C_0 h^{1-n}R(h)^{2n-1}\Big\langle\frac{s}{h}\Big\rangle.$$

Proof. We first note that it suffices to prove the statement for $|s| \le 2h$. Indeed, this is because we may apply the arguments from Lemma 5.4 and decompose $\mathbb{1}_{[t-s,t]}(P) = \sum_{k=0}^{k_0-1}\mathbb{1}_{[t_k,t_{k+1}]}(P)$, with $|t_{k+1} - t_k| \le 2h$. This allows us to obtain the result for $|s| \le \varepsilon_0$.

Suppose $|s| \le 2h$. Let $\tilde{U} \supset B(U, 2R(h))$, $j \in \mathcal{J}(h)$, and $A := Op_h(\chi_{\mathcal{T}_j})(A_1 \otimes A_2)$. Note that

$$[\mathbf{P_t}, A] = [\mathbf{P_t}, Op_h(\chi_{\mathcal{T}_j})](A_1 \otimes A_2) + Op_h(\chi_{\mathcal{T}_j})[P - t, A_1] \otimes A_2 \in \Psi_\delta(M) \tag{7.5}$$

with seminorms bounded by those of $\chi_{\mathcal{T}_j}$, A_1, and A_2. We next apply Lemma 4.1 with $A := Op_h(\chi_{\mathcal{T}_j})(A_1 \otimes A_2)$, $\mathbf{P_t}$ in place of P_t, $k = n$, $M \times M$ in place of M, and $u := \mathbb{1}_{[t-s,t]}(P)Op_h(\chi_{\tilde{U}})$, where the latter is viewed as a kernel on $M\times M$. Here, $\chi_{\tilde{U}} \in S_\delta(T^*M)$ with $\chi_{\tilde{U}} \equiv 1$ on $B(U, R(h))$, $\operatorname{supp}\chi_{\tilde{U}} \subset \tilde{U}$. Let $\tilde{\chi}_{\mathcal{T}_j} \in \mathcal{V}$ with $\operatorname{supp}\tilde{\chi}_{\mathcal{T}_j} \subset \mathcal{T}_j$ and $\tilde{\chi}_{\mathcal{T}_j} \equiv 1$ on $\operatorname{supp}\chi_{\mathcal{T}_j}$. Then, since $\mathrm{MS_h}(A) \subset \mathcal{T}_j$, by Lemma 4.1 there exist $C_0 > 0$ and $C > 0$, such that

$$h^{\frac{n-1}{2}}\Big|\int_\Delta Op_h(\chi_{\mathcal{T}_j})A_1\mathbb{1}_{[t-s,t]}(P)A_2 d\sigma_\Delta\Big| \le C_0R(h)^{\frac{2n-1}{2}}\Big(\|Op_h(\tilde{\chi}_{\mathcal{T}_j})u\|_{L^2(M)} + \frac{C}{h}\|Op_h(\tilde{\chi}_{\mathcal{T}_j})\mathbf{P}_t u\|_{L^2(M)}\Big).$$

Note that we omit the analogous error terms appearing in the estimate of Lemma 4.1 since these error terms can be dealt with by applying the bounds in (5.7) and (5.8) in combination with (7.1).

Next, since $Op_h(\tilde{\chi}_{\mathcal{T}_j}) = Op_h(\tilde{\chi}_{\tilde{\mathcal{T}}_j})\otimes Op_h(\tilde{\chi}_{\rho_j})$, where $\tilde{\chi}_{\rho_j}$ and $\tilde{\chi}_{\tilde{\mathcal{T}}_j}$ are bounded in $S_\delta(T^*M; [0, 1])$ by the seminorms in the set $\mathcal{V}$, we obtain

$$\begin{aligned}&h^{\frac{n-1}{2}}R(h)^{-\frac{2n-1}{2}}\Big|\int_\Delta Op_h(\chi_{\mathcal{T}_j})A_1\mathbb{1}_{[t-s,t]}(P)A_2 d\sigma_\Delta\Big|\\ &\le C_0\|Op_h(\tilde{\chi}_{\tilde{\mathcal{T}}_j})uOp_h(\tilde{\chi}_{\rho_j})\|_{HS} + C_0Ch^{-1}\|Op_h(\tilde{\chi}_{\tilde{\mathcal{T}}_j})P_tuOp_h(\tilde{\chi}_{\rho_j})\|_{HS} \le C_0h^{\frac{1-n}{2}}R(h)^{\frac{2n-1}{2}},\end{aligned}$$

where u is now viewed as an operator. In the last line we used Lemma 7.2 and the existence of $C > 0$ such that $\mu_t\Big((\operatorname{supp}\tilde{\chi}_{\rho_j}) \cap p^{-1}(t)\Big) \le CR(h)^{2n-1}$. This finishes the proof when $|s| \le 2h$. □

Lemma 7.4. *Let a, b, ε_0, τ_0, $\mathcal{V}_1, \mathcal{V}_2$ R_0, τ, δ, $R(h)$ and α as in Lemma 4.4. Let $N^*\Delta \cap (U \times T^*M)$ be $\mathbf{T}$ non-looping for $\mathbf{p}$ in the window $[a, b]$ via τ-coverings and let $C_{\rm np}$ be the constant $C_{\rm nl}$ in*

Definition 2.1. Then, there is $C_0 = C_0(n, P, \mathcal{V}_1, \mathcal{V}_2, C_{\rm np}, \varepsilon_0) > 0$ *and for all* $K > 0$ *there is* $h_0 > 0$ *such that for all* $0 < h \leq h_0$, $A_1 \in \mathcal{V}_1$, $A_2 \in \mathcal{V}_2$ *with* $\mathrm{MS_h}(A_2) \subset U$, $|s| \leq \varepsilon_0$, *and* $t \in [a - Kh, b + Kh]$,

$$h^{n-1}\Big|\int_\Delta A_1 \mathbb{1}_{[t-s,t]}(P) A_2 d\sigma_\Delta\Big|^2 \leq C_0 \frac{1}{T(h)}\Big\langle\frac{T(h)s}{h}\Big\rangle \|\mathbb{1}_{[t-s,t]}(P) Op_h(\chi_{\tilde{U}})\|^2_{L^2},$$

where $\tilde{U}(h) \supset B(U, 2R(h))$, $\chi_{\tilde{U}} \in S_\delta$, $\chi_{\tilde{U}} \equiv 1$ *on* $B(U, R(h))$, *and* $\operatorname{supp} \chi_{\tilde{U}} \subset \tilde{U}$.

Proof. Decomposing $\mathbb{1}_{[t-s,t]}(P) = \sum_{k=0}^{k_0-1} \mathbb{1}_{[t_k,t_{k+1}]}(P)$, with $|t_{k+1} - t_k| \leq 2h$ and using the proof of Lemma 5.4 to obtain the result for $|s| \leq \varepsilon_0$, it suffices to prove the statement for $|s| \leq 2h$.

From now on we assume $|s| \leq 2h$. Since $N^*\Delta \cap (U \times T^*M)$ is $\mathbf{T}$ non-looping in the window $[a, b]$ via τ_0-coverings, for all $t \in [a - Kh, b + Kh]$, there is a partition of indices $\mathcal{J}_t(h) = \mathcal{G}_{t,0}(h) \sqcup \mathcal{G}_{t,1}(h)$ as described in Definition 2.1 (with $H = \Delta$). Let $t_0 = t_0$, $t_1 = 1$, $T_0(h) = T(h)$ and $T_1(h) = 1$. Then, there is $C_{\rm np} > 0$ such that for all $t \in [a - Kh, b + Kh]$

$$\sum_{\ell=0}^{1} \sqrt{\frac{|\mathcal{G}_{t,\ell}(h)| t_\ell}{T_\ell}} \leq \frac{C_{\rm np} R(h)^{\frac{1-2n}{2}}}{\sqrt{T(h)}}, \qquad \sum_{\ell=0}^{1} \sqrt{|\mathcal{G}_{t,\ell}(h)| t_\ell T_\ell} \leq C_{\rm np} R(h)^{\frac{1-2n}{2}} \sqrt{T(h)}. \tag{7.6}$$

Next, we argue as in (5.5), and then apply a combination of Lemma 4.3 and Lemma 4.4 with $A := A_1 \otimes A_2$, $\mathbf{P_t}$ in place of P_E, $2n$ in place of n, $M \times M$ in place of M, $k = n$, and $u := \mathbb{1}_{[t-s,t]}(P) Op_h(\chi_{\tilde{U}})$, where u is viewed as a kernel on $M \times M$. Then, there is $C_0 > 0$ so that

$$h^{\frac{n-1}{2}}\Big|\int_\Delta A_1 \mathbb{1}_{[t-s,t]}(P) A_2 d\sigma_\Delta\Big| \leq C_0 R(h)^{\frac{2n-1}{2}} \Big(\sum_{\ell=0}^{1} \frac{(|\mathcal{G}_{t,\ell}(h)|\tilde{t}_\ell)^{\frac12}}{(\tau \tilde{T}_\ell)^{\frac12}} \|u\|_{L^2} + \sum_\ell \frac{(|\mathcal{G}_{t,\ell}(h)|\tilde{t}_\ell \tilde{T}_\ell)^{\frac12}}{h} \|\mathbf{P}_t u\|_{L^2}\Big),$$

where $\tilde{t}_\ell$ and $\tilde{T}_\ell$ are as in (5.4). We have used that, since $\mathrm{MS_h}(A) \subset U \times T^*M$ and the tubes are a covering for $\mathbf{p}^{-1}([a, b]) \cap N^*\Delta \cap (U \times T^*M)$, then $\mathrm{MS_h}(A) \cap \Lambda^\tau_{\Sigma_t^\Delta}(R(h)/2) \subset \bigcup_{j \in \mathcal{J}_t(h)} \mathcal{T}_j$. Also, note that we omit the analogous error terms appearing in the estimate of Lemma 4.4 since these error terms can be dealt with by applying the bounds in (5.7) and (5.8) in combination with (7.1).

