Wavenumber-explicit analysis for the Helmholtz h-BEM: error estimates and iteration counts for the Dirichlet problem

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Abstract

We consider solving the exterior Dirichlet problem for the Helmholtz equation with the h-version of the boundary element method (BEM) using the standard second-kind combined-field integral equations. We prove a new, sharp bound on how the number of GMRES iterations must grow with the wavenumber k in order to have the error in the iterative solution bounded independently of k as $k \to \infty$ when the boundary of the obstacle is analytic and has strictly positive curvature. To our knowledge, this result is the first-ever sharp bound on how the number of GMRES iterations depends on the wavenumber for an integral equation used to solve a scattering problem. We also prove new bounds on how h must decrease with k to maintain k-independent quasi-optimality of the Galerkin solutions as $k \to \infty$ when the obstacle is nontrapping.

Keywords: Helmholtz equation, high frequency, boundary integral equation, boundary element method, GMRES, pollution effect, semiclassical

AMS Subject Classifications: 35J05, 35J25, 65N22, 65N38, 65R20

1 Introduction

This paper is concerned with the wavenumber-explicit numerical analysis of boundary integral equations (BIEs) for the Helmholtz equation

$$\Delta u + k^2 u = 0, \tag{1.1}$$

where $k > 0$ is the *wavenumber*, posed in the exterior of a 2- or 3-dimensional bounded obstacle Ω with Dirichlet boundary conditions on $\Gamma := \partial\Omega$.

We consider the standard second-kind combined-field integral equation formulations of this problem: the so-called "direct" formulation (arising from Green's integral representation)

$$A'_{k,\eta} v = f_{k,\eta} \tag{1.2}$$

and the so-called "indirect" formulation (arising from an ansatz of layer potentials not related to Green's integral representation)

$$A_{k,\eta}\phi = g_k, \tag{1.3}$$

where

$$A'_{k,\eta} := \frac{1}{2}I + D'_k - \mathrm{i}\eta S_k, \qquad A_{k,\eta} := \frac{1}{2}I + D_k - \mathrm{i}\eta S_k, \tag{1.4}$$

$\eta \in \mathbb{R}\backslash\{0\}$ is an arbitrary coupling parameter, S_k is the single-layer operator, D_k is the double-layer operator, and D'_k is the adjoint double-layer operator (1.8), (1.9).

For simplicity of exposition, we focus on the direct equation (1.2), but the main results also hold for the indirect equation (1.3) (see Remark 1.25 below). The contribution to Equation (1.2)

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from the Dirichlet boundary conditions is contained in the right-hand side $f_{k,\eta}$; our results are independent of the particular form of $f_{k,\eta}$, and so we can simplify the presentation by restricting attention to the particular exterior Dirichlet problem corresponding to scattering by a point source or plane wave, i.e. the *sound-soft scattering problem* (Definition 1.7 below).

We consider solving the equation (1.2) in $L^2(\partial\Omega)$ using the Galerkin method; this method seeks an approximation v_N to the solution v from a finite-dimensional approximation space $\mathcal{V}_N$ (where N is the dimension, i.e. the total number of degrees of freedom). In the majority of the paper Γ is C^2, in which case $\mathcal{V}_N$ will be the space of piecewise polynomials of degree p, for some fixed $p \geq 0$, on shape-regular meshes of diameter h, with h decreasing to zero; this is the so-called *h–version* of the Galerkin method, and we denote $\mathcal{V}_N$ and v_N by $\mathcal{V}_h$ and v_h, respectively, and note that $N \sim h^{-(d-1)}$, where d is the dimension. To find the Galerkin solution v_h, one must solve a linear system of dimension N; in practice this is usually done using Krylov-subspace iterative methods such as the generalized minimal residual method (GMRES).

For the numerical analysis of this situation when k is large, there are now, roughly speaking, two main questions:

Q1. How must h decrease with k in order to maintain accuracy of the Galerkin solution as $k \to \infty$?

Q2. How does the number of GMRES iterations required to achieve a prescribed accuracy grow with k?

The goal of this paper is to prove rigorous results about these two questions, and then compare them with the results of numerical experiments.

We now give short summaries of the main results. These results depend on the choice of the coupling parameter η; for the results on Q1 we need $|\eta| \sim k$ and for the results on Q2 we need $\eta \sim k$, where we use the notation $a \sim b$ to mean that there exists $C_1, C_2 > 0$, independent of h and k, such that $C_1 b \leq a \leq C_2 b$. We also use the notation $a \lesssim b$ to mean that there exists $C > 0$, independent of h and k, such that $a \leq Cb$.

Summary of main results regarding Q1 and their context. Numerical experiments indicate that, in many cases, the condition $hk \lesssim 1$ is sufficient for the Galerkin method to be quasi-optimal (with the constant of quasi-optimality independent of k; i.e., (1.20) below holds); see [42, §5]. This feature can be described by saying that the h-BEM does not suffer from the pollution effect (in constrast to the h-FEM; see, e.g., [7], [51, Chapter 4]). The best existing result in the literature is that k-independent quasi-optimality of the Galerkin method applied to the integral equation (1.2) holds when $hk^{(d+1)/2} \lesssim 1$ for 2- and 3-d $C^{2,\alpha}$ obstacles that are star-shaped with respect to a ball [42, Theorem 1.4]. In this paper we improve this result by showing that the k-independent quasioptimality holds for 2-d *nontrapping* obstacles when $hk^{3/2} \lesssim 1$, for 3-d nontrapping obstacles when $hk^{3/2} \log k \lesssim 1$, and for 2- and 3-d smooth (i.e. C^∞) convex obstacles with strictly positive curvature when $hk^{4/3} \lesssim 1$ (see Theorem 1.15 below).

The ideas behind the proofs of these results are summarised in Remark 1.18 below, but we highlight here that all the integral-operator bounds used in these arguments are sharp up to a factor of $\log k$. Therefore, to lower these thresholds on h for which quasi-optimality is proved, one would need to use different arguments than in the present paper. We also highlight that recent experiments by Marburg [59], [10], [60] give examples where pollution occurs, and therefore determining the sharp threshold on h for k-independent quasi-optimality to hold in general is an exciting open question.

Summary of main results regarding Q2 and their context. There has been a large amount of research effort expended on understanding empirically how iteration counts for integral-equation formulations of scattering problems involving the Helmholtz or Maxwell equations depend on k; see, e.g, [1], [4], [14], [15], [84], and the references therein.

To our knowledge, however, there are no sharp k-explicit bounds in the literature, for any integral-equation formulation of a Helmholtz or Maxwell scattering problem, on the number of iterations GMRES requires to achieve a prescribed accuracy. The main reason, in this current setting of the Helmholtz sound-soft scattering problem, is that the operator $A'_{k,\eta}$ is non-normal for all obstacles other than the circle and sphere [13], [12], and so one cannot use the well-known bounds on GMRES iterations in terms of the condition number of $A'_{k,\eta}$ (see, e.g., the review in [73, §6]).

In this paper, we prove that, for 2- and 3-d analytic obstacles with strictly positive curvature, the number of GMRES iterations growing like $k^{1/3}$ is sufficient to have the error in the iterative solution bounded independently of k (see Theorem 1.21 below). Numerical experiments in §6 show that the numbers of GMRES iterations for the sphere and an ellipsoid grow slightly less than $k^{1/3}$ ($k^{0.29}$ for the sphere and $k^{0.28}$ for an ellipsoid), and thus our bound is effectively sharp.

The ideas behind the proof are summarised in Remark 1.23 below. The focus of this paper is in proving results for $A'_{k,\eta}$, i.e. the standard second-kind integral formulation, but we highlight in Remark 5.5 below how a bound on the number of GMRES iterations of $k^{1/2}$ when $d = 2$ and $k^{1/2} \log k$ when $d = 3$ can be obtained for a modification of $A'_{k,\eta}$, the so-called *star-combined integral equation* [76]. Moreover, whereas our bound on the number of iterations of $k^{1/3}$ for $A'_{k,\eta}$ holds for analytic obstacles with strictly positive curvature, the bounds for the star-combined operator hold for a much wider class of obstacles, namely piecewise-smooth Lipschitz obstacles that are star-shaped with respect to a ball.

Discussion of these results in the context of using semiclassical analysis in the numerical analysis of the Helmholtz equation. In the last 10 years, there has been growing interest in using results about the k-explicit analysis of the Helmholtz equation from *semiclassical analysis* (a branch of *microlocal analysis*) to design and analyse numerical methods for the Helmholtz equation[1] . The activity has occurred in, broadly speaking, four different directions:

1. The use of the results of Melrose and Taylor [64] on the rigorous $k \to \infty$ asymptotics of the solution of the Helmholtz equation in the exterior of a smooth convex obstacle with strictly positive curvature to design and analyse k-dependent approximation spaces for integral-equation formulations [30], [41], [5], [33], [34], [32],

2. The use of the Melrose-Taylor results [64], along with the work of Ikawa [52] on scattering from several convex obstacles, to analyse algorithms for multiple scattering problems [35], [2].

3. The use of bounds on the Helmholtz solution operator (also known as *resolvent estimates*) due to Vainberg [82] (using the propagation of singularities results of Melrose–Sjöstrand [63]) and Morawetz [68] to prove bounds on both $\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}$ and the inf-sup constant of the domain-based variational formulation [23], [75], [9], [25], and also to analyse preconditioning strategies [40].

4. The use of identities originally due to Morawetz [68] to prove coercivity of $A'_{k,\eta}$ [77] and to introduce new coercive formulations of Helmholtz problems [76], [67].

This paper concerns a fifth direction, namely proving sharp k-explicit bounds on S_k, D_k and D'_k as operators from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$ using estimates on the restriction of eigenfunctions of the Laplacian to hypersurfaces from [81], [16], [79], [47], [27], and [80] (and recapped in §2.3 below). We then use these results, in conjunction with the results in Points 3 and 4 above, to obtain answers to Q1 and Q2. Our $L^2(\partial\Omega) \to H^1(\partial\Omega)$ bounds include the sharp $L^2(\partial\Omega) \to L^2(\partial\Omega)$ bounds on S_k, D_k and D'_k from [46, Appendix A], and [38]; indeed, the presence of these $L^2(\partial\Omega) \to L^2(\partial\Omega)$ bounds and the realisation that they could be extended to $L^2(\partial\Omega) \to H^1(\partial\Omega)$ bounds were the motivation for this paper.

1.1 Formulation of the problem

1.1.1 Geometric definitions.

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded Lipschitz open set, such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega}$ is connected. Let $H^1_{\mathrm{loc}}(\Omega_+)$ denote the set of functions v such that $\chi v \in H^1(\Omega_+)$ for

[1] A closely-related activity is the design and analysis of numerical methods for the Helmholtz equation based on proving *new* results about the $k \to \infty$ asymptotics of Helmholtz solutions for polygonal obstacles; see [22], [50], [49], [20], and [48].

every $\chi \in C^\infty_{\mathrm{comp}}(\overline{\Omega_+}) := \{\chi|_{\Omega_+} : \chi \in C^\infty(\mathbb{R}^d)$ is compactly supported$\}$. Let γ^+ denote the trace operator from Ω_+ to $\partial\Omega$. Let n be the outward-pointing unit normal vector to Ω (i.e. n points *out* of Ω and *in* to Ω_+), and let ∂_n^+ denote the normal derivative trace operator from Ω_+ to $\partial\Omega$ that satisfies $\partial_n^+ u = n \cdot \gamma^+(\nabla u)$ when $u \in H^2_{\mathrm{loc}}(\Omega_+)$. (We also call $\gamma^+ u$ the Dirichlet trace of u and $\partial_n^+ u$ the Neumann trace.)