The proof follows from applying the bounds in (5.5) in combination with (7.1). □

Lemma 7.5. *Let* $\ell_i \in \mathbb{R}$, $\mathcal{V}_i \subset \Psi^{\ell_i}_\delta(M)$ *bounded for* $i = 1, 2$. *Then, there are* $N_0 > 0$, $C > 0$, $h_0 > 0$ *such that for all* $A_1 \in \mathcal{V}_1$ *and* $A_2 \in \mathcal{V}_2$, $s \in \mathbb{R}$ *and* $0 < h < h_0$

$$\Big|\int A_1 \mathbb{1}_{(-\infty,s]}(P) A_2 d\sigma_\Delta\Big| \leq C h^{-\frac{n}{2}} \langle s\rangle^{N_0} \|\mathbb{1}_{(-\infty,s]}(P)\|_{L^2}.$$

Proof. We apply Lemma 4.5 with $H = \Delta$, $A = A_1 \otimes A_2$, and $u = \mathbb{1}_{(-\infty,s]}(P)$. Then, for $r > \frac{n+2(\ell_1+\ell_2)}{2m}$, there is $C > 0$ such that for all $N > 0$ there is $C_N > 0$ such that

$$h^{\frac{n}{2}}\Big|\int_\Delta A_1 \mathbb{1}_{(-\infty,s]}(P) A_2 d\sigma_\Delta\Big| \leq C(\|\mathbb{1}_{(-\infty,s]}(P)\|_{L^2} + \|\mathbf{P}^r \mathbb{1}_{(-\infty,s]}(P)\|_{L^2}) + C_N h^N \|\mathbb{1}_{(-\infty,s]}(P)\|_{L^2}.$$

It follows from (7.1) that the last term can be bounded by $C(1 + |s|^r)\|\mathbb{1}_{(-\infty,s]}(P)\|_{L^2}$. □

7.1. **Proofs of Theorems 2 and 6.** We claim that for $E\in[a-Kh,b+Kh]$ and $A_1\in\mathcal{V}_1$, and $A_2\in\mathcal{V}_2$ with $\mathrm{MS_h}(A_2)\subset U$,

$$h^{n-1}\Big|\int_\Delta A_1\Big(\mathbb{1}_{(-\infty,E]}(P)-\big(\rho_{h,t_0}*\mathbb{1}_{(-\infty,\cdot]}(P)\big)(E)\Big)A_2d\sigma_\Delta\Big|\leq C_0/T(h). \tag{7.7}$$

We start by showing under the same assumptions that

$$h^{n-1}\Big|\int_\Delta A_1\Big(\big(\rho_{h,T(h)}*\mathbb{1}_{(-\infty,\cdot]}(P)\big)(E)-\mathbb{1}_{(-\infty,E]}(P)\Big)A_2d\sigma_\Delta\Big|\leq C_0/T(h), \tag{7.8}$$

$$h^{n-1}\Big|\int_\Delta A_1\Big(\big(\rho_{h,T(h)}*\mathbb{1}_{(-\infty,\cdot]}(P)\big)(E)-\big(\rho_{h,t_0}*\mathbb{1}_{(-\infty,\cdot]}(P)\big)(E)\Big)A_2d\sigma_\Delta\Big|\leq C_0/T(h). \tag{7.9}$$

for some t_0 independent of h. At the end of the section we will derive Theorems 2 and 6 from (7.7).

7.1.1. *Proof of* (7.8). Let $\tilde{U},U_0\subset T^*M$ with $B(U_0,2R(h))\subset U\subset B(U_0,4R(h))\subset\tilde{U}$. Then, let $\chi_{\tilde{U}},\chi_{U_0},\chi_{\tilde{U}\setminus U_0}\in S_\delta(T^*M;[0,1])$ with $\chi_{\tilde{U}}\equiv1$ on U, $\operatorname{supp}\chi_{\tilde{U}}\subset B(U_0,3R(h))$, $\chi_{U_0}\equiv1$ on $B(U_0,R(h))$, $\operatorname{supp}\chi_{U_0}\subset U$, $\chi_{\tilde{U}\setminus U_0}\equiv1$ on $\operatorname{supp}\chi_{\tilde{U}}(1-\chi_{U_0})$, $\operatorname{supp}\chi_{\tilde{U}\setminus U_0}\subset\tilde{U}\setminus U_0$. By Lemma 7.2 and (1.12) there exists $C_0>0$ such that for $|s|\leq2h$,

$$h^{n-1}\big\|\mathbb{1}_{[t-s,t]}(P)Op_h(\chi_{\tilde{U}\setminus U_0})\big\|_{HS}^2\leq C_0\mu_{p^{-1}(t)}(p^{-1}(t)\cap(\tilde{U}\setminus U_0))\leq C_0C_U/T(h). \tag{7.10}$$

Note that when $U=T^*M$ this is an empty statement. Then, for $|s|\leq2h$, by Lemma 7.4

$$\begin{aligned}&h^{n-1}\operatorname{tr}\Big(\mathbb{1}_{[t-s,t]}(P)Op_h(\chi_{U_0})\Big)^2\Big(\frac{1}{T(h)}\Big\langle\frac{T(h)s}{h}\Big\rangle\Big)^{-1}\leq C_0\|\mathbb{1}_{[t-s,t]}(P)Op_h(\chi_{\tilde{U}})\|_{L^2}^2\\&\leq C_0\operatorname{tr}\mathbb{1}_{[t-s,t]}(P)Op_h(\chi_{U_0})+C_0\|\mathbb{1}_{[t-s,t]}(P)Op_h(\chi_{\tilde{U}\setminus U_0})\|_{HS}^2+C_Nh^N.\end{aligned}$$

Then, applying the quadratic formula with $x=\operatorname{tr}\mathbb{1}_{[t-s,t]}(P)Op_h(\chi_{U_0})$, for $|s|\leq2h$ we have

$$0\leq h^{n-1}\operatorname{tr}\mathbb{1}_{[t-s,t]}(P)Op_h(\chi_{U_0})\leq\frac{C_0}{T(h)}\Big\langle\frac{T(h)s}{h}\Big\rangle+\frac{C_UC_0}{T(h)}+c_Nh^N.$$

Next, for $|s|\leq\varepsilon_0$, splitting $\mathbb{1}_{[t-s,t]}(P)=\sum_{k=0}^{k_0-1}\mathbb{1}_{[t_k,t_{k+1}]}(P)$ as before, we have by Lemma 7.4 and Lemma 7.5 that there exists $N_0>0$ such that

$$h^{n-1}\Big|\int_\Delta A_1\mathbb{1}_{[t-s,t]}(P)A_2d\sigma_\Delta\Big|\leq C_0\frac{1}{T(h)}\Big\langle\frac{T(h)s}{h}\Big\rangle, \tag{7.11}$$

$$h^{\frac{n}{2}}\Big|\int_\Delta A_1\mathbb{1}_{(-\infty,s]}(P)A_2d\sigma_\Delta\Big|\leq C\langle s\rangle^{N_0}\|\mathbb{1}_{(-\infty,s]}(P)\|_{L^2}\leq Ch^{-\frac{n}{2}}(1+|s|^{2N_0}), \tag{7.12}$$

where to get the last inequality, we use Lemma 7.5 with $U=M$, $A_1=A_2=\mathrm{Id}$.

In particular, combining (7.11) and (7.12) together with Lemma 5.3 implies (7.8) holds.