Definition 1.1 (Star-shaped, and star-shaped with respect to a ball)
(i) Ω *is* star-shaped with respect to the point $x_0 \in \Omega$ *if, whenever* $x \in \Omega$*, the segment* $[x_0, x] \subset D$.

(ii) Ω *is* star-shaped with respect to the ball $B_a(x_0)$ *if it is star-shaped with respect to every point in* $B_a(x_0)$.

(iii) Ω *is* star-shaped with respect to a ball *if there exists* $a > 0$ *and* $x_0 \in \Omega$ *such that* Ω *is star-shaped with respect to the ball* $B_a(x_0)$.

Definition 1.2 (Nontrapping) *We say that* $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ *is* nontrapping *if* $\partial\Omega$ *is smooth* (C^∞) *and, given* R *such that* $\overline{\Omega} \subset B_R(0)$*, there exists a* $T(R) < \infty$ *such that all the billiard trajectories (in the sense of Melrose–Sjöstrand [63, Definition 7.20]) that start in* $\Omega_+ \cap B_R(0)$ *at time zero leave* $\Omega_+ \cap B_R(0)$ *by time* $T(R)$.

Definition 1.3 (Smooth hypersurface) *We say that* $\Gamma \subset \mathbb{R}^d$ *is a* smooth hypersurface *if there exists* $\widetilde{\Gamma}$ *a compact embedded smooth* $d-1$ *dimensional submanifold of* $\mathbb{R}^d$*, possibly with boundary, such that* Γ *is an open subset of* $\widetilde{\Gamma}$ *and the boundary of* Γ *can be written as a disjoint union*

$$\partial\Gamma = \left(\bigcup_{\ell=1}^{n} Y_\ell\right) \cup \Sigma,$$

where each Y_ℓ *is an open, relatively compact, smooth embedded manifold of dimension* $d-2$ *in* $\widetilde{\Gamma}$, Γ *lies locally on one side of* Y_ℓ*, and* Σ *is closed set with* $d-2$ *measure* 0 *and* $\Sigma \subset \overline{\bigcup_{l=1}^{n} Y_l}$. *We then refer to the manifold* $\widetilde{\Gamma}$ *as an extension of* Γ.

For example, when $d = 3$, the interior of a 2-d polygon is a smooth hypersurface, with Y_i the edges and Σ the set of corner points.

Definition 1.4 (Curved) *We say a smooth hypersurface is* curved *if there is a choice of normal so that the second fundamental form of the hypersurface is everywhere positive definite.*

Recall that the principal curvatures are the eigenvalues of the matrix of the second fundamental form in an orthonormal basis of the tangent space, and thus "curved" is equivalent to the principal curvatures being everywhere strictly positive (or everywhere strictly negative, depending on the choice of the normal).

Definition 1.5 (Piecewise smooth) *We say that a hypersurface* Γ *is* piecewise smooth *if* $\Gamma = \cup_{i=1}^N \overline{\Gamma}_i$ *where* Γ_i *are smooth hypersurfaces and* $\Gamma_i \cap \Gamma_j = \emptyset$.

Definition 1.6 (Piecewise curved) *We say that a piecewise-smooth hypersurface* Γ *is* piecewise curved *if* Γ *is as in Definition 1.5 and each* Γ_j *is curved.*

1.1.2 The boundary value problem and integral equation formulation

Definition 1.7 (Sound-soft scattering problem) *Given* $k > 0$ *and an incident plane wave* $u^I(x) = \exp(\mathrm{i}kx \cdot \widehat{a})$ *for some* $\widehat{a} \in \mathbb{R}^d$ *with* $|\widehat{a}| = 1$*, find* $u^S \in C^2(\Omega_+) \cap H^1_{\mathrm{loc}}(\Omega_+)$ *such that the total field* $u := u^I + u^S$ *satisfies the Helmholtz equation (1.1) in* Ω_+, $\gamma^+ u = 0$ *on* $\partial\Omega$*, and* u^S *satisfies the Sommerfeld radiation condition*

$$\frac{\partial u^S}{\partial r}(x) - \mathrm{i}k\, u^S(x) = o\left(\frac{1}{r^{(d-1)/2}}\right)$$

as $r := |x| \to \infty$*, uniformly in* $\widehat{x} := x/r$.

The incident field in the sound-soft scattering problem of Definition 1.7 is a plane wave, but this could be replaced by a point source or, more generally, a solution of the Helmholtz equation in a neighbourhood of Ω; see [18, Definition 2.11].

Obtaining the direct integral equation (1.2)**.** If u satisfies the sound-soft scattering problem of Definition 1.7 then Green's integral representation implies that

$$u(x) = u^I(x) - \int_{\partial\Omega} \Phi_k(x,y)\partial_n^+ u(y)\,\mathrm{d}s(y), \quad x \in \Omega_+, \tag{1.5}$$

(see, e.g., [18, Theorem 2.43]), where $\Phi_k(x,y)$ is the fundamental solution of the Helmholtz equation given by

$$\Phi_k(x,y) = \frac{\mathrm{i}}{4}H_0^{(1)}\big(k|x-y|\big),\ d=2, \qquad \Phi_k(x,y) = \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|},\ d=3 \tag{1.6}$$

(note that we have chosen the sign of $\Phi_k(x,y)$ so that $-(\Delta+k^2)\Phi_k(x,y) = \delta(x-y)$). Taking the exterior Dirichlet and Neumann traces of (1.5) on $\partial\Omega$ and using the jump relations for the single- and double-layer potentials (see, e.g., [18, Equation 2.41]) we obtain the integral equations

$$S_k\partial_n^+ u = \gamma^+ u^I \quad \text{ and } \quad \left(\frac{1}{2}I + D_k'\right)\partial_n^+ u = \partial_n^+ u^I, \tag{1.7}$$

where S_k and D_k' are the single- and adjoint-double-layer operators defined by

$$S_k\phi(x) := \int_{\partial\Omega}\Phi_k(x,y)\phi(y)\,\mathrm{d}s(y), \quad D_k'\phi(x) := \int_{\partial\Omega}\frac{\partial\Phi_k(x,y)}{\partial n(x)}\phi(y)\,\mathrm{d}s(y), \tag{1.8}$$

for $\phi \in L^2(\partial\Omega)$ and $x \in \partial\Omega$. Later we will also need the definition of the double-layer potential,

$$D_k\phi(x) := \int_{\partial\Omega}\frac{\partial\Phi_k(x,y)}{\partial n(y)}\phi(y)\,\mathrm{d}s(y) \quad \text{ for } \phi \in L^2(\partial\Omega) \text{ and } x \in \partial\Omega. \tag{1.9}$$

The first equation in (1.7) is not uniquely solvable when $-k^2$ is a Dirichlet eigenvalue of the Laplacian in Ω, and the second equation in (1.7) is not uniquely solvable when $-k^2$ is a Neumann eigenvalue of the Laplacian in Ω (see, e.g., [18, Theorem 2.25]). The standard way to resolve this difficulty is to take a linear combination of the two equations, which yields the integral equation (1.2) where $A'_{k,\eta}$ is defined by (1.4),

$$f_{k,\eta} := \partial_n^+ u^I - \mathrm{i}\eta\,\gamma^+ u^I, \tag{1.10}$$

and we use the notation that $v = \partial_n^+ u$ (this makes denoting the Galerkin solution below easier since we then have v_h instead of $(\partial_n^+ u)_h$).

The space $L^2(\partial\Omega)$ is a natural space for the practical solution of second-kind integral equations since it is self-dual, and, for $\eta \in \mathbb{R}\setminus\{0\}$, $A'_{k,\eta}$ is a bounded invertible operator from $L^2(\partial\Omega)$ to itself [18, Theorem 2.27]. Furthermore the right-hand side $f_{k,\eta}$ is in $L^2(\partial\Omega)$ (since $u^I \in C^\infty(\overline{\Omega_+})$) and thus we consider the equation (1.2) as an equation in $L^2(\partial\Omega)$.

The Galerkin method. Given a finite-dimensional approximation space $\mathcal{V}_N \subset L^2(\partial\Omega)$, the Galerkin method for the integral equation (1.2) is

$$\text{find } v_N \in \mathcal{V}_N \text{ such that } \big(A'_{k,\eta}v_N, w_N\big)_{L^2(\partial\Omega)} = \big(f_{k,\eta}, w_N\big)_{L^2(\partial\Omega)} \text{ for all } w_N \in \mathcal{V}_N. \tag{1.11}$$

Let $\mathcal{V}_N = \operatorname{span}\{\phi_i : i = 1,\ldots,N\}$, let $v_N \in \mathcal{V}_N$ be equal to $\sum_{j=1}^N V_j\phi_j$, and define $\mathbf{v} \in \mathbb{C}^N$ by $\mathbf{v} := (V_j)_{j=1}^N$. Then, with $\mathbf{A}_{ij} := (A'_{k,\eta}\phi_j, \phi_i)_{L^2(\partial\Omega)}$ and $\mathbf{f}_i := (f_{k,\eta}, \phi_i)_{L^2(\partial\Omega)}$, the Galerkin method (1.11) is equivalent to solving the linear system $\mathbf{A}\mathbf{v} = \mathbf{f}$.

We consider the h–version of the Galerkin method, and we then denote $\mathcal{V}_N$ and v_N by $\mathcal{V}_h$ and v_h respectively. The main results for Q1 and Q2 will be stated under the following assumption on $\mathcal{V}_h$.

Assumption 1.8 (Assumptions on $\mathcal{V}_h$) *$\mathcal{V}_h$ is a space of piecewise polynomials of degree p for some fixed $p \geq 0$ on shape-regular meshes of diameter h, with h decreasing to zero (see, e.g., [72, Chapter 4] for specific realisations). Furthermore* *(a) if $w \in H^1(\partial\Omega)$ then*

$$\inf_{w_h \in \mathcal{V}_h} \|w - w_h\|_{L^2(\partial\Omega)} \lesssim h \|w\|_{H^1(\partial\Omega)}, \tag{1.12}$$

(b)

$$\|w_h\|^2_{L^2(\partial\Omega)} \sim h^{d-1} \|\mathbf{w}\|_2^2, \tag{1.13}$$

where $\|\cdot\|_2$ denotes the l_2 (i.e. euclidean) vector norm.

Remark 1.9 (For what situations is Assumption 1.8 proved?) *Part (a) is proved for subspaces consisting of piecewise-constant basis functions in [72, Theorem 4.3.19] when $\partial\Omega$ is a polyhedron or curved (in the sense of Assumptions 4.3.17 and 4.3.18, respectively, in [72]) and in [78, Theorem 10.4] when Ω is a piecewise-smooth Lipschitz domain. Part (a) is proved for subspaces consisting of continuous piecewise-polynomials of degree $p \geq 1$ (in the sense of [72, Definition 4.1.36]) in [72, Theorem 4.3.28].*

Part (b) is proved for subspaces consisting of piecewise-linear basis function in [78, Lemma 10.5] when $\partial\Omega$ is piecewise-smooth and Lipschitz, and for more general subspaces in [72, Theorem 4.4.7].

1.2 Statement of the main results and discussion

We split the statement of the main results into three sections

- k-explicit bounds on S_k, D_k, and D'_k as mappings from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$ (§1.2.1).
- Results concerning Q1 (§1.2.2).
- Results concerning Q2 (§1.2.3).

1.2.1 k-explicit bounds on S_k, D_k, and D'_k

Theorem 1.10 (Bounds on $\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$, $\|D_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$, $\|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$)
(a) If $\partial\Omega$ is a piecewise-smooth hypersurface (in the sense of Definition 1.5), then, given $k_0 > 1$,

$$\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \lesssim k^{1/2} \log k, \tag{1.14}$$

for all $k \geq k_0$. Moreover, if $\partial\Omega$ is piecewise curved (in the sense of Definition 1.6), then, given $k_0 > 1$, the following stronger estimate holds for all $k \geq k_0$

$$\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \lesssim k^{1/3} \log k. \tag{1.15}$$

(b) If $\partial\Omega$ is a piecewise smooth, $C^{2,\alpha}$ hypersurface, for some $\alpha > 0$, then, given $k_0 > 1$,

$$\|D_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \lesssim k^{5/4} \log k$$

for all $k \geq k_0$. Moreover, if $\partial\Omega$ is piecewise curved, then, given $k_0 > 1$, the following stronger estimates hold for all $k \geq k_0$

$$\|D_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \lesssim k^{7/6} \log k.$$

(c) If Ω is convex and $\partial\Omega$ is C^∞ and curved (in the sense of Definition 1.4) then, given $k_0 > 0$,

$$\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \lesssim k^{1/3}, \tag{1.16}$$

$$\|D_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \lesssim k$$

for all $k \geq k_0$.

Remark 1.11 (Sharpness of the bounds in Theorem 1.10) *In Section 3 we show that, modulo the factor $\log k$, all of the bounds in Theorem 1.10 are sharp (i.e. the powers of k in the bounds are optimal). The sharpness (modulo the factor $\log k$) of the $L^2(\partial\Omega) \to L^2(\partial\Omega)$ bounds in Theorem 2.10 was proved in [46, §A.2-A.3]. Earlier work in [17, §4] proved the sharpness of some of the $L^2(\partial\Omega) \to L^2(\partial\Omega)$ bounds in 2-d; we highlight that Section 3 and [46, §A.2-A.3] contain the appropriate generalisations to multidimensions of some of the arguments of [17, §4] (in particular [17, Theorems 4.2 and 4.4]).*

Remark 1.12 (Comparison to previous results) *The only previously-existing bounds on the $L^2(\partial\Omega) \to H^1(\partial\Omega)$-norms of S_k, D_k, and D'_k were the following:*

$$\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \lesssim k^{(d-1)/2} \tag{1.17}$$

when $\partial\Omega$ is Lipschitz, and

$$\|D_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \lesssim k^{(d+1)/2} \tag{1.18}$$

when $\partial\Omega$ is $C^{2,\alpha}$ [42, Theorem 1.6].

We see that (1.17) is a factor of $\log k$ sharper than the bound (1.14) when $d = 2$, but otherwise all the bounds in Theorem 1.10 are sharper than (1.17) and (1.18).

Remark 1.13 (Bounds for general dimension and $k \in \mathbb{R}$) *We have restricted attention to 2- and 3-dimensions because these are the most practically interesting ones. From a semiclassical point of view, it is natural work in $d \geq 1$, and the results of Theorem 1.10 apply for any $d \geq 1$ (although when $d = 1$ it is straightforward to get sharper bounds). We have also restricted attention to the case when k is positive and bounded away from 0. Nevertheless, the methods used to prove the bounds in Theorem 1.10 show that if one replaces $\log k$ by $\log\langle k\rangle$ (where $\langle\cdot\rangle = (2 + |\cdot|^2)^{1/2}$) and includes an extra factor of $\log\langle k^{-1}\rangle$ when $d = 2$, then the resulting bounds hold for all $k \in \mathbb{R}$*

This paper is concerned with second-kind Helmholtz BIEs posed in $L^2(\partial\Omega)$, but there is also a large interest in both first- and second-kind Helmholtz BIEs posed in the trace spaces $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ (see, e.g., [72, §3.9], [78, §7.6]). The k-explicit theory of Helmholtz BIEs in the trace spaces is much less developed than the theory in $L^2(\partial\Omega)$, so we therefore highlight that the $L^2(\partial\Omega) \to H^1(\partial\Omega)$ bounds in Theorem 1.10 can be converted to $H^{s-1/2}(\partial\Omega) \to H^{s+1/2}(\partial\Omega)$ bounds for $|s| \leq 1/2$.

Corollary 1.14 (Bounds from $H^{s-1/2}(\partial\Omega) \to H^{s+1/2}(\partial\Omega)$ for $|s| \leq 1/2$) *Theorem 1.10 is valid with all the norms from $L^2(\partial\Omega) \to H^1(\partial\Omega)$ replaced by norms from $H^{s-1/2}(\partial\Omega) \to H^{s+1/2}(\partial\Omega)$ for $|s| \leq 1/2$.*

1.2.2 Results concerning Q1

Theorem 1.15 (Sufficient conditions for the Galerkin method to be quasi-optimal) *Let u be the solution of the sound-soft scattering problem of Definition 1.7, let $|\eta| \sim k$, and let $\mathcal{V}_h$ satisfy Part (a) of Assumption 1.8.*

(a) If either *(i) Ω is nontrapping,* or *(ii) Ω is star-shaped with respect to a ball and $\partial\Omega$ is $C^{2,\alpha}$ and piecewise smooth, then given $k_0 > 0$, there exists a $C > 0$ (independent of k and h) such that if*

$$hk^{3/2} \leq C, \quad d = 2, \quad hk^{3/2}\log k \leq C, \quad d = 3, \tag{1.19}$$

then the Galerkin equations (1.11) have a unique solution which satisfies

$$\|v - v_h\|_{L^2(\partial\Omega)} \lesssim \inf_{w_h\in\mathcal{V}_h} \|v - w_h\|_{L^2(\partial\Omega)} \tag{1.20}$$

for all $k \geq k_0$.

(b) In case (ii) above, if $\partial\Omega$ is piecewise curved, then given $k_0 > 0$, there exists a $C > 0$ (independent of k and h) such that if

$$hk^{4/3}\log k \leq C, \quad d = 2, 3 \tag{1.21}$$

then (1.20) holds.

(c) If Ω is convex and $\partial\Omega$ is C^∞ and curved then given $k_0 > 0$, there exists a $C > 0$ (independent of k and h) such that if

$$hk^{4/3} \leq C, \quad d = 2, 3 \tag{1.22}$$

then (1.20) holds.

Having established quasi-optimality, it is then natural to ask how the best approximation error $\inf_{w_h\in\mathcal{V}_h} \|v - w_h\|_{L^2(\partial\Omega)}$ depends on k, h, and $\|v\|_{L^2(\partial\Omega)}$.

Theorem 1.16 (Bounds on the best approximation error) *Let u be the solution of the sound-soft scattering problem of Definition 1.7 and let $\mathcal{V}_h$ satisfy Assumption 1.8.*

(a) If $\partial\Omega$ is $C^{2,\alpha}$ and piecewise smooth, then, given $k_0 > 0$,

$$\inf_{w_h\in\mathcal{V}_h} \|v - w_h\|_{L^2(\partial\Omega)} \lesssim hA(k)\|v\|_{L^2(\partial\Omega)} \tag{1.23}$$

with $A(k) = k^{5/4}\log k$, for all $k \geq k_0$.

(b) If $\partial\Omega$ is piecewise curved, then, given $k_0 > 0$, (1.23) holds with $A(k) = k^{7/6}\log k$, for all $k \geq k_0$.

(c) If Ω is convex and $\partial\Omega$ is C^∞ and curved, then, given $k_0 > 0$, then (1.23) holds with $A(k) = k$, for all $k \geq k_0$.

Combining Theorems 1.15 and 1.16 we can obtain bounds on the relative error of the Galerkin method. For brevity, we only state the ones corresponding to cases (a) and (c) in Theorems 1.15 and 1.16.

Corollary 1.17 (Bound on the relative errors in the Galerkin method) *Let u be the solution to the sound-soft scattering problem, let $|\eta| \sim k$, and let $\mathcal{V}_h$ satisfy Part (a) of Assumption 1.8.*

(a) If either *(i) Ω is nontrapping,* or *(ii) Ω is star-shaped with respect to a ball and $\partial\Omega$ is $C^{2,\alpha}$ and piecewise smooth, then given $k_0 > 0$, there exists a $C > 0$ (independent of k and h) such that if h and k satisfy (1.19) then the Galerkin equations (1.11) have a unique solution which satisfies*

$$\frac{\|v - v_h\|_{L^2(\partial\Omega)}}{\|v\|_{L^2(\partial\Omega)}} \lesssim \begin{cases} k^{-1/4}\log k, & d = 2, \\ k^{-1/4}, & d = 3, \end{cases}$$

for all $k \geq k_0$.

(b) If Ω is convex and $\partial\Omega$ is C^∞ and curved, then given $k_0 > 0$, there exists a $C > 0$ (independent of k and h) such that if $hk^{4/3} \leq C$ the Galerkin equations (1.11) have a unique solution which satisfies

$$\frac{\|v - v_h\|_{L^2(\partial\Omega)}}{\|v\|_{L^2(\partial\Omega)}} \lesssim \frac{1}{k^{1/3}}$$

for all $k \geq k_0$.

Remark 1.18 (The main ideas behind the proofs of Theorems 1.15 and 1.16) *The proof of Theorem 1.15 uses the classic projection-method analysis of second-kind integral equations (see, e.g., [6]), with $A'_{k,\eta}$ be treated as a compact perturbation of the identity. In [42], this argument was used to reduce the question of finding k-explicit bounds on the mesh threshold h for k-independent quasi-optimality to hold to finding k-explicit bounds on*

$$\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}, \quad \|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}, \quad \text{and} \quad \|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}.$$

We use the sharp bounds on the first two of these norms from Theorem 1.10, and the sharp bounds on the third of these norms from [9, Theorem 1.13] (for nontrapping obstacles) and [23, Theorem 4.3] (for obstacles that are star-shaped with respect to a ball).

The bounds of Theorem 1.16 are proved by showing that

$$\|v\|_{H^1(\partial\Omega)} \lesssim A(k)\|v\|_{L^2(\partial\Omega)}, \tag{1.24}$$

and then using the approximation theory result (1.12). The bound (1.24) is obtained from the integral equation (1.2) using the second-kind-structure of the equation and the the $L^2(\partial\Omega) \to H^1(\partial\Omega)$ bounds on S_k and D'_k from Theorem 1.10.