7.1.2. *Proof of* (7.9). Using Lemma 6.3, the proof of (7.9) amounts to understanding

$$A_1\big((\rho_{h,\widetilde{T}(h)}-\rho_{h,t_0})*\mathbb{1}_{(-\infty,\,\cdot\,]}(P)\big)(E)A_2=A_1f_{t_0,\widetilde{T}(h),h}(P_E)A_2+O(h^\infty)_{H^{-N}_{\rm scl}\to H^N_{\rm scl}},$$

where $f_{S,T,h}$ is given by (6.2), and $\tilde{T}(h)=\frac{T(h)}{2}$. In particular, for $E\in[a-Kh,b+Kh]$, we consider $\operatorname{tr}A_1f_{t_0,\tilde{T},h}(P_E)A_2$. For this, we let $\{\mathcal{T}_j\}_{j\in\mathcal{J}(h)}$ be a $(\mathfrak{D},\tau,R(h))$-good covering of $\mathbf{p}^{-1}([a,b])\cap N^*\Delta\cap(U\times T^*M)$ and $\mathcal{V}\subset S_\delta(T^*M\times M;[0,1])$ a bounded subset. Let $\{\chi_{\mathcal{T}_j}\}_{j\in\mathcal{J}(h)}\subset\mathcal{V}$ be a partition associated to $\{\mathcal{T}_j\}_{j\in\mathcal{J}_E(h)}$.

Lemma 7.6. *Let $\mathcal{I}\subset\mathcal{J}_E(h)$, $\mathcal{V}_1\subset\Psi^{\ell_1}(M)$, $\mathcal{V}_2\subset\Psi^{\ell_2}_\delta(M)$ bounded subsets. Then, there exist $C_0>0$ and $h_0>0$ such that for all $A_1\in\mathcal{V}_1$, $A_2\in\mathcal{V}_2$, $0<h<h_0$*

$$\Big|\int_\Delta\sum_{j\in\mathcal{I}}Op_h(\chi_{\mathcal{T}_j})A_1f_{t_0,\tilde{T},h}(P_E)A_2d\sigma_\Delta\Big|\le C_0h^{1-n}R(h)^{2n-1}|\mathcal{I}|.$$

Proof. We first note that $f_{t_0,\widetilde{T}(h),h}(P_E)=\varrho_h*\partial_s\mathbb{1}_{(-\infty,\,\cdot\,]}(P)(E)$, where $\varrho_h(s):=f_{t_0,\widetilde{T}(h),h}(-s)$. Then, since $\widehat{f_{t_0,\widetilde{T}(h)}}(0)=0$, we have $\int_{\mathbb{R}}\partial_s\varrho_h(s)ds=0$. In particular, by the estimates (6.3), Lemma 5.3 applies with $\sigma_h=h^{-1}$. Note that by Lemma 7.3, for $t\in[a-Kh,b+Kh]$, and $|s|\le1$,

$$\Big|\int_\Delta Op_h(\chi_{\mathcal{T}_j})A_1(\mathbb{1}_{(-\infty,t]}-\mathbb{1}_{(-\infty,t-s]})A_2d\sigma_\Delta\Big|\le Ch^{1-n}R(h)^{2n-1}\Big\langle\frac{s}{h}\Big\rangle. \tag{7.13}$$

Also, by Lemma 7.5, there exists N_0 such that for $s\in\mathbb{R}$,

$$\Big|\int_\Delta Op_h(\chi_{\mathcal{T}_j})A_1\mathbb{1}_{(-\infty,s]}(P)A_2d\sigma_\Delta\Big|\le Ch^{-n}\langle s\rangle^{N_0}. \tag{7.14}$$

The proof follows from Lemma 5.3 using (7.13) and (7.14), and by summing in $j\in\mathcal{I}$. □

Lemma 7.7. *Let $\mathcal{V}_1,\mathcal{V}_2$ as in Lemma 7.6 and suppose $\mathcal{T}_j$ is a tube such that $\tilde{\mathcal{T}}_j$, its corresponding tube in T^*M, satisfies $\varphi_t(\tilde{\mathcal{T}}_j)\cap\tilde{\mathcal{T}}_j=\emptyset$ for $|t|\in[t_0,T(h)]$. Then for all $N>0$ there is $C_N>0$ such that for all $A_1\in\mathcal{V}_1$, and $A_2\in\mathcal{V}_2$,*

$$\Big|\int_\Delta Op_h(\chi_{\mathcal{T}_j})A_1f_{t_0,\tilde{T},h}(P_E)A_2d\sigma_\Delta\Big|\le C_Nh^N.$$

Proof. Note that the assumption on $\tilde{\mathcal{T}}_j$ implies $\exp(tH_{\mathbf{p}})(\mathcal{T}_j)\cap N^*\Delta=\emptyset$ for $|t|\in[t_0,T(h)]$. Therefore, the same application of Egorov's theorem as in Lemma 6.4, completes the proof. □

Since U is $\mathbf{T}$ non-periodic in the window $[a,b]$ via τ-coverings, for all $E\in[a-Kh,b+Kh]$, there is a splitting $\mathcal{J}_E(h)=\mathcal{B}_E(h)\cup\mathcal{G}_E(h)$ such that $\varphi_t(\tilde{\mathcal{T}}_j)\cap\tilde{\mathcal{T}}_j=\emptyset$ for $|t|\in[t_0,T(h)]$ for $j\in\mathcal{G}_E(h)$, and $|\mathcal{B}_E(h)|R(h)^{2n-1}\le T^{-1}(h)$. We write, using $\mathrm{MS_h}(A_1\otimes A_2)\cap\Lambda^\tau_{\Sigma^\Delta_t}(R(h)/2)\subset\bigcup_{j\in\mathcal{J}_{h,E}}\mathcal{T}_j$,

$$\int_\Delta A_1f_{t_0,\tilde{T},h}(P_E)A_2d\sigma_\Delta=\sum_{j\in\mathcal{G}_E(h)\cup\mathcal{B}_E(h)}\int_\Delta Op_h(\chi_{\mathcal{T}_j})A_1f_{t_0,\tilde{T},h}(P_E)A_2d\sigma_\Delta+O(h^\infty).$$

Applying Lemma 7.7 to the sum over $\mathcal{G}_E(h)$ and Lemma 7.6 to the sum over $\mathcal{B}_E(h)$, we have

$$\Big|\int_\Delta A_1f_{t_0,\tilde{T},h}(P_E)A_2d\sigma_\Delta\Big|\le Ch^{1-n}|\mathcal{B}_E(h)|R(h)^{2n-1}+O(h^\infty)\le C/T(h)$$

for any $E \in [a - Kh, b + Kh]$. In particular (7.9) holds.

7.1.3. *Completion of the proof of Theorem 6.* In order to complete the proof of Theorem 6, we take $A_1 = \mathrm{Id}$ and $A_2 = A^t$ and apply (7.7) to obtain the theorem. □

7.1.4. *Proof of Theorem 2.* We assume $W \subset M$ is $\mathbf{T}$ non-periodic and let $P = Q$ as in (2.13). Then $|d\pi_M H_p| > c > 0$ on $|\xi|_g > \frac{1}{2} > 0$ so we may apply (7.7) for $E > \frac{1}{2}$. Let $0 < \delta < \frac{1}{2}$. Let $\chi_h \in C_c^\infty(M)$ as in [9, (19)] i.e. such that $\chi_h \equiv 1$ in a neighborhood of ∂W, $\operatorname{supp} \chi_h \subset \{d(x, \partial W) < 2h^\delta\}$, $|\partial_x^\alpha \chi| \le C_\alpha h^{-|\alpha|\delta}$, $\mathrm{vol}_M(\operatorname{supp} \chi_h) \le Ch^{\delta(n - \dim_{\mathrm{box}} \partial W)}$.

Let $R(h) \ge h^\delta$, and $T(h) = \mathbf{T}(R(h))$. Then, put $A_1 = 1$ and $A_2 = (1 - \chi_h)1_W$ in (7.7) to obtain

$$\Big| \int_\Delta \Big(\mathbb{1}_{(-\infty,1]}(P) - \rho_{h,t_0} * \mathbb{1}_{(-\infty,\cdot]}(P)(1) \Big)(1 - \chi_h) 1_W d\sigma_\Delta \Big| \le C_0 h^{1-n}/T(h).$$

Next, since $\rho_{h,t_0} * \mathbb{1}_{(-\infty,\cdot]}(P)(1)(x,x) = \frac{\mathrm{vol}_{\mathbb{R}^n}(B^n)}{(2\pi h)^n} + O(h^{-n+2})$ (apply Theorem 3 with $\mathbf{T} = 1$),

$$\Big| \int_W (1 - \chi_h(x)) \Big(\Pi_h(1, x, x) - (2\pi h)^{-n} \mathrm{vol}_{\mathbb{R}^n}(B^n) \Big) \, \mathrm{dv}_g(x) \Big| \le C_0 h^{1-n}/T(h).$$

Also, since $\Pi_h(1, x, x) = (2\pi h)^{-n} \mathrm{vol}_{\mathbb{R}^n}(B^n)| = O(h^{1-n})$ (apply Theorem 3 with $\mathbf{T} = \operatorname{inj} M$),

$$\Big| \int_W \chi_h(x) \Big(\Pi_h(1, x, x) - (2\pi h)^{-n} \mathrm{vol}_{\mathbb{R}^n}(B^n) \Big) \, \mathrm{dv}_g(x) \Big| \le C h^{1-n+\delta(n - \dim_{\mathrm{box}}(\partial W))},$$

where we used $\mathrm{vol}(\operatorname{supp} \chi_h) \le h^{\delta(n - \dim_{\mathrm{box}}(\partial W))}$. In particular,

$$\Big| \int_W \Pi_h(1, x, x) \, \mathrm{dv}_g(x) - (2\pi h)^{-n} \mathrm{vol}_{\mathbb{R}^n}(B^n) \, \mathrm{vol}_M(W) \Big| \le C h^{1-n} \Big(T(h)^{-1} + C h^{\delta(n - \dim_{\mathrm{box}} \partial W)} \Big). \quad \square$$

Appendix A. Index of Notation

In general we denote points in T^*M by ρ. When position and momentum need to be distinguished we write $\rho = (x, \xi)$ for $x \in M$ and $\xi \in T_x^*M$. The natural projection is $\pi_M : T^*M \to M$. Sets of indices are denoted in calligraphic font (e.g., $\mathcal{J}$). Next, we list symbols that are used repeatedly in the text along with the location where they are first defined.

ρ_σ	(1.7)	$E_{H_1,H_2}^{A_1,A_2}$	(1.17)	$\mathcal{K}_\alpha$	(2.8)
$E_\lambda^{t_0}$	(1.8)	$\Lambda_A^\tau(r)$	(2.2)	$\lvert H_p r_H \rvert$	(2.10)
$\Lambda_{\max}$	(1.11)	$\mathcal{Z}$	(2.1)	$\mathfrak{I}_0$	(2.11)
$T_e(h)$	(1.11)	τ_{inj}	(2.3)	$\rho_{h,T}$	(1.16)
$\Sigma_{[a,b]}$	(1.14)	$\mathcal{J}_E(h)$	(2.5)	P_E	(4.2)

For $U \subset V \subset T^*M$ we write $B_V(U, R) = \{\rho \in V : d(U, \rho) < R\}$ and $B(U, R) = B_{T^*M}(U, R)$. For $A \subset T^*M$ we write μ_A for the Liouville measure induced on A. The injectivity radius of M is denoted by $\operatorname{inj} M$. For the definitions of the semiclassical objects $\Psi^\ell(M)$, $\Psi_\delta^\ell(M)$, $S^\ell(T^*M)$, $S_\delta^\ell(T^*M)$, $\mathrm{WF_h}$, $\mathrm{MS_h}$, $H_{\mathrm{scl}}^N(M)$, we refer the reader to [11, Appendix A.2]. For the definition of $[t, T]$ non-self looping, see (2.6), that of $(\mathfrak{D}, \tau, r)$ good covers, see (2.4). Non-periodic, non-looping, and non-recurrent are defined in Definitions 1.7, 1.10, and 1.11 respectively. For non-looping via coverings and non-recurrent via coverings, see Definitions 2.1 and 2.2.

Appendix B. Examples

In this section, we verify our dynamical conditions in some concrete examples (some of which are displayed in Tables 1 and 2). In particular, we verify that certain subsets of manifolds are non-periodic (see Definition 1.2), that various pairs of submanifolds (H_1, H_2) are non-looping (see Definition 1.3), and that certain submanifolds are non-recurrent either via coverings (see Definition 2.2) or simply non-recurrent (see Definition 1.5). Recall also that if (H_1, H_1) is a non-looping pair, then H_1 is non-looping and hence also non-recurrent. Once these conditions are verified, one can directly apply the relevant theorems (Theorem 2, 3, 4, and 5).

B.1. **Manifolds without conjugate points and generalizations.** Let Ξ denote the collection of maximal unit speed geodesics for (M, g). For m a positive integer, $R > 0$, $T \in \mathbb{R}$, and $x \in M$ define

$$\Xi_x^{m,R,T} := \big\{\gamma \in \Xi : \gamma(0) = x,\ \exists \text{ at least } m \text{ conjugate points to } x \text{ in } \gamma(T-R, T+R)\big\},$$

where we count conjugate points with multiplicity. Next, for a set $W \subset M$ write

$$\mathcal{C}_W^{m,R,T} := \bigcup_{x\in W} \{\gamma(T) : \gamma \in \Xi_x^{m,R,T}\}.$$

Note that if $\mathbf{T}(R) \to \infty$ as $R \to 0^+$, then saying $y \in \mathcal{C}_x^{n-1,R,\mathbf{T}(R)}$ for R small indicates that x behaves like a point that is maximally conjugate to y. Note that if (M, g) has no conjugate points, then $\mathcal{C}_x^{m,r,T} = \emptyset$ for all $x \in M$ and $r < |T|$.

Lemma B.1.1. *Let $\alpha > 0$, $t_0 > 0$ and $\mathbf{T}(R) = \alpha \log R^{-1}$. Then there are $C_{\rm nl} > 0$ and $c > 0$ such that if $H_1, H_2 \subset M$ of co-dimension k_1, k_2, and*

$$d\big(H_1, \mathcal{C}_{H_2}^{k_1+k_2-n-1,R,\mathbf{T}(R)}\big) > R$$

for all $R < e^{-t_0/\alpha}$, then (H_1, H_2) is a $(t_0, c\log R^{-1})$ non-looping pair with constant $C_{\rm nl}$, for $p(x, \xi) = |\xi|_{g(x)}$.

Proof. By [8, Proposition 2.2, Lemma 4.1] there exist $\tau > 0$, $\delta > 0$, $C_{\rm nl} > 0$, $C > 0$, such that the pair (H_1, H_2) is a $(t_0, T(h))$ non-looping via (τ, h^δ) coverings with constant $C_{\rm nl}$ in the window $[a, b]$ for any $0 < a < b$, where $T(h) = c \log h^{-1}$ for some $c > 0$ depending on (M, g, α). Combining this result with Lemma 3.4 completes the proof. □

Remark B.1.2. We note that [8, Proposition 2.2] was only proved for $H_1 = H_2$. However, the same argument works for the general case.

B.1.1. *Product Manifolds.* Let (M_i, g_i), $i = 1, 2$, be two compact Riemannian manifolds. Let $M = M_1 \times M_2$ endowed with the product metric $g = g_1 \oplus g_2$. By [11, Lemma 1.1] we have $\mathcal{C}_x^{n-1,r,T} = \emptyset$ for $0 < r < |T|$. Therefore, by Lemma B.1.1 for every $\alpha, t_0 > 0$ there is $C_{\rm nl}$ such that every $x \in M$ is $(t_0, \alpha \log R^{-1})$ non-looping with constant $C_{\rm nl}$ for $|\xi|_{g(x)}$. Note that, integrating over M, and using

$$\mu_{S^*M}(A) = \int_M \mu_{S^*_xM}(A \cap S^*_xM)\, \mathrm{dv}_g,$$

this also implies M is $\alpha \log R^{-1}$ non-periodic. We point out that although $\mathcal{C}_x^{n-1,r,T}$ is empty for $0 < r < |T|$, M may, and often does, have conjugate points. For example, this is the case when $M^1 = S^{n_1}$ with $n_1 \geq 2$.

B.1.2. *Flow invariance of non-looping condition.* In this section, we show that non-looping properties of a pair (H_1, H_2) are inherited by their flow-outs $H^t := \pi(\varphi_t(SN^*H))$. Note, for example, that a geodesic sphere is given by H^t when $H = \{x\}$ is a point for some $t > 0$.

Lemma B.1.3. *Suppose (H_1, H_2) is a $(t_0, \mathbf{T})$ non-looping pair. Then, for all $s, t \in \mathbb{R}$ there exists $C > 0$ such that (H_1^t, H_2^s) is a $(t_0+|t|+|s|, \tilde{\mathbf{T}})$ non-looping pair where $\tilde{\mathbf{T}}(R) = \mathbf{T}(CR) - (|t|+|s|)$.*

Proof. First, note that $SN^*H_j^t = \varphi_t(SN^*H_j) \cup \varphi_{-t}(SN^*H_j)$ for $j = 1, 2$. Let $T > 0$ and suppose $\rho \in B(\mathcal{L}^{R,1}_{H_1^t, H_2^s}(t_0, T), R)$. Then, there is $q_1 \in \mathcal{L}^{R,1}_{H_1^t, H_2^s}(t_0, T)$ such that $d(q_1, \rho) < R$. In particular, there are $q_2 \in T^*M$ and $t_0 \leq |t_1| \leq T$ such that $d(q_1, q_2) < R$ and $d(\varphi_{t_1}(q_2), SN^*H_2^s) < R$.