Remark 1.19 (Comparison to previous results) *Theorems 1.15 and 1.16 and Corollary 1.17 sharpen previous results in [42]: the mesh thresholds for quasi-optimality in Theorem 1.15 are sharper than the corresponding ones in [42], and the results are valid for a wider class of obstacles.*

This sharpening is due to the new, sharp bounds on $L^2(\partial\Omega) \to H^1(\partial\Omega)$*-norms of* S_k, D_k, *and* D'_k *from Theorem 1.10, and the widening of the class of obstacles is due to the bound on* $\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}$ *for nontrapping obstacles from [9, Theorem 1.13]. In more detail: Theorem 1.4 of [42] is the analogue of our Theorem 1.15 except that the former is only valid when* Ω *is star-shaped with respect to a ball and* $C^{2,\alpha}$ *and the mesh threshold is* $hk^{(d+1)/2} \leq C$*. Comparing this result to Theorem 1.15 we see that we've sharpened the threshold in the* $d = 3$ *case, expanded the class of obstacles to nontrapping ones, and added the additional results (b) and (c). Theorem 1.16 on the best approximation error is again proved using the* $L^2(\partial\Omega) \to H^1(\partial\Omega)$*-bounds in Theorem 1.10 and thus we see similar improvements over the corresponding theorem in [42] ([42, Theorem 1.3]).*

As discussed in Remark 1.18, both the present paper and [42] use the classic projection-method argument to obtain k*-explicit results about quasi-optimality of the* h*-BEM. There are two other sets of results about quasi-optimality of the* h*-BEM in the literature:*

(a) results that use coercivity [30], [76], [77], and

(b) results that give sufficient conditions for quasi-optimality to hold in terms of how well the spaces $\mathcal{V}_h$ *approximate the solution of certain adjoint problems [8], [56], [62].*

These two sets of results are discussed in detail in [42, pages 181–182] and [42, §4.2] respectively, and neither give results as strong as those in Theorem 1.15.

Finally, in this paper we have only considered the h*-BEM; a thorough* k*-explicit analysis of the* hp*-BEM for the exterior Dirichlet problem was conducted in [56] and [62]. In particular, this analysis, combined with the bound on* $\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}$ *for nontrapping obstacles from [9, Theorem 1.13], proves that* k*-independent quasi-optimality can be obtained for nontrapping obstacles through a choice of* h *and* p *that keeps the total number of degrees of freedom proportional to* k^{d-1} *[56, Corollaries 3.18 and 3.19].*

Remark 1.20 (How sharp are the quasioptimality results?) *Numerical experiments in [42, §5] show that for a wide variety of obstacles (including certain trapping obstacles) the* h*-BEM is quasi-optimal with constant independent of* k *(i.e. (1.20) holds), when* $hk \sim 1$*. The closest we can get to proving this is the result for strictly convex obstacles in Theorem 1.15 part (c), with the threshold being* $hk^{4/3} \leq C$*. The recent results of Marburg [59], [10], [60], however, give examples of cases where* $hk \sim 1$ *is not sufficient to keep the error bounded as* $k \to \infty$*.*

1.2.3 Result concerning Q2

We now consider solving the linear system $\mathbf{A}\mathbf{v} = \mathbf{f}$ with the generalised minimum residual method (GMRES) introduced by Saad and Schultz in [71]; for details of the implementation of this algorithm, see, e.g., [70], [43].

Theorem 1.21 (A bound on the number of GMRES iterations) *Let* Ω *be a 2- or 3-d convex obstacle whose boundary* $\partial\Omega$ *is analytic and curved. Let* $\mathcal{V}_h$ *satisfy Part (b) of Assumption 1.8, let the Galerkin matrix corresponding to (1.11) be denoted by* $\mathbf{A}$*, and consider GMRES applied to* $\mathbf{A}\mathbf{v} = \mathbf{f}$

There exist constants $\eta_0 > 0$ *and* $k_0 > 0$ *(with* $\eta_0 = 1$ *if* Ω *is a ball) such that if* $k \geq k_0$ *and* $\eta_0 k \leq \eta \lesssim k$*, then, given* $0 < \varepsilon < 1$*, there exists a* C *(independent of* k*,* η*, and* ε*) such that if*

$$m \geq Ck^{1/3} \log\left(\frac{12}{\varepsilon}\right), \tag{1.25}$$

then the m*th GMRES residual* $\mathbf{r}_m := \mathbf{A}\mathbf{v}_m - \mathbf{f}$ *satisfies*

$$\frac{\|\mathbf{r}_m\|_2}{\|\mathbf{r}_0\|_2} \leq \varepsilon,$$

where $\|\cdot\|_2$ *denotes the* l_2 *(i.e. euclidean) vector norm. Furthermore, when* Ω *is a ball (i.e.* $\partial\Omega$ *is a circle or sphere), then the constant* $\eta_0 = 1$*.*

In other words, Theorem 1.21 states that, for convex, analytic, curved Ω, the number of iterations growing like $k^{1/3}$ is a sufficient condition for GMRES to maintain accuracy as $k \to \infty$.

Remark 1.22 (How sharp is the result of Theorem 1.21?) *Numerical experiments in §6 show that for the sphere the number of GMRES iterations grows like $k^{0.29}$, and for an ellipsoid they grow like $k^{0.28}$. The bound in Theorem 1.21 is therefore effectively sharp (at least for the range of k considered in the experiments).*

Remark 1.23 (The main ideas behind the proof of Theorem 1.21) *The two ideas behind Theorem 1.21 are that:*

(a) A sufficient (but not necessary) condition for iterative methods to be well behaved is that the numerical range *(also known as the* field of values*) of the matrix is bounded away from zero, and in this case the Elman estimate [37, 36] and its refinement due to Beckermann, Goreinov, and Tyrtyshnikov [11] can be used to bound the number of GMRES iterations in terms of (i) the distance of the numerical range to the origin, and (ii) the norm of the matrix.*

(b) When Ω is convex, C^3, piecewise analytic, and $\partial\Omega$ is curved, [77] proved that $A'_{k,\eta}$ is coercive for sufficiently large k (with $\eta \sim k$) . The k-dependence of the coercivity constant, along with the k-dependence of $\|A'_{k,\eta}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}$ then give the information needed about the numerical range of the Galerkin matrix $\mathbf{A}$ required in (a).

Remark 1.24 (Comparison to previous results) *The bound $m \gtrsim k^{2/3}$ when $\partial\Omega$ is a sphere was stated in [77, §1.3]; this bound was obtained using the original Elman estimate (see Remark 5.4 below), and the fact that the sharp bound $\|A'_{k,\eta}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \lesssim k^{1/3}$ was known for the circle and sphere; see [18, §5.4]. To our knowledge, there are no other k-explicit bounds in the literature, for any Helmholtz BIE, on the number of GMRES iterations required to achieve a prescribed accuracy. The closest related work is [24], which uses a second-kind integral equation to solve the Helmholtz equation in a half-plane with an impedance boundary condition. The special structure of this integral equation allows a two-grid iterative method to be used, and [24] prove that there exists $C > 0$ such that if $kh \le C$, then, after seven iterations, the difference between the solution and the Galerkin solution computed via the iterative method is bounded independently of k and h.*

Remark 1.25 (Translating the results to the indirect equation (1.3)) *Instead of using Green's integral representation (1.5) to formulate the sound-soft scattering problem as the integral equation (1.2), one can pose the ansatz that the scattered field satisfies*

$$u^S(x) = \int_{\partial\Omega} \frac{\partial \Phi_k(x,y)}{\partial n(y)} \phi(y)\,\mathrm{d}s(y) - \mathrm{i}\eta \int_{\partial\Omega} \Phi_k(x,y)\phi(y)\,\mathrm{d}s(y)$$

for $x \in \Omega_+$, $\phi \in L^2(\partial\Omega)$, and $\eta \in \mathbb{R}\setminus\{0\}$. Imposing the boundary condition $\gamma^+ u^S = -\gamma^+ u^I$ on $\partial\Omega$ and using the jump relations for the single- and double-layer potentials leads to the integral equation (1.3) where $A_{k,\eta}$ is defined by (1.4) and $g = -\gamma^+ u^I$. One can use (2.53) below to show that $A_{k,\eta}$ and $A'_{k,\eta}$ are adjoint with respect to the real-valued $L^2(\partial\Omega)$ inner product (see, e.g., [18, Equation 2.37, Remark 2.24, §2.6]), and so their norms are equal, the norms of their inverses are equal, and if one is coercive then so is the other (with the same coercivity constant). These facts imply that the results of Theorems 1.15 and Theorem 1.21 hold for the indirect equation (1.3).

The bounds on the best approximation error in Theorem 1.16 hold for the indirect equation (1.3) with (a) $A(k) = k^{3/2}$ for $d = 2$, $A(k) = k^{3/2}\log k$ for $d = 3$, (b) $A(k) = k^{4/3}\log k$, and (c) $A(k) = k^{4/3}$. These powers of k are all slightly higher than those for the direct equation; the reason for this is essentially that we have more information about the unknown in the direct equation (since it is $\partial_n^+ u$) than about the unknown in the indirect equation (one can express ϕ in terms of the difference of solutions to interior and exterior boundary value problems – see [18, Page 132] – but it is harder to make use of this fact than for the direct equation).

Remark 1.26 (Translating the results to the general exterior Dirichlet problem) *The results of Theorems 1.15 and 1.21 are independent of the right-hand side of the integral equation (1.2), and therefore hold for the general Dirichlet problem with Dirichlet data in $H^1(\partial\Omega)$ (this assumption is needed so that $A'_{k,\eta}$ can still be considered as an operator on $L^2(\partial\Omega)$; see, e.g., [18, §2.6]). The results of Theorem 1.16 and Corollary 1.17, however, do not immediately hold for the general Dirichlet problem, since they use the particular form of the right-hand side in (1.10).*

Outline of the paper In §2 we prove Theorem 1.10 (the $L^2(\partial\Omega) \to H^1(\partial\Omega)$ bounds) and Corollary 1.14, and in §3 we show that these bounds are sharp in their k-dependence. In §4 we prove Theorems 1.15 and 1.16 (the results concerning Q1). In §5 we prove Theorem 1.21 (the result concerning Q2), and then in §6 we give numerical experiments showing that Theorem 1.21 is sharp in its k-dependence.

2 Proof of Theorem 1.10 (the $L^2(\partial\Omega) \to H^1(\partial\Omega)$ bounds) and Corollary 1.14

In this section we prove Theorem 1.10 and Corollary 1.14. The vast majority of the work will be in proving Parts (a) and (b) of Theorem 1.10, with Part (c) of Theorem 1.10 following from the results in [38, Chapter 4], and Corollary 1.14 following from the results of [42].

The outline of this section is as follows: In §2.1 we discuss some preliminaries from the theory of semiclassical pseudodifferential operators, with our default references the texts [85] and [31]. In §2.2 we recap facts about function spaces on piecewise-smooth hypersurfaces. In §2.3 we recap restriction bounds on quasimodes – these results are central to our proof of Theorem 1.10. In §2.4 we prove of Parts (a) and (b) of Theorem 1.10, in §2.5 we prove Part (c) of Theorem 1.10 §2.5, and in §2.6 we prove Corollary §2.6.

We drop the $\lesssim$ notation in this section and state every bound with a constant C; we do this because later in the proof it will be useful to be able to indicate whether or not the constant in our estimates depends on the order s of the Sobolev space, or on a particular hypersurface Γ (we do this via the subscript s and Γ – see, e.g., (2.17) below).