Now, either $\varphi_{-t}(q_1) \in SN^*H_1$ or $\varphi_t(q_1) \in SN^*H_1$. We consider the case $\varphi_t(q_1) \in SN^*H_1$, the other begin similar. Then, there exist $C_t, C_s > 0$ such that

$$d(\varphi_t(q_1), \varphi_t(q_2)) < C_t R, \qquad d(\varphi_{-t+t_1 \pm s} \circ \varphi_t(q_2), SN^*H_2) < C_s R.$$

In particular, letting $C = \max(C_t, C_s)$, $\varphi_t(q_1) \in \mathcal{L}^{CR}_{H_1,H_2}(t_0 + |t| + |s|, T - (|t| + |s|))$, and, since $d(\varphi_t(\rho), \varphi_t(q_1)) < CR$,

$$\varphi_t(\rho) \in B\big(\mathcal{L}^{CR,1}_{H_1,H_2}(t_0 + |t| + |s|, T - (|t| + |s|)), CR\big).$$

Repeating this argument when $\varphi_{-t}(q_1) \in SN^*H_1$, we obtain

$$B_{SN^*H_1^t}(\mathcal{L}^{R,1}_{H_1^t,H_2^s}(t_0, T), R) \subset \bigcup_{\pm} \varphi_{\pm t}\big(B_{SN^*H_1}(\mathcal{L}^{CR,1}_{H_1,H_2}(t_0 + |t| + |s|, T - (|t| + |s|)), CR)\big).$$

In particular, there is $C > 0$ such that

$$\mu_{SN^*H_1^t}\Big(B_{SN^*H_1^t}(\mathcal{L}^{R,1}_{H_1^t,H_2^s}(t_0, T), R)\Big) \leq \sum_{\pm} C\mu_{SN^*H_1}\Big(B_{SN^*H_1}(\mathcal{L}^{CR,1}_{H_1,H_2}(t_0 + |t| + |s|, T - (|t| + |s|)), CR)\Big).$$

Therefore, since (H_1, H_2) is a $(t_0, \mathbf{T})$ non-looping pair, (H_1^t, H_2^s) is a $(t_0 + |t| + |s|, \tilde{\mathbf{T}})$ non-looping pair with $\tilde{\mathbf{T}}(R) = \mathbf{T}(CR) - |t| - |s|$. □

Now, by Lemma B.1.1, in the case $d\big(y, \mathcal{C}_x^{n-1,R,\mathbf{T}(R)}\big) > R$, for $R < e^{-t_0/\alpha}$ and $\mathbf{T}(R) = \alpha \log R^{-1}$, we have (x, y) is a $(t_0, c \log R^{-1})$ non-looping pair. Hence, by Lemma B.1.3 that the geodesic spheres generated by x and y form a non-looping pair with resolution function $\mathbf{T}(R) = \tilde{C} \log R^{-1}$ for some $\tilde{C} > 0$.

B.2. **Surfaces of revolution.** Consider $M = S^2$ with the metric a $\iota^* g$ where

$$g(s, \theta) = ds^2 + \alpha^2(s) d\theta^2, \tag{B.1}$$

and $\iota : [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}/2\pi\mathbb{Z} \to S^2$, with $\iota(s, \theta) = (\cos(s)\cos(\theta), \cos(s)\sin(\theta), \sin(s))$. Here, α is a smooth function satisfying $\alpha(\pm\frac{\pi}{2}) = 0$ and $\pm\alpha'(\pm\pi/2) = 1$. This assumption implies g is a smooth Riemannian metric. Furthermore, we assume $-s\alpha'(s) > 0$ for $s \neq 0$ and $\alpha''(0) < 0$. Note that the round sphere is given by $\alpha(s) = \cos(s)$.

For a unit speed geodesic, $t \mapsto (s(t), \theta(t))$ with $(s(0), \theta(0)) = (0, 0)$, $\dot\theta(0) > 0$, $\dot s(0) > 0$, we have by the Clairaut formula (see e.g. [3, Proposition 4.7])

$$\big(\dot s(t)\big)^2 + \alpha^2(s(t))\big(\dot\theta(t)\big)^2 = 1 \qquad \text{and} \qquad \dot\theta(t) = \alpha(s_+)\alpha^{-2}(s(t))$$

where s_+ is the maximal value of s on the geodesic. In particular, putting $t(s_+)$ for the first time when $s(t) = s_+$, we have $s : [0, t(s_+)] \to [0, s_+]$ is invertible,

$$t(s) = \int_0^s \frac{\alpha(w)}{\sqrt{\alpha^2(w) - \alpha^2(s_+)}} dw, \qquad \theta(t(s_+)) = \int_0^{t(s_+)} \frac{\alpha(s_+)}{\alpha^2(s(t))} dt$$

and, changing variables to $w = s(t)$ and using $\dot s(t) = \sqrt{1 - \frac{\alpha^2(s_+)}{\alpha^2(s(t))}}$, we have

$$\theta(t(s_+)) = \int_0^{s_+} \frac{\alpha(s_+)}{\alpha(w)} \frac{1}{\sqrt{\alpha^2(w) - \alpha^2(s_+)}} dw.$$

We then define $\theta_+(s_+) := 2\theta(t(s_+))$. If we instead suppose $\dot\theta > 0$ and $\dot s < 0$, we can define $\theta_-(s_-)$ analogously where s_- is the minimal s value on the trajectory. Now, there is a smooth function

$$s_- : [0, \pi/2] \to [-\pi/2, 0]$$

such that if s_+ is the maximal s value of a trajectory, then $s_-(s_+)$ is the minimal s value. Moreover, $\partial_{s_+} s_- < 0$.

Finally, note that for a trajectory with maximal s value s_+, $s(0) = 0$, $\dot s \neq 0$, if T is the second return time to $s(0) = 0$, then

$$\Theta_0(s_+) = \theta(T) - \theta(0), \qquad \Theta_0(s_+) := \theta_+(s_+) + \theta_-(s_-(s_+)).$$

Note that apriori, $\theta(T) - \theta(0)$ could depend on the precise geodesic whose maximal s value is s_+. However, the integrable torus, $\mathbb{T}_{s_+}$, consisting of all such geodesics has the same $\theta(T) - \theta(0)$ up to sign.

In the next lemmas, we reduce the study of dynamical properties on (M, g) to the Poincaré section $\{s(0) = 0, \dot s(0) > 0\} \subset TM$. The function $\Theta_0 : (0, \pi/2] \to \mathbb{R}$ is the change in θ after a return to the Poincaré section. In particular, $\mathbb{T}_{s_+}$ is a periodic torus (i.e. all its trajectories are periodic) if and only if for some $p, q \in \mathbb{Z}$, $q \neq 0$,

$$\Theta_0(s_+) = 2\pi\, p/q.$$

Lemma B.2.1. *Suppose there exists $b > 0$ such that*

$$\partial_{s_+}\Theta_0(s_+) \neq 0, \qquad s_+ \geq b.$$

Then, there are $C_{\rm np}, c > 0$ such that every subset $U \subset \{s > b\} \cup \{s < s_-(b)\}$ is $\mathbf{T}$ non-periodic for $\mathbf{T}(R) = cR^{-1/3}$ with constant $C_{\rm np}$.

Proof. Suppose $\rho \in S^*M$ with $s_+(\rho) > b$, and let $t \in \mathbb{R}$ be such that

$$\varphi_t(B_{S^*M}(\rho, R)) \cap B_{S^*M}(\rho, R) \neq \emptyset. \tag{B.2}$$

Then, there is $|t_1| \leq R$ such that $d(\varphi_{t+t_1}(\rho), \rho) < (1 + C(|t| + |t_1|))R$. Now, for some $0 \leq t_2 \leq c$, we have $s(\varphi_{t_2}(\rho)) = 0$ and

$$d(\varphi_{t+t_1+t_2}(\rho), \varphi_{t_2}(\rho)) < (1 + C(|t| + |t_1| + t_2))R.$$

Let s_+ be the maximal s value for the trajectory through ρ. Then, there are $p, q \in \mathbb{Z}$ with $|p|, |q| \leq C(1 + |t|)$, $|q| \geq c(1 + |t|)$ such that

$$\Big|\Theta_0(s_+) - 2\pi\, p/q\Big| < C(1 + C(|t| + |t_1| + t_2))R/q \leq CR. \tag{B.3}$$

We have shown that if $\rho \in S^*M$ is such that (B.2) holds, then $\rho \in \bigcup_{s_+\in A(t)} \mathbb{T}_{s_+}$, where

$$A(t) := \{s_+ \in (b, \tfrac{\pi}{2}] :\ \exists p, q \in \mathbb{Z},\ |p|, |q| \leq C(1 + |t|),\ \text{(B.3) holds}, \}.$$

Next, we claim

$$|A(t)| \leq C(1 + |t|)^2 R. \tag{B.4}$$

Indeed, $\#\{r \in [0, 1] :\ \exists p, q \in \mathbb{Z},\ r = p/q,\quad |p|, |q| \leq C(1 + |t|)\} \leq C(1 + |t|)^2$ and hence, the volume of possible values of $\Theta_0(s_+)$ such that (B.3) holds is bounded by $C(1 + |t|)^2 R$. The claim in (B.4) then follows from the assumption $\partial_{s_+}\Theta_0(s_+) \neq 0$ on $s_+ \geq b$.

Our next goal is to show that the bound in (B.4) translates to a bound on the set of ρ with (B.2). To see this, note that $\mathbb{T}_{s_+} = \{|\xi_\theta| = \alpha(s_+)\} \cap S^*M$ where we work in the cotangent bundle with coordinates $(s, \theta, \xi_s, \xi_\theta)$. Therefore, when $\alpha(s_+) < \alpha(s_0)$, the intersection $\mathbb{T}_{s_+} \cap S^*_{(s_0,\theta)}M$ is transversal for any θ. In particular, for any $\varepsilon > 0$ and $s_0 \geq 0$, there exists $C_\varepsilon > 0$ such that for any $A \subset [s_0 + \varepsilon, \pi/2]$

$$\mu_{S^*_{(s_0,\theta)}M}\Big(\bigcup_{s_+\in A} \mathbb{T}_{s_+} \cap S^*_{(s_0,\theta)}M\Big) \leq C_\varepsilon |A|.$$

Moreover, since there is $T > 0$ such that the restriction of the map $(t, q) \mapsto \varphi_t(q)$

$$[-T, T] \times \Big(\bigcup_{\substack{s_+\geq s_0+\varepsilon\\ \theta\in[0,2\pi]}} S^*_{(s_0,\theta)}M \cap \mathbb{T}_{s_+}\Big) \to \bigcup_{s_+\geq s_0+\varepsilon} \mathbb{T}_{s_+}$$

is a surjective local diffeomorphism,

$$\mu_{S^*M}\Big(\bigcup_{s_+\in A} \mathbb{T}_{s_+} \cap S^*M\Big) \leq C_\varepsilon |A|. \tag{B.5}$$

In particular, by (B.4), since $b > 0$, there exists $C_b > 0$ such that

$$\mu_{S^*M}\Big(\bigcup_{s_+\in A(t)} \mathbb{T}_{s_+} \cap S^*M\Big) \leq C_b\, |A(t)| \leq C_b (1 + |t|)^2 R.$$

Hence, for $U \subset \{s > b\} \cup \{s < s_-(b)\}$,

$$\mu_{S^*M}\Big(B_{S^*M}\big(\mathcal{P}^R_U(t_0, \mathbf{T}(R)), R\big)\Big) \leq C(1 + |\mathbf{T}(R)|)^2 R.$$

So, provided $\mathbf{T}(R) \leq R^{-1/3}$, U is $\mathbf{T}(R)$ non-periodic with constant $C_{\rm np} = C/2$. □

Lemma B.2.2. *Suppose x_0 is a pole, and $x_1 = (s_1, \theta_1)$ for $-\pi/2 < s_1 < \pi/2$. Then, there is $C_{\rm nl} > 0$ such that (x_0, x_1) is a $\mathbf{T}(R) = R^{-1}$ non-looping pair.*

Proof. Suppose x_0 is the pole with $s=\pi/2$. Suppose $\rho\in S^*_{x_1}M$ and there exists $\rho_1\in S^*_{x_1}M$ such that $d(\rho,\rho_1)<R$ and $\varphi_t(B(\rho_1,R))\cap B(S^*_{x_0}M,R)\neq\emptyset$. Then, there is $\rho_2\in B(\rho_1,R)$ such that $s_+(\rho_2)>\pi/2-R$. Therefore, there is $C>0$ such that $s_+(\rho)>\pi/2-CR$ and (since $|s_1|<\pi/2$),

$$\mu_{S^*_{x_1}M}\Big(\bigcup_{s_+>\pi/2-CR}\mathbb{T}_{s_+}\cap S^*_{x_1}M\Big)\le CR.$$

In particular, for any $t_0>0$, $T>0$,

$$\mu_{S^*_{x_1}M}\Big(B(\mathcal{L}^{R,1}_{x_1,x_0}(t_0,T),R)\Big)\le CR$$

and hence (x_0,x_1) is a $\mathbf{T}(R)=R^{-1}$ non-looping pair. □

Lemma B.2.3. *Suppose the assumptions of Lemma B.2.1 hold and $x_0=(s_0,\theta_0)$ with $s\in(-\pi/2,s_-(b))\cup(b,\pi/2)$. Then there is $\delta>0$ such that x_0 is $\mathbf{T}(R)=R^{-\delta}$ non-looping.*

Proof. The proof is identical to [11, Lemma 5.1]. □

B.2.1. *Perturbed spheres.* Next, we construct examples which have large (positive measure) periodic sets as well as large non-periodic sets. In particular, we find examples where the assumptions of Lemma B.2.1 hold and such that there is $c>0$ with the property that the flow is periodic on $-c<s<c$. If $s_0>0$, we will call (s_0,θ_0) *aperiodic* if

$$\partial_{s_+}\Theta_0(s_+)\neq 0 \text{ on } \{s_+\ge s_0\}.$$

In the case $s_0<0$, we require the same condition on $\{\alpha(s_+)\le\alpha(s)\}$. We define the *aperiodic set* to be the set of aperiodic points and Theorem 2 holds for any U inside this set.

In order to do this, we make a small perturbation of the round metric ($\alpha(s)=\cos s$). First, we compute

$$\begin{aligned}\partial_{s_+}\theta_+ = 2\alpha'(s_+)\int_a^{s_+}\big[\alpha^2(w)-2\alpha^2(s_+)\big]\frac{2(\alpha'(w))^2+\alpha(w)\alpha''(w)}{\sqrt{\alpha^2(w)-\alpha^2(s_+)}\alpha^3(w)(\alpha'(w))^2}dw\\ -2\alpha'(s_+)\frac{\alpha^2(b)-2\alpha^2(s_+)}{\sqrt{\alpha^2(b)-\alpha^2(s_+)}\alpha^2(b)\alpha'(b)}+2\alpha'(s_+)\int_0^b\frac{\alpha(w)}{(\alpha^2(w)-\alpha^2(s_+))^{3/2}}dw.\end{aligned}$$

Let $0<a<b<\pi/2$ and $\alpha_\varepsilon=\alpha_0+\varepsilon(f_++f_-)$, with $\operatorname{supp} f_+\subset(a,b)$ and $\operatorname{supp} f_-\subset(-\pi/2,0)$. We have for $s_+\ge b$,

$$\partial_\varepsilon\partial_{s_+}\theta_+\Big|_{\varepsilon=0}=-2\alpha_0'(s_+)\int_0^b f_+(w)\frac{2\alpha_0^2(w)+\alpha_0^2(s_+)}{(\alpha_0^2(w)-\alpha_0^2(s_+))^{5/2}}dw.$$

Arguing identically for θ_-, if $\alpha_\varepsilon=\alpha_0+\varepsilon(f_++f_-)$ with $\operatorname{supp} f_-\subset(s_-(b),s_-(a))$ and $\operatorname{supp} f_+\subset(0,\pi/2)$, then

$$\partial_\varepsilon\partial_{s_-}\theta_-\Big|_{\varepsilon=0}=-2\alpha_0'(s_-)\int_{-b}^0 f_-(w)\frac{2\alpha_0^2(w)+\alpha_0^2(s_-)}{(\alpha_0^2(w)-\alpha_0^2(s_-))^{5/2}}dw.$$

To construct an example where the assumptions of Lemma B.2.1 hold, let $\alpha_0(s)=\cos(s)$ so that α_0 induces the standard round metric. Let $0<a<b<\frac{\pi}{2}$, f_+ not identically 0 and $f_+\ge 0$

with $\operatorname{supp} f_+ \subset (a,b)$, and let $f_- \geq 0$ with $\operatorname{supp} f_- \subset (s_-(b), s_-(a))$. Then, we have for $s_+ \geq b$, and $\Theta_{0,\varepsilon}$ corresponding to the perturbed metric with α_ε,

$$\partial_\varepsilon \partial_{s_+}\Big(\Theta_{0,\varepsilon}(s_+)\Big) > 0, \qquad s_+ \geq b.$$

In particular, we may choose $\varepsilon_0 > 0$ small enough such that for $0 < \varepsilon < \varepsilon_0$ and $\alpha = \alpha_\varepsilon$, we have $-s\alpha_\varepsilon'(s) > 0$ when $s \neq 0$, and

$$\partial_{s_+}\Big(\Theta_{0,\varepsilon}(s_+)\Big) > 0, \qquad s_+ \geq b.$$

Moreover, since α_0 is the round metric on the sphere, the flow is periodic for trajectories not leaving $(s_-(a), a)$. (See Figure 1)

B.2.2. *The spherical pendulum.* We now recall the spherical pendulum on S^2 whose Hamiltonian is given in the (s,θ) coordinates by

$$q(s,\theta,\xi_s,\xi_\theta) = \xi_s^2 + \cos^{-2}(s)\xi_\theta^2 + 2\sin s - E.$$