2.1 Semiclassical Preliminaries

2.1.1 Symbols and quantization

We define the symbol class $S^m(\mathbb{R}^{2d})$ by

$$S^m(\mathbb{R}^{2d}) := \big\{a(x,\xi;k) \in C^\infty(\mathbb{R}^{2d}_{x,\xi}) \,|\, \text{for all } \alpha,\, \beta \text{ there exists } C_{\alpha\beta} \text{ with } |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}\big\}.$$

We write $S^{-\infty} = \bigcap_m S^m$. We say that $a \in S^{\text{comp}}$ if $a \in S^{-\infty}$ with $\operatorname{supp} a \subset K$ for some compact set $K \subset \mathbb{R}^{2d}$ independent of k.

For an element $a \in S^m$, we define its quantization to be the operator

$$u \mapsto a(x, k^{-1}D)u := \frac{k^d}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \exp\big(\mathrm{i}k\langle x-y,\xi\rangle\big)\, a(x,\xi)\, u(y)\, \mathrm{d}y\mathrm{d}\xi \tag{2.1}$$

for $u \in \mathcal{S}(\mathbb{R}^d)$. These operators can be defined by duality on $u \in \mathcal{S}'(\mathbb{R}^d)$. We denote the set of pseudodifferential operators of order m by

$$\Psi^m(\mathbb{R}^d) := \{a(x, k^{-1}D) \,|\, a \in S^m\}.$$

We denote $\Psi^{-\infty}(\mathbb{R}^d) = \cap_m \Psi^m(\mathbb{R}^d)$. We say that $A \in \Psi^{\text{comp}}(\mathbb{R}^d)$ if

$$A = a(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty})$$

for some $a \in S^{\text{comp}}$. Here, we say that an operator B is $\mathcal{O}_{\Psi^{-\infty}}(k^{-\infty})$ if for any $N > 0$, there exists C_N such that

$$\|B\|_{H^{-N}\to H^N} \leq C_N k^{-N}.$$

Suppose that $A \in \Psi^m(\mathbb{R}^d)$ has $A = a(x, k^{-1}D)$. Then we call a the *full symbol* of A. The *principle symbol* of $A \in \Psi^m(\mathbb{R}^d)$, is defined by

$$\sigma(A) := a \mod k^{-1}S^{m-1}.$$

Lemma 2.1 *[31, Proposition E.16] Let $a \in S^{m_1}$ and $b \in S^{m_2}$. Then we have*

$$\begin{aligned} a(x,k^{-1}D)b(x,k^{-1}D) &= (ab)(x,k^{-1}D) + k^{-1}r_1(x,k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}) \\ [a(x,k^{-1}D),b(x,k^{-1}D)] &:= a(x,k^{-1}D)b(x,k^{-1}D) - b(x,k^{-1}D)a(x,k^{-1}D) \\ &= \frac{1}{ik}\{a,b\}(x,k^{-1}D) + k^{-2}r_2(x,k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}) \end{aligned}$$

where $r_1 \in S^{m_1+m_2-1}$, $r_2 \in S^{m_1+m_2-2}$, $\operatorname{supp} r_i \subset \operatorname{supp} a \cap \operatorname{supp} b$, and the Poisson bracket $\{a,b\}$ is defined by

$$\{a,b\} := \sum_{j=1}^{d} \partial_{\xi_j} a \partial_{x_j} b - \partial_{\xi_j} b \partial_{x_j} a.$$

2.1.2 Action on semiclassical Sobolev spaces

We define the Semiclassical Sobolev spaces $H^s_k(\mathbb{R}^d)$ to be the space $H^s(\mathbb{R}^d)$ equipped with the norm

$$\|u\|^2_{H^s_k(\mathbb{R}^d)} = \|\langle k^{-1}D\rangle^s u\|^2_{L^2(\mathbb{R}^d)},$$

where $\langle \xi \rangle := (1+|\xi|^2)^{1/2} \in S^1$ and $D := -\mathrm{i}\partial$. Note that for s an integer, this norm is equivalent to

$$\|u\|^2_{H^s_k(\mathbb{R}^d)} = \sum_{|\alpha|\le s} \|(k^{-1}\partial)^\alpha u\|^2_{L^2(\mathbb{R}^d)}.$$

The definition of the semiclassical Sobolev spaces on a smooth compact manifold of dimension $d-1$ Γ, i.e. $H^s_k(\Gamma)$ for $|s| \le 1$, follows from the definition of $H^s_k(\mathbb{R}^{d-1})$ (see, e.g., [61, Page 98]). Because solutions of the Helmholtz equation $(-k^{-2}\Delta - 1)u = 0$ oscillate at frequency k, scaling derivatives by k^{-1} makes the norms uniform in the number of derivatives.

With these definitions in hand, we have the following lemma on boundedness of pseudodifferential operators.

Lemma 2.2 *[31, Proposition E.22] Let $A \in \Psi^m(\mathbb{R}^d)$. Then $\|A\|_{H^s_k(\mathbb{R}^d)\to H^{s-m}_k(\mathbb{R}^d)} \le C$.*

2.1.3 Ellipticity

For $A \in \Psi^m(\mathbb{R}^d)$, we say that $(x,\xi) \in \mathbb{R}^{2d}$ is in the *elliptic set of* A, denoted $\operatorname{ell}(A)$, if there exists U a neighborhood of (x,ξ) such that for some $\delta > 0$,

$$\inf_U |\sigma(A)(x,\xi)| \ge \delta.$$

We then have the following lemma

Lemma 2.3 *[31, Proposition E.31] Suppose that $A \in \Psi^{m_1}(\mathbb{R}^d)$, $b \in S^{\mathrm{comp}}$ with $\operatorname{supp} b \subset \operatorname{ell}(A)$. Then there exists $R_1, R_2 \in \Psi^{\mathrm{comp}}(\mathbb{R}^d)$ with*

$$R_1 A = b(x,k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}), \qquad AR_2 = b(x,k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).$$

Moreover, if $b \in S^{m_2}$ and there exists $M > 0$, $\delta > 0$

$$\inf_{\operatorname{supp} b} |\sigma(A)| \langle \xi \rangle^{-m_1} > \delta,$$

then the same conclusions hold with $R_i \in \Psi^{m_2-m_1}(\mathbb{R}^d)$.

2.1.4 Pseudodifferential operators on manifolds

Since we only use the notion of a pseudodifferential operator on a manifold in passing (in Lemma 2.15 and §2.5 below), we simply note that it is possible to define pseudodifferential operators on manifolds (see, e.g., [85, Chapter 14]). The analogues of Lemmas 2.1, 2.2, and 2.3 all hold in this setting. Moreover, the principal symbol map can still be defined although its definition is somewhat more involved.

2.2 Function spaces on piecewise-smooth hypersurfaces

Definition 2.4 (Extendable Sobolev space $\overline{H^s}(\Gamma)$ on a smooth hypersurface) *Let Γ be a smooth hypersurface of $\mathbb{R}^d$ (in the sense of Definition 1.3) and let $\widetilde{\Gamma}$ be an extension of Γ. Given $s \in \mathbb{R}$, we say that $u \in \overline{H^s}(\Gamma)$ if there exists $\underline{u} \in H^s_{\mathrm{comp}}(\widetilde{\Gamma})$ such that $\underline{u}|_\Gamma = u$.*

Let $(U_j, \psi_j)_{j\in J}$ be an atlas of $\widetilde{\Gamma}$ such that $U_j \cap \partial\Gamma \cap \partial\widetilde{\Gamma} = \emptyset$ for all $j \in J$, and let

$$J_\Gamma := \big\{ j \in J,\ U_j \cap \Gamma \neq \emptyset \big\} \quad \text{and} \quad J_\partial := \big\{ j \in J,\ U_j \cap \partial\Gamma \neq \emptyset \big\}$$

(observe that if $\partial\Gamma = \emptyset$ then $J_\partial = \emptyset$). Let $(\chi_j)_{j\in J}$ be a partition of unity of $\widetilde{\Gamma}$ subordinated to $(U_j)_{j\in J}$. Given $\chi \in C^\infty_c(\mathrm{Int}(\widetilde{\Gamma}))$ such that $\chi = 1$ in a neighborhood of Γ, we define

$$\|u\|_{\overline{H^s}(\Gamma)} = \sum_{j\in J_\Gamma\setminus J_\partial} \|(\chi_j u)\circ\psi_j^{-1}\|_{H^s(\mathbb{R}^{d-1})} + \inf_{\underline{u}\in H^s_{\mathrm{comp}}(\widetilde{\Gamma}),\underline{u}|_\Gamma = u} \sum_{j\in J_\partial} \|(\chi_j\chi\underline{u})\circ\psi_j^{-1}\|_{H^s(\mathbb{R}^{d-1})}. \tag{2.2}$$

We make two remarks:

1. The definition of the norm $\overline{H^s}(\Gamma)$ depends on $\widetilde{\Gamma}$, χ, and the choice of charts (U_j, ψ_j) and partition of unity (χ_j). One can however prove that two different choices of charts (U_j, ψ_j) and partition of unity (χ_j) lead to equivalent norms $\overline{H^s}(\Gamma)$. In what follows, (U_j, ψ_j, χ_j) shall be traces on $\widetilde{\Gamma}$ of charts and partition of unity on $\mathbb{R}^d$.
2. When Γ is a compact embedded submanifold without boundary, the norm on $\overline{H^s}(\Gamma)$ coincides with usual $H^s(\Gamma)$ norm.

Definition 2.5 (Sobolev space $\dot{H}^s(\Gamma)$ on a smooth hypersurface) *Let Γ be a smooth hypersurface of $\mathbb{R}^d$ (in the sense of Definition 1.3) and let $\widetilde{\Gamma}$ be an extension of Γ. Given $s \in \mathbb{R}$, We say that $u \in \dot{H}^s(\Gamma)$ if there exists $\underline{u} \in H^s_{\mathrm{comp}}(\widetilde{\Gamma})$ such that $\underline{u}|_\Gamma = u$ and $\operatorname{supp}\underline{u} \subset \overline{\Gamma}$. Then,*

$$\|u\|_{\dot{H}^s(\Gamma)} := \|\underline{u}\|_{H^s(\widetilde{\Gamma})} = \|u\|_{\overline{H^s}(\Gamma)}.$$

Since Γ has C^0 boundary, one can show [21, Theorem 3.3, Lemma 3.15] that the dual of $\overline{H^s}(\Gamma)$ is given by $\dot{H}^{-s}(\Gamma)$ with the dual pairing inherited from that of $H^s_{\mathrm{comp}}(\widetilde{\Gamma})$ and $H^{-s}_{\mathrm{comp}}(\widetilde{\Gamma})$.

For piecewise-smooth $\partial\Omega$, it is useful to consider the following "piecewise-H^s" spaces.

Definition 2.6 (Sobolev space $\overline{H^s}(\partial\Omega)$) *Let Ω be a bounded Lipschitz open set such that its open complement is connected and $\partial\Omega$ is a piecewise smooth hypersurface (in the sense of Definition 1.5); i.e., $\partial\Omega = \cup_{i=1}^N \overline{\Gamma}_i$ where Γ_i are smooth hypersurfaces. With $|s| \le 1$, we say that $u \in \overline{H^s}(\partial\Omega)$ if*

$$u = \sum_{i=1}^N u_i, \quad \textit{for } u_i \in \overline{H^s}(\Gamma_i), \quad \textit{and we let} \quad \|u\|_{\overline{H^s}(\partial\Omega)} := \sqrt{\sum_{i=1}^N \|u_i\|^2_{\overline{H^s}(\Gamma_i)}}.$$

We similarly define the norms $\bar{H}^s_k(\Gamma)$ and $\dot{H}^s_k(\Gamma)$ replacing $\|\cdot\|_{H^s(\mathbb{R}^{d-1})}$ in (2.2) with the weighted-norm $\|\cdot\|_{H^s_k(\mathbb{R}^{d-1})}$.