This Hamiltonian describes the movement of a pendulum of mass 1 moving without friction on the surface of a sphere of radius 1. When $E > 2$, up to reparametrization of the integral curves, the dynamics for the spherical pendulum are equivalent to those for the Hamiltonian $p = |\xi|^2_{\iota^* g}$ and g is given by

$$g = (E - 2\sin(s))ds^2 + (E - 2\sin(s))\cos^2(s)d\theta^2.$$

Making a further change of variables in the s variable, we can put the metric in the form (B.1) and, moreover, by [25] for $E \geq \frac{14}{\sqrt{17}}$, $|\partial_{s_+}\Theta_0| > c > 0$ for $s_+ \in (0, \pi/2]$. Note that the failure of this condition at the torus $\mathbb{T}_0$ is due to the fact that this torus is singular, consisting of the two curves $\{s = 0, \theta \in \mathbb{R}/2\pi\mathbb{Z}, \xi_r = 0, |\xi_\theta| = \alpha(0)\}$. In fact, it is easy to see that $|\Theta_0(s_+)| > cs_+^{1/2}$ for s_+ near 0. This, together with Lemmas B.2.1 and [11, Lemma 5.1] are enough to obtain the results in Table 2 and that Theorem 2 applies to the spherical pendulum with $U = M$.

B.3. **Submanifolds of manfiolds with Anosov geodesic flow.** We next recall some examples when (M,g) has Anosov geodesic flow. The geodesic flow is Anosov if there is $\mathbf{B} > 0$ such that for all $\rho \in T^*M$ there is a splitting

$$T_\rho T^*M = E_+(\rho) \oplus E_-(\rho) \oplus \mathbb{R}H_p(\rho)$$

such that

$$|d\varphi_t(\mathbf{v})| \leq \mathbf{B}e^{\mp\frac{t}{\mathbf{B}}}|\mathbf{v}|, \qquad \mathbf{v} \in E_\pm(\rho), \quad t \to \pm\infty,$$

where $|\cdot|$ is the norm induced by a Riemannian metric on T^*M. Here, $E_+(\rho)$ is called the stable space and $E_-(\rho)$, the unstable space.

We also note (see [19, 31]) that a manifold with non-positive sectional curvature has no conjugate points and that

$$\text{negative sectional curvature} \quad \Rightarrow \quad \text{Anosov geodesic flow} \quad \Rightarrow \quad \text{no conjugate points.}$$

Note that these implications are *not* equivalences. Indeed, there exist manifolds with Anosov geodesic flow containing sets with strictly positive sectional curvature as well as manifolds with no conjugate points which do not have Anosov geodesici flow.

One of the main goals of [8] was to prove that various submanifolds of manifolds with the Anosov or non-focal property are non-recurrent via coverings. We will review only some of these results here, referring the reader to [8] for further examples. In what follows we present several dynamical lemmas which yield the statements from Table 2.

Define for a submanifold $H \subset M$, and for every $\rho \in SN^*H$

$$m_\pm(H,\rho) := \dim(E_\pm(\rho) \cap T_\rho SN^*H).$$

Note that in two dimensions $m_\pm(H,\rho) \neq 0$ is equivalent to H being tangent to, and having the same curvature as, a stable/unstable horosphere with conormal ρ. In fact, in any dimension, a generic $H \subset M$ satisfies $m_\pm(H,\rho) = 0$ for all $\rho \in SN^*H$.

Lemma B.3.1. *Let $H \subset M$ be a smooth submanifold . Suppose (M,g) is a manifold with Anosov geodesic flow and for all $\rho \in SN^*H$*

$$m_+(H,\rho) + m_-(H,\rho) < n-1 \qquad \text{or} \qquad m_-(H,\rho)m_+(H,\rho) = 0.$$

Then there are $c,\delta,\tau > 0$ such that for all $0 < a < b$, H is $c\log h^{-1}$ non-recurrent via $(\tau, R(h))$ coverings for the symbol $p(x,\xi) = |\xi|_{g(x)}$ in the window $[a,b]$.

Proof. The proof of this result is that of [8, Theorem 6], see [8, Section 5.1]. □

Lemma B.3.2. *Suppose (M,g) is a manifold with Anosov geodesic flow and $H_1, H_2 \subset M$ are a smooth submanifolds such that for $i = 1,2$, $\sup_{\rho\in SN^*H_i} m_\pm(H_i,\rho) = 0$. Then there are $c, t_0 > 0$ such that for all $0 < a < b$, (H_1,H_2) is a $(t_0, c\log R)$ non-looping pair for $p(x,\xi) = |\xi|_{g(x)}$ in the window $[a,b]$.*

Proof. By [8, Proposition 2.2, Lemma 5.1] (in particular, adapting the arguments in [8, "Treatment of $D \in \{D_i\}_{i\in\mathcal{I}_K}$", page 38]) there exist $\tau > 0$, $\delta > 0$, $C_{\rm nl} > 0$, $C > 0$, such that the pair (H_1,H_2) is a $(t_0, T(h))$ non-looping via (τ, h^δ) coverings with constant $C_{\rm nl}$ in the window $[a,b]$ for any $0 < a < b$, where $T(h) = c\log h^{-1}$ for some $c > 0$ depending on (M,g,α). Combining this result with Lemma 3.4 yields the claim. □

Recall that a stable/unstable horosphere is defined by the property that $T_\rho SN^*H = E_\pm(\rho)$ for all $\rho \in SN^*H$.

Lemma B.3.3. *Suppose (M,g) is a manifold with Anosov geodesic flow, $H_\pm \subset M$ is a compact subset of a stable/unstable horosphere and $H_2 \subset M$ is a submanifold with $m_\pm(H_2,\rho) < n-1$ for all $\rho \in SN^*H_2$. Then, there are $c, t_0 > 0$ such that for all $0 < a < b$, $(H_\pm, H_2)$ is a $(t_0, c\log R)$ non-looping pair for $p(x,\xi) = |\xi|_{g(x)}$ in the window $[a,b]$.*

For simplicity, we prove only Lemma B.3.3 but point out that the arguments similar to those in [8, Lemma 5.1] can be used to obtain much more general statements.

Proof. We consider the case H_+. The other case following identically. By Lemma 3.4 it suffices to show (H_+,H_2) is a non-looping pair via coverings. Thus, by [8, Proposition 2.2] and Lemma 3.4 it suffices to show there exists $\alpha > 0$ such that for all $(t,\rho) \in [t_0,T_0] \times SN^*H_+$ such that $d(\varphi_t(\rho), SN^*H_2) \leq e^{-\alpha|t|}/\alpha$, there exists $\mathbf{w} \in T_\rho SN^*H_+$ for which the restriction

$$d\psi_{(t,\rho)} : \mathbb{R}\partial_t \times \mathbb{R}\mathbf{w} \to T_{\psi(t,\rho)}\mathbb{R}^{n+1}$$

has left inverse $L_{(t,\rho)}$ with $\|L_{(t,\rho)}\| \leq \alpha e^{\alpha|t|}$. Here, $\psi : \mathbb{R} \times SN^*H_+ \to \mathbb{R}^{n+1}$ is given by $\psi(t,\rho) = F \circ \varphi_t(\rho)$ and $F : T^*M \to \mathbb{R}^{n+1}$ is a defining function for $SN^*H_2 = F^{-1}(0)$.

Note that $T_\rho SN^*H_+ = E_+(\rho)$ and there is $\mathbf{D} > 0$ such $d\varphi_t : E_+(\rho) \to E_+(\varphi_t(\rho))$ is invertible with inverse satisfying

$$\|(d\varphi_t)^{-1}\| \leq e^{-\mathbf{D}|t|}/\mathbf{D}.$$

Since H_2 is compact, and $m_+(H_2, q) < n-1$ for all $q \in SN^*H_2$, there is $c > 0$ such that for all $q \in SN^*H_2$ there is $\mathbf{u} \in E_+(q)$ with $|\mathbf{u}| = 1$ such that $|dF\mathbf{u}| \geq c|\mathbf{u}|$.

Since $\rho \mapsto E_+(\rho)$ is ν-Hölder continuous for some $\nu > 0$ [29, Theorem 19.1.6], there is $C_M > 0$ and $\tilde{\mathbf{u}} \in E_+(\tilde{q})$ with

$$d(\tilde{\mathbf{u}}, \mathbf{u}) < C_M d(q, \tilde{q})^\nu, \qquad |\tilde{\mathbf{u}}| = 1.$$

Therefore,

$$|dF\tilde{\mathbf{u}}| \geq (c - Cd(q, \tilde{q})^\nu)|\tilde{\mathbf{u}}|.$$

Let $\tilde{q} = \varphi_t(\rho)$, so that $d(q, \tilde{q}) < e^{-\alpha t}/\alpha$ and set $\mathbf{w} = (d\varphi_t)^{-1}(\tilde{\mathbf{u}})$. The claim follows provided $\alpha > 1$ is large enough (depending on $\mathbf{D}, \nu, c, C$). □

Lemma B.3.4. *Suppose (M, g) has Anosov geodesic flow and non-positive curvature. Then if $H \subset M$ is a totally geodesic submanifold, $m_\pm(H, \rho) \equiv 0$.*

Proof. We need only show that for a totally geodesics submanifold $m_+(H,\rho) = m_-(H,\rho) = 0$. It is easier to work on the tangent space side, so we will do so, denoting $E^\sharp_\pm(\rho^\sharp)$ for the dual stable and unstable bundles.

Suppose $\rho^\sharp \in SNH$. Then, arguing as in [8, Proof of Theorem 4.C], and using that H is totally geodesic, we have for all $v \in T_{\rho^\sharp}SNH$

$$-\langle \tilde{\nabla}_{d\pi v} N, d\pi v\rangle = \langle \rho^\sharp, \Pi_H(d\pi v, d\pi v)\rangle = 0.$$

Here $N : (-\varepsilon, \varepsilon) \to NH$ is a smooth vectorfield with $N(0) = \rho^\sharp$ and $N'(0) = v$, $\tilde{\nabla}$ is the Levi-Civita connection on M, and Π_H is the second fundamental form to H. On the other hand, by [8, (5.46)], for $v_\pm \in E^\sharp_\pm(\rho^\sharp)$,

$$|-\langle \tilde{\nabla}_{d\pi v_\pm} N, d\pi v_\pm\rangle| = |\langle \rho^\sharp, \Pi_{\mathcal{W}_\pm}(d\pi v, d\pi v)\rangle| > 0,$$

where $\mathcal{W}_\pm$ is a stable/unstable horosphere with normal vector $\rho^\sharp$. Therefore, $T_{\rho^\sharp}SNH \cap E^\sharp_\pm(\rho^\sharp) = \emptyset$ and in particular $m_\pm(H,\rho) = 0$. □

References

[1] V. G. Avakumović. Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten. *Math. Z.*, 65:327–344, 1956.

[2] P. H. Bérard. On the wave equation on a compact Riemannian manifold without conjugate points. *Math. Z.*, 155(3):249–276, 1977.

[3] A. L. Besse. *Manifolds all of whose geodesics are closed*, volume 93 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin-New York, 1978. With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan.

[4] Y. Bonthonneau. The Θ function and the Weyl law on manifolds without conjugate points. *Doc. Math.*, 22:1275–1283, 2017.

[5] R. W. Bruggeman. *Fourier coefficients of automorphic forms*, volume 865 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981. Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien [Mathematical Textbooks and Monographs, Part II: Mathematical Monographs], 48.

[6] K. Burns and G. P. Paternain. On the growth of the number of geodesics joining two points. In *International Conference on Dynamical Systems (Montevideo, 1995)*, volume 362 of *Pitman Res. Notes Math. Ser.*, pages 7–20. Longman, Harlow, 1996.

[7] Y. Canzani. Monochromatic random waves for general riemannian manifolds. In *Frontiers in Analysis and Probability*. Springer, 2020.

[8] Y. Canzani and J. Galkowski. Improvements for eigenfunction averages: an application of geodesic beams. *arXiv:1809.06296, to appear in J. Differential Geom.*, 2019.

[9] Y. Canzani and J. Galkowski. On the growth of eigenfunction averages: microlocalization and geometry. *Duke Math. J.*, 168(16):2991–3055, 2019.

[10] Y. Canzani and J. Galkowski. Growth of high L^p norms for eigenfunctions: an application of geodesic beams. *arXiv:2003.04597 to appear in Anal. PDE*, 2020.

[11] Y. Canzani and J. Galkowski. Eigenfunction concentration via geodesic beams. *J. Reine Angew. Math.*, 775:197–257, 2021.

[12] Y. Canzani and J. Galkowski. Logarithmic improvements in the Weyl law and exponential bounds on the number of closed geodesics are predominant. *arXiv:2204.11921*, 2022.

[13] Y. Canzani and B. Hanin. Scaling limit for the kernel of the spectral projector and remainder estimates in the pointwise Weyl law. *Anal. PDE*, 8(7):1707–1731, 2015.

[14] Y. Canzani and B. Hanin. C^∞ scaling asymptotics for the spectral projector of the Laplacian. *J. Geom. Anal.*, 28(1):111–122, 2018.

[15] J. Chazarain. Formule de Poisson pour les variétés riemanniennes. *Invent. Math.*, 24:65–82, 1974.

[16] Y. Colin de Verdière. Spectre du laplacien et longueurs des géodésiques périodiques. II. *Compositio Math.*, 27(2):159–184, 1973.

[17] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29(1):39–79, 1975.

[18] S. Dyatlov and C. Guillarmou. Microlocal limits of plane waves and Eisenstein functions. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(2):371–448, 2014.

[19] P. Eberlein. When is a geodesic flow of Anosov type? I. *Journal of Differential Geometry*, 8:437–463, 1973.

[20] L. Gå rding. On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators. *Math. Scand.*, 1:237–255, 1953.

[21] J. Galkowski. Defect measures of eigenfunctions with maximal L^∞ growth. *Ann. Inst. Fourier (Grenoble)*, 69(4):1757–1798, 2019.

[22] A. Good. *Local analysis of Selberg's trace formula*, volume 1040 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[23] D. A. Hejhal. Sur certaines séries de Dirichlet associées aux géodésiques fermées d'une surface de Riemann compacte. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(8):273–276, 1982.

[24] L. Hörmander. The spectral function of an elliptic operator. *Acta Math.*, 121:193–218, 1968.

[25] E. Horozov. On the isoenergetical nondegeneracy of the spherical pendulum. *Phys. Lett. A*, 173(3):279–283, 1993.

[26] A. Iosevich and E. Wyman. Weyl law improvement for products of spheres. *Anal. Math.*, 47(3):593–612, 2021.

[27] V. J. Ivriĭ. The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary. *Funktsional. Anal. i Prilozhen.*, 14(2):25–34, 1980.

[28] H. Iwaniec. Nonholomorphic modular forms and their applications. In *Modular forms (Durham, 1983)*, Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., pages 157–196. Horwood, Chichester, 1984.

[29] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[30] B. Keeler. A logarithmic improvement in the two-point weyl law for manifolds without conjugate points. *arXiv:1905.05136*, 2019.

[31] W. Klingenberg. Riemannian manifolds with geodesic flow of Anosov type. *Ann. of Math. (2)*, 99:1–13, 1974.

[32] H. Koch, D. Tataru, and M. Zworski. Semiclassical L^p estimates. *Ann. Henri Poincaré*, 8(5):885–916, 2007.
[33] N. V. Kuznecov. The Petersson conjecture for cusp forms of weight zero and the Linnik conjecture. Sums of Kloosterman sums. *Mat. Sb. (N.S.)*, 111(153)(3):334–383, 479, 1980.
[34] B. M. Levitan. On the asymptotic behavior of the spectral function of a self-adjoint differential equation of the second order. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 16:325–352, 1952.
[35] S. Minakshisundaram and A. . Pleijel. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. *Canad. J. Math.*, 1:242–256, 1949.
[36] Y. Safarov and D. Vassiliev. *The asymptotic distribution of eigenvalues of partial differential operators*, volume 155 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1997. Translated from the Russian manuscript by the authors.
[37] Y. G. Safarov. Asymptotic of the spectral function of a positive elliptic operator without the nontrap condition. *Functional Analysis and Its Applications*, 22(3):213–223, 1988.
[38] R. T. Seeley. Complex powers of an elliptic operator. pages 288–307, 1967.
[39] C. D. Sogge. *Fourier integrals in classical analysis*, volume 105 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.
[40] C. D. Sogge and S. Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Math. J.*, 114(3):387–437, 2002.
[41] A. V. Volovoy. Improved two-term asymptotics for the eigenvalue distribution function of an elliptic operator on a compact manifold. *Comm. Partial Differential Equations*, 15(11):1509–1563, 1990.
[42] A. V. Volovoy. Verification of the Hamilton flow conditions associated with Weyl's conjecture. *Ann. Global Anal. Geom.*, 8(2):127–136, 1990.
[43] A. Weinstein. Fourier integral operators, quantization, and the spectra of Riemannian manifolds. In *Géométrie symplectique et physique mathématique (Colloq. Internat. CNRS, No. 237, Aix-en-Provence, 1974)*, pages 289–298. 1975. With questions by W. Klingenberg and K. Bleuler and replies by the author.
[44] H. Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Math. Ann.*, 71(4):441–479, 1912.
[45] S. Zelditch. Kuznecov sum formulae and Szegő limit formulae on manifolds. *Comm. Partial Differential Equations*, 17(1-2):221–260, 1992.
[46] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

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