The following lemma shows that, when S_k, D_k, and D'_k map $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$, to bound the $H^1(\partial\Omega)$ norms of $S_k\phi$, $D_k\phi$, and $D'_k\phi$, it is sufficient to bound their $\overline{H^1}(\partial\Omega)$ norms.

Lemma 2.7 *Let Ω be a bounded Lipschitz open set such that its open complement is connected and $\partial\Omega$ is a piecewise smooth hypersurface (in the sense of Definition 1.5). If $u \in H^1(\partial\Omega)$ then*

$$\|u\|_{H^1(\partial\Omega)} \le \|u\|_{\overline{H^1}(\partial\Omega)} \tag{2.3}$$

Proof. Recall that $H^1(\partial\Omega)$ can be defined as the completion of $C^\infty_{\mathrm{comp}}(\partial\Omega) := \{u|_{\partial\Omega} : u \in C^\infty_0(\mathbb{R}^d)\}$ with respect to the norm

$$\int_{\partial\Omega} \Big(|\nabla_{\partial\Omega} u|^2 + |u|^2 \Big) \mathrm{d}s \tag{2.4}$$

[18, Pages 275-276] where $\nabla_{\partial\Omega}$ is the *surface gradient*, defined in terms of a parametrisation of the boundary by, e.g., [18, Equations (A.13) and (A.14)]. By the definition of the $\overline{H^1}(\Gamma_i)$ norm from Definition 2.4, u restricted to Γ_i satisfies

$$\begin{aligned}\|u\|^2_{\overline{H^1}(\Gamma_i)} &= \int_{\Gamma_i}\Big(|\nabla_{\Gamma_i}u|^2+|u|^2\Big)\mathrm{d}s(\Gamma_i)+\int_{\widetilde{\Gamma}_i\setminus\Gamma_i}\Big(|\nabla_{\widetilde{\Gamma}_i}\underline{u}|^2+|\underline{u}|^2\Big)\mathrm{d}s(\widetilde{\Gamma}_i),\\ &\geq \int_{\Gamma_i}\Big(|\nabla_{\Gamma_i}u|^2+|u|^2\Big)\mathrm{d}s(\Gamma_i).\end{aligned}$$

Then,

$$\|u\|^2_{H^1(\partial\Omega)}=\int_{\partial\Omega}\Big(|\nabla_{\partial\Omega}u|^2+|u|^2\Big)\mathrm{d}s=\sum_{i=1}^N\int_{\Gamma_i}\Big(|\nabla_{\Gamma_i}u|^2+|u|^2\Big)\mathrm{d}s(\Gamma_i)\leq\sum_{i=1}^N\|u\|^2_{\overline{H^1}(\Gamma_i)}=\|u\|^2_{\overline{H^1}(\partial\Omega)}$$

and the proof is complete. ■

Observe that Lemma 2.7 also holds when $H^1(\partial\Omega)$ and $\overline{H^1}(\partial\Omega)$ are replaced by $H^1_k(\partial\Omega)$ and $\overline{H^1_k}(\partial\Omega)$ respectively.

2.3 Recap of restriction estimates for quasimodes

Theorem 2.8 *Let $U\subset\mathbb{R}^d$ be open and precompact with Γ a smooth hypersurface (in the sense of Definition 1.3) satisfying $\overline{\Gamma}\subset U$. Given $k_0>0$, there exists $C>0$ (independent of k) so that if $\|u\|_{L^2(U)}=1$ with*

$$(-k^{-2}\Delta-1)u=\mathcal{O}_{L^2(U)}(k^{-1}),\tag{2.5}$$

then, for all $k\geq k_0$,

$$\|u\|_{L^2(\Gamma)}\leq\begin{cases}Ck^{1/4},\\ Ck^{1/6}, & \Gamma\ \textit{curved},\end{cases}\tag{2.6}$$

and

$$\|\partial_\nu u\|_{L^2(\Gamma))}\leq Ck\tag{2.7}$$

where ∂_ν is a choice of normal derivative to Γ.

In the context of the wave equation on smooth Riemannian manifolds with restriction to a submanifold, the estimates (2.6) along with their L^p generalizations appear in the work of Tataru [81] who also notes that the L^2 bounds are a corollary of an estimate of Greenleaf and Seeger [44]. The semiclassical version was studied by Burq, Gérard and Tzvetkov in [16], Tacy [79] and Hassell-Tacy [47].

Estimates like (2.7) first appeared in the work of Tataru [81] in the form of regularity estimates for restrictions of solutions to hyperbolic equations. Semiclassical analogs of this estimate were proved in Christianson–Hassell–Toth [27] and Tacy [80].

Remark 2.9 (Smoothness of Γ required for the quasimode estimates) *The $k^{1/4}$-bound in (2.6) is valid when Γ is only $C^{1,1}$, and the $k^{1/6}$-bound is valid when Γ is $C^{2,1}$ and curved. Therefore, with some extra work it should be possible to prove that the bounds on S_k in Theorem 1.10 hold with the assumption "piecewise smooth" replaced by "piecewise $C^{1,1}$" and "piecewise $C^{2,1}$ and curved" respectively. On the other hand, the bound (2.29) is not known in the literature for lower regularity Γ.*

2.4 Proof of Parts (a) and (b) of Theorem 1.10

When proving these results, it is more convenient to work in semiclassical Sobolev spaces, i.e. to prove the bounds from $L^2(\partial\Omega)$ to $H^1_k(\partial\Omega)$. We therefore now restate Theorem 1.10 as Theorem 2.10 below, working in these spaces.

Theorem 2.10 (Restatement of Theorem 1.10 as bounds from $L^2(\partial\Omega) \to H^1_k(\partial\Omega)$)
(a) If $\partial\Omega$ is a piecewise-smooth hypersurface (in the sense of Definition 1.5), then, given $k_0 > 1$, there exists $C > 0$ (independent of k) such that

$$\|S_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} \leq C\, k^{-1/2}\, \log k. \tag{2.8}$$

for all $k \geq k_0$. Moreover, if $\partial\Omega$ is piecewise curved (in the sense of Definition 1.6), then, given $k_0 > 1$, the following stronger estimate holds for all $k \geq k_0$

$$\|S_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} \leq Ck^{-2/3} \log k. \tag{2.9}$$

(b) If $\partial\Omega$ is a piecewise smooth, $C^{2,\alpha}$ hypersurface, for some $\alpha > 0$, then, given $k_0 > 1$, there exists $C > 0$ (independent of k) such that

$$\|D_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} \leq Ck^{1/4} \log k. \tag{2.10}$$

Moreover, if $\partial\Omega$ is piecewise curved, then, given $k_0 > 1$, there exists $C > 0$ (independent of k) such that the following stronger estimates hold for all $k \geq k_0$

$$\|D_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} \lesssim k^{1/6} \log k.$$

(c) If Ω is convex and $\partial\Omega$ is C^∞ and curved (in the sense of Definition 1.4) then, given $k_0 > 1$, there exists C such that, for $k \geq k_0$,

$$\|S_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} \leq Ck^{-2/3},$$
$$\|D_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} + \|D'_k\|_{L^2(\partial\Omega)\to H^1_k(\partial\Omega)} \leq C.$$

This theorem is actually stronger than Theorem 1.10 in that it now contains the $L^2(\partial\Omega) \to L^2(\partial\Omega)$ estimates originally proved in [39, Theorem 1.2], [46, Appendix A], and [38, Theorems 4.29, 4.32].

In §2.4.2 below, we give an outline of the proof of Parts (a) and (b). This outline, however, requires the definitions of S_k, D_k, and D'_k in terms of the free resolvent, given in the next subsection.

2.4.1 S_k, D_k, and D'_k written in terms of the free resolvent $R_0(k)$

We now recall the definitions of S_k, D_k, and D'_k in terms of the free resolvent $R_0(k)$, these expressions are well-known in the theory of BIEs on Lipschitz domains [29], [61, Chapters 6 and 7]. We then specialise these to the case when $\partial\Omega$ is a piecewise-smooth hypersurface (in the sense of Definition 1.5)

Let $R_0(k)$ be the free (outgoing) resolvent at k; i.e. for $\psi \in C^\infty_{\rm comp}(\mathbb{R}^d)$ we have

$$\big(R_0(k)\psi\big)(x) := \int_{\mathbb{R}^d} \Phi_k(x,y)\psi(y)\,{\rm d}y,$$

where $\Phi_k(x,y)$ is the (outgoing) fundamental solution defined by (1.6) for $d = 2, 3$. Recall that $R_0(k) : H^s_{\rm comp}(\mathbb{R}^d) \to H^{s+2}_{\rm loc}(\mathbb{R}^d)$; see, e.g., [61, Equation 6.10].

With Ω a bounded Lipschitz open set with boundary $\partial\Omega$ and $1/2 < s < 3/2$, let $\gamma^+ : H^s_{loc}(\Omega_+) \to H^{s-1/2}(\partial\Omega)$ and $\gamma^- : H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$, be the trace maps [29, Lemma 3.6], [61, Theorem 3.38]. When $\gamma^+ u = \gamma^- u$ we write both as γu (so that $\gamma : H^s_{\rm loc}(\mathbb{R}^d) \to H^{s-1/2}(\partial\Omega)$), and we then let $\gamma^* : H^{-s+1/2}(\partial\Omega) \to H^{-s}_{\rm comp}(\mathbb{R}^d)$ be the adjoint of γ [61, Page 201]. Then S_k can be written as

$$S_k = \gamma R_0(k)\gamma^* \tag{2.11}$$

[61, Page 202 and Equation 7.5], [29, Proof of Theorem 1]. With ∂_n^* denoting the adjoint of the normal derivative trace (see, e.g., [61, Equation 6.14]), we have that the double-layer potential, $\mathcal{D}_k$, is defined by

$$\mathcal{D}_k := R_0(k)\partial_n^*$$

[61, Page 202]. Recalling that the normal vector n to point out of Ω and into Ω_+, we have that the traces of $\mathcal{D}_k$ from $\Omega_\pm$ to Γ are given by

$$\gamma^\pm \mathcal{D}_k = \pm\frac{1}{2}I + D_k$$

[61, Equation 7.5 and Theorem 7.3] and thus

$$D_k = \mp\frac{1}{2}I + \gamma^\pm R_0(k)\partial_n^*. \tag{2.12}$$

Similarly, the result about the normal-derivative traces of the single-layer potential $\mathcal{S}_k$ implies that

$$\partial_n^\pm \mathcal{S}_k = \mp\frac{1}{2}I + D_k'$$

so

$$D_k' = \pm\frac{1}{2}I + \partial_n^\pm R_0(k)\gamma^*. \tag{2.13}$$

When $\partial\Omega$ is Lipschitz, $S_k : L^2(\partial\Omega) \to H^1(\partial\Omega)$ by [83, Theorem 1.6] (see also, e.g., [65, Chapter 15, Theorem 5], [66, Proposition 3.8]), and when $\partial\Omega$ is $C^{2,\alpha}$ for some $\alpha > 0$, then $D_k, D_k' : L^2(\partial\Omega) \to H^1(\partial\Omega)$ by [53, Theorem 4.2] (see also [28, Theorem 3.6]).

We now consider the case when $\partial\Omega$ is a piecewise-smooth hypersurface (in the sense of Definition 1.5) and use the notation that $\widetilde{\Gamma}_i$ are the compact embedded smooth manifolds of $\mathbb{R}^d$ such that, for each i, Γ_i is an open subset of $\widetilde{\Gamma}_i$. Let L_i be a vector field whose restriction to $\widetilde{\Gamma}_i$ is equal to ∂_{ν_i}, the normal to $\widetilde{\Gamma}_i$ that is outward pointing with respect to $\partial\Omega$. Let $\gamma_i : H^s_{\rm loc}(\mathbb{R}^d) \to H^{s-1/2}(\Gamma_i)$ denote restriction to Γ_i. We note that γ_i^* is the inclusion map $f \mapsto f\delta_{\Gamma_i}$ where δ_{Γ_i} is $d-1$ dimensional Hausdorff measure on Γ. Finally, we let $\gamma_i^\pm$ denote restrictions from the interior and exterior respectively, where "interior" and "exterior" are defined via considering Γ_i as a subset of $\partial\Omega$. With these notations, we have that

$$D_k = \mp\frac{1}{2}I + \sum_{i,j} \gamma_i^\pm R_0(k) L_j^* \gamma_j^* \tag{2.14}$$

and

$$D_k' = \pm\frac{1}{2}I + \sum_{i,j} \gamma_i^\pm L_i R_0(k) \gamma_j^*; \tag{2.15}$$

the advantage of these last two expressions over (2.12) and (2.13) is that they involve γ_i and L_i instead of ∂_n^* and $\partial_n^\pm$.

In the rest of this section, we use the formulae (2.11), (2.14), and (2.15) as the definitions of S_k, D_k, and D_k'. Note that we slightly abuse notation by omitting the sums in (2.14) and (2.15) and instead writing

$$D_k = \pm\frac{1}{2}I + \gamma^\pm R_0(k) L\gamma^*, \quad D_k' = \mp\frac{1}{2}I + \gamma^\pm L R_0(k)\gamma^*. \tag{2.16}$$

2.4.2 Outline of the proof of Parts (a) and (b) of Theorem 2.10

The proof of Parts (a) and (b) of Theorem 2.10 will follow in two steps. In Lemma 2.11, we obtain estimates on frequencies $\le Mk$ and in Lemma 2.19 we complete the proof by estimating the high frequencies ($\ge Mk$).

To estimate the low frequency components, we spectrally decompose the resolvent using the Fourier transform. We are then able to reduce the proof of the low-frequency estimates to the estimates on the restriction of eigenfunctions (or more precisely quasimodes) to $\partial\Omega$ that we recalled in §2.3. To understand this reduction, we proceed schematically; from the description of S_k in terms of the free resolvent, (2.11), the spectral decomposition of S_k via the Fourier transform is schematically

$$S_k f = \int_0^\infty \frac{1}{r^2 - (k + {\rm i}0)^2} \big\langle f, \gamma u(r)\big\rangle_{L^2(\partial\Omega)} \gamma u(r)\, {\rm d}r$$

where $u(r)$ is a generalized eigenfunction of $-\Delta$ with eigenvalue r^2. Using this decomposition we see that estimating S_k amounts to estimating the restriction of the generalized eigenfunction $u(r)$ to $\partial\Omega$.

At very high frequency, we compare the operators S_k, D_k, and D_k' with the corresponding operators when $k = 1$ (recall that the mapping properties of boundary integral operators with $k = 1$ have been extensively studied on rough domains; see, e.g. [65, Chapter 15], [61], [66]). By using a description of the resolvent at very high frequency as a pseudodifferential operator, we are able to see that these differences gain additional regularity and hence to obtain estimates on them easily.

The new ingredients in our proof compared to [39] and [46] are that we have H^s norms in Lemma 2.11 and Lemma 2.19 rather than the L^2 norms appearing in the previous work.

2.4.3 Proof of Parts (a) and (b) of Theorem 2.10

Low-frequency estimates. Following the outline in §2.4.2, our first task is to estimate frequencies $\leq kM$. We start by proving a conditional result that assumes a certain estimate on restriction of the Fourier transform of surface measures to the sphere of radius r (Lemma 2.11). In Lemma 2.13 we then show that the hypotheses in Lemma 2.11 are a consequence of restriction estimates for quasimodes. In Lemma 2.16 we show how the low-frequency estimates on S_k, D_k, and D_k' follow from Lemma 2.11.

In this section we denote the sphere of radius r by S_r^{d-1} and we denote the surface measure on S_r^{d-1} by $\mathrm{d}\sigma$.

Lemma 2.11 *Suppose that for $\Gamma \subset \mathbb{R}^d$ any precompact smooth hypersurface, $s \geq 0$, $f \in \dot{H}^{-s}(\Gamma)$, and some α, $\beta > 0$,*

$$\int_{S_r^{d-1}} |\widehat{L^* f\delta_\Gamma}|^2(\xi)\mathrm{d}\sigma(\xi) \leq C_\Gamma \langle r\rangle^{2\alpha+2s}\|f\|^2_{\dot{H}^{-s}(\Gamma)}, \tag{2.17}$$

$$\int_{S_r^{d-1}} |\widehat{f\delta_\Gamma}|^2(\xi)\mathrm{d}\sigma(\xi) \leq C_\Gamma \langle r\rangle^{2\beta+2s}\|f\|^2_{\dot{H}^{-s}(\Gamma)}. \tag{2.18}$$

Let Γ_1, $\Gamma_2 \subset \mathbb{R}^d$ be compact embedded smooth hypersurfaces. Recall that L_i is a vector field with $L_i = \partial_\nu$ on Γ_i for some choice of normal ν on Γ_i and $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ in neighborhood of 0. With the frequency cutoff $\psi(k^{-1}D)$ defined as in (2.1), we then define for $f \in \dot{H}^{-s_1}(\Gamma_1)$, $g \in \dot{H}^{-s_2}(\Gamma_2)$, $s_i \geq 0$,

$$Q_S(f,g) := \int_{\mathbb{R}^d} R_0(k)(\psi(k^{-1}D)f\delta_{\Gamma_1})\bar{g}\delta_{\Gamma_2}\mathrm{d}x\,, \quad Q_D(f,g) := \int_{\mathbb{R}^d} R_0(k)(\psi(k^{-1}D)L_1^*(f\delta_{\Gamma_1}))\bar{g}\delta_{\Gamma_2}\mathrm{d}x$$

$$Q_{D'}(f,g) := \int_{\mathbb{R}^d} R_0(k)(\psi(k^{-1}D)f\delta_{\Gamma_1})\overline{L_2^*(g\delta_{\Gamma_2})}\,\mathrm{d}x.$$

Then there exists C_{Γ_1,Γ_2} so that for $k > 1$,

$$|Q_S(f,g)| \leq C_{\Gamma_1,\Gamma_2,\psi}\langle k\rangle^{2\beta-1+s_1+s_2}\log\langle k\rangle\|f\|_{\dot{H}^{-s_1}(\Gamma_1)}\|g\|_{\dot{H}^{-s_2}(\Gamma_2)} \tag{2.19}$$

$$|Q_D(f,g)| + |Q_{D'}(f,g)| \leq C_{\Gamma_1,\Gamma_2,\psi}\langle k\rangle^{\alpha+\beta-1+s_1+s_2}\log\langle k\rangle\|f\|_{\dot{H}^{-s_1}(\Gamma_1)}\|g\|_{\dot{H}^{-s_2}(\Gamma_2)} \tag{2.20}$$

The key point is that, modulo the frequency cutoff $\psi(k^{-1}D)$, $Q_S(f,g)$, $Q_D(f,g)$, and $Q_{D'}(f,g)$ are given respectively by $\langle S_k f, g\rangle_\Gamma$, $\langle D_k f, g\rangle_\Gamma$, and $\langle D_k' f, g\rangle_\Gamma$,where f is supported on Γ_1 and g on Γ_2.

Proof of Lemma 2.11. We follow [39] [46] to prove the lemma. First, observe that due to the compact support of $f\delta_{\Gamma_i}$, (2.17) and (2.18) imply that for $\Gamma \subset \mathbb{R}^d$ precompact,

$$\int_{S_r^{d-1}} \left|\nabla_\xi\, \widehat{L^* f\delta_\Gamma}(\xi)\right|^2 \mathrm{d}\sigma(\xi) \leq C\,\langle r\rangle^{2\alpha+2s}\|f\|^2_{\dot{H}^{-s}(\Gamma)}\,, \tag{2.21}$$

$$\int_{S_r^{d-1}} \left|\nabla_\xi\, \widehat{f\delta_\Gamma}(\xi)\right|^2 \mathrm{d}\sigma(\xi) \leq C\,\langle r\rangle^{2\beta+2s}\|f\|^2_{\dot{H}^{-s}(\Gamma)}\,. \tag{2.22}$$

Indeed, $\nabla_\xi \widehat{f\delta_\Gamma} = \widehat{xf\delta_\Gamma}$ and since Γ is compact,

$$\|xf\|_{\dot{H}^{-s}(\Gamma)} \leq C\|f\|_{\dot{H}^{-s}(\Gamma)}.$$

Also, $\nabla_\xi \widehat{L^*(f\delta_\Gamma)} = \mathcal{F}(xL^*(f\delta_\Gamma))$. Thus,

$$xL^*(f\delta_\Gamma) = L^*(xf\delta_\Gamma) + [x, L^*]f\delta_\Gamma$$

and $[x, L^*] \in C^\infty$. Therefore, using compactness of Γ,

$$\|xf\|_{\dot{H}^{-s}(\Gamma)} + \|[x, L^*]f\|_{\dot{H}^{-s}(\Gamma)} \leq C\|f\|_{\dot{H}^{-s}(\Gamma)}.$$

Now, $g\delta_{\Gamma_2} \in H^{\min(-s,-1/2-\epsilon)}(\mathbb{R}^d)$, $L_2^*(g\delta_{\Gamma_2}) \in H^{\min(-s-1,-3/2-\epsilon)}(\mathbb{R}^d)$ and since $\psi \in C_c^\infty(\mathbb{R})$,

$$R_0(k)(\psi(k^{-1}|D|)L^*(f\delta_{\Gamma_1})) \in C^\infty(\mathbb{R}^d), \qquad R_0(k)(\psi(k^{-1}|D|))f\delta_{\Gamma_1}) \in C^\infty(\mathbb{R}^d).$$

By Plancherel's theorem,

$$Q_S(f,g) = \int_{\mathbb{R}^d} \psi(k^{-1}|\xi|)\frac{\widehat{f\delta_{\Gamma_1}}(\xi)\overline{\widehat{g\delta_{\Gamma_2}}(\xi)}}{|\xi|^2 - (k+\mathrm{i}0)^2}\mathrm{d}\xi, \qquad Q_D(f,g) = \int_{\mathbb{R}^d} \psi(k^{-1}|\xi|)\frac{\widehat{L_1^* f\delta_{\Gamma_1}}(\xi)\,\overline{\widehat{g\delta_{\Gamma_2}}(\xi)}}{|\xi|^2 - (k+\mathrm{i}0)^2}\mathrm{d}\xi,$$

$$\text{and} \quad Q_{D'}(f,g) = \int_{\mathbb{R}^d} \psi(k^{-1}|\xi|)\frac{\widehat{f\delta_{\Gamma_1}}(\xi)\,\overline{\widehat{L_2^* g\delta_{\Gamma_2}}(\xi)}}{|\xi|^2 - (k+\mathrm{i}0)^2}\mathrm{d}\xi,$$

where $k + \mathrm{i}0$ is understood as the limit of $k + \mathrm{i}\epsilon$ as $\epsilon \to 0^+$.

Therefore, to prove the lemma, we only need to estimate

$$\int_{\mathbb{R}^d} \psi(k^{-1}|\xi|)\frac{F(\xi)\,G(\xi)}{|\xi|^2 - (k+\mathrm{i}0)^2}\mathrm{d}\xi \tag{2.23}$$

where, by (2.17), (2.18), (2.21), and (2.22),

$$\|F\|_{L^2(S_r^{d-1})} + \|\nabla_\xi F\|_{L^2(S_r^{d-1})} \leq C\langle r\rangle^{\delta_1+s_1}\|f\|_{\dot{H}^{-s_1}(\Gamma_1)}, \qquad \text{and}$$
$$\|G\|_{L^2(S_r^{d-1})} + \|\nabla_\xi G\|_{L^2(S_r^{d-1})} \leq C\langle r\rangle^{\delta_2+s_2}\|g\|_{\dot{H}^{-s_2}(\Gamma_2)}.$$

Consider first the integral in (2.23) over $\big||\xi| - |k|\big| \geq 1$. Since $\big||\xi|^2 - k^2\big| \geq \big||\xi|^2 - |k|^2\big|$, by the Schwartz inequality, (2.17), and (2.18), this piece of the integral is bounded by

$$\begin{aligned}
&\int_{||\xi|-|k||\geq 1} \left|\psi(k^{-1}|\xi|)\frac{F(\xi)\,G(\xi)}{|\xi|^2 - k^2}\right|\mathrm{d}\xi \\
&\qquad \leq \int_{Mk\geq|r-|k||\geq 1} \frac{1}{r^2 - |k|^2}\int_{S_r^{d-1}} F(r\theta)\,G(r\theta)\mathrm{d}\sigma(\theta)\mathrm{d}r \\
&\qquad \leq C\|f\|_{H^{-s_1}(\Gamma_1)}\|g\|_{H^{-s_2}(\Gamma_2)}\int_{M|k|\geq|r-|k||\geq 1} \langle r\rangle^{\delta_1+\delta_2+s_1+s_2}\left|\,r^2 - |k|^2\,\right|^{-1}\mathrm{d}r \\
&\qquad \leq C\|f\|_{H^{-s_1}(\Gamma_1)}\|g\|_{H^{-s_2}(\Gamma_2)}|k|^{\delta_1+\delta_2-1+s_1+s_2}\int_{M|k|\geq|r-|k||\geq 1} |r - |k||^{-1}\,\mathrm{d}r \\
&\qquad \leq C\,|k|^{\delta_1+\delta_2-1+s_1+s_2}\log|k|\,\|f\|_{\dot{H}^{-s_1}(\Gamma_1)}\|g\|_{\dot{H}^{-s_2}(\Gamma_2)}.
\end{aligned} \tag{2.24}$$

Since $k > 1$, we write

$$\frac{1}{|\xi|^2 - (k+\mathrm{i}0)^2} = \frac{1}{|\xi| + (k+\mathrm{i}0)}\,\frac{\xi}{|\xi|}\cdot\nabla_\xi\log(|\xi| - (k+\mathrm{i}0)),$$

where the logarithm is well defined since $\mathrm{Im}(|\xi| - (k+\mathrm{i}0)) < 0$. Let $\chi(r) = 1$ for $|r| \leq 1$ and vanish for $|r| \geq 3/2$. We then use integration by parts, together with (2.17), (2.18), (2.21), and (2.22) to bound

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \chi(|\xi| - |k|)\,\psi(k^{-1}|\xi|)\,\frac{1}{|\xi| + k + \mathrm{i}0}\,F(\xi)\,G(\xi)\,\frac{\xi}{|\xi|}\cdot\nabla_\xi\log(|\xi| - (k+\mathrm{i}0))\,\mathrm{d}\xi\right| \\
&\qquad\qquad \leq C\,|k|^{\delta_1+\delta_2-1+s_1+s_2}\,\|f\|_{\dot{H}^{-s_1}(\Gamma_1)}\|g\|_{\dot{H}^{-s_2}(\Gamma_2)}.
\end{aligned}$$

Now, taking $\delta_1 = \delta_2 = \beta$ gives (2.19), and taking $\delta_1 = \alpha$ and $\delta_2 = \beta$ gives (2.20). ■

Remark 2.12 *The estimate* (2.24) *is the only term where the* $\log|k|$ *appears, which leads to the* $\log k$ *factors in the bounds of Theorem 1.10 (without which these bounds would be sharp).*

The proofs of the estimates (2.17) and (2.18) are contained in the following lemma.

Lemma 2.13 *Let* $\Gamma \subset \mathbb{R}^d$ *be a precompact smooth hypersurface. Then estimate* (2.18) *holds with* $\beta = 1/4$ *and for* $L = \partial_\nu$ *on* Γ, *estimate* (2.17) *holds with* $\alpha = 1$. *Moreover, if* Γ *is curved then* (2.18) *holds with* $\beta = 1/6$.

To prove this lemma, we need to understand certain properties of the operator T_r defined by

$$T_r\phi := \int_{S_r^{d-1}} \phi(\xi)e^{i\langle x,\xi\rangle}d\sigma(\xi). \tag{2.25}$$

Indeed, with $A : H^s(\mathbb{R}^d) \to H^{s-1}(\mathbb{R}^d)$, to estimate

$$\int_{S_r^{d-1}} |\widehat{A^*(f\delta_\Gamma)}(\xi)|^2 d\sigma(\xi),$$

we write

$$\begin{aligned} \langle \widehat{A^*(f\delta_\Gamma)}(\xi), \phi(\xi)\rangle_{S_r^{d-1}} &= \int_{S_r^{d-1}}\int_{\mathbb{R}^d} A^*(f(x)\delta_\Gamma)\overline{\phi(\xi)e^{i\langle x,\xi\rangle}}dx d\sigma(\xi) \\ &= \int_\Gamma f A T_r\phi \, dx = \langle f, AT_r\phi\rangle_\Gamma, \end{aligned} \tag{2.26}$$

with T_r defined by (2.25).

Before proving Lemma 2.13 we prove two lemmas (Lemma 2.14 and 2.15) collecting properties of T_r.

Lemma 2.14 *Let* T_r *be defined by* (2.25) *and* $\chi \in C_c^\infty(\mathbb{R}^d)$. *Then,*

$$\|\chi T_r\phi\|_{L^2(\mathbb{R}^d)} \le C\|\phi\|_{L^2(S_r^{d-1})}.$$

Proof of Lemma 2.14. We estimate $B := (\chi T_r)^*\chi T_r : L^2S_r^{d-1} \to L^2S_r^{d-1}$. This operator has kernel

$$B(\xi,\eta) = \int_{\mathbb{R}^d} \chi^2(y)\exp\left(i\langle y, \xi-\eta\rangle\right)dy = \widehat{\chi^2}(\eta-\xi).$$

Now, for $\eta \in S_r^{d-1}$, and any $N > 0$,

$$\int_{S_r^{d-1}} |\widehat{\chi^2}(\eta-\xi)|\, d\sigma(\xi) \le \int_{B(0,r/2)} \langle\xi'\rangle^{-N}\left[1-\frac{|\xi'|^2}{r^2}\right]^{-1/2} d\xi' + C\langle r\rangle^{-N} \le C.$$

Thus, by Schur's inequality, B is bounded on $L^2S_r^{d-1}$ uniformly in r. Therefore,

$$\|\chi T_r\phi\|^2_{L^2(\mathbb{R}^d)} \le C\|\phi\|^2_{L^2(S_r^{d-1})}.$$

∎

Lemma 2.15 *With* T_r *be defined by* (2.25), *let* $\widetilde{\Gamma}$ *denote an extension of* Γ, $\chi \in C_c^\infty(\mathbb{R}^d)$ *and* $A \in \Psi^1(\mathbb{R}^d)$ *with* $\chi \equiv 1$ *in a neighborhood of* $\widetilde{\Gamma}$. *Then for* $s \in \mathbb{R}$,

$$\|\chi AT_r\phi\|_{\overline{H}_r^s(\Gamma)} \le C_s\|\chi AT_r\phi\|_{L^2(\widetilde{\Gamma})}.$$

Proof of Lemma 2.15. Since $\widehat{T_r\phi}$ is supported on $|\xi| \le 2r$, $\chi T_r\phi$ is compactly microlocalized in the sense that for $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $[-2,2]$ with support in $[-3,3]$,

$$\psi(r^{-1}|D|)\chi AT_r\phi = \chi AT_r\phi + \mathcal{O}_{\Psi^{-\infty}}(r^{-\infty})\chi T_r\phi.$$

(Note that $\psi(r^{-1}|D|)$ can be defined using (2.1) since $\psi(t)$ is constant near $t = 0$.)

Let $\gamma_{\widetilde{\Gamma}}$ denote restriction to $\widetilde{\Gamma}$, and $\gamma|_\Gamma$ restriction to Γ. Let $\chi_\Gamma \in C_c^\infty(\widetilde{\Gamma})$ with $\chi_\Gamma \equiv 1$ on Γ. Then for $\psi_1 \in C_c^\infty(\mathbb{R})$ with $\psi_1 \equiv 1$ on $[-4, 4]$,

$$\chi_\Gamma \psi_1(r^{-1}|D'|_g)\chi_\Gamma \gamma_{\widetilde{\Gamma}} \chi A T_r \phi = \chi_\Gamma^2 \gamma_{\widetilde{\Gamma}} \chi A T_r \phi + \mathcal{O}_{\Psi^{-\infty}}(r^{-\infty})\gamma_{\widetilde{\Gamma}} \chi T_r \phi$$

where $\psi_1(r^{-1}|D'|_g)$ is a pseudodifferential operator on $\widetilde{\Gamma}$ with symbol $\psi_1(|\xi'|_g)$ and $|\cdot|_g$ denotes the metric induced on $T^*\widetilde{\Gamma}$ from $\mathbb{R}^d$. Hence, for $r > 1$,

$$\|\gamma_\Gamma \chi A T_r \phi\|_{\overline{H}_r^s(\Gamma)} \leq C_s \|\chi A T_r \phi\|_{L^2(\widetilde{\Gamma})}.$$

■

We are now in a position to prove Lemma 2.13.

Proof of Lemma 2.13. The key observation for the proof of Lemma 2.13 is that for $\chi \in C_c^\infty(\mathbb{R}^d)$, $\chi T_r \phi$ is a quasimode of the Laplacian with $k = r$ in the sense of (2.5) in Theorem 2.8. To see this, observe first that $-\Delta T_r \phi = r^2 T_r \phi$ by the definition (2.25). Therefore,

$$-\Delta \chi T_r \phi = r^2 \chi T_r \phi + [-\Delta, \chi] T_r \phi.$$

Now, observe that for $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \tilde{\chi} \subset \{\chi \equiv 1\}$, $\tilde{\chi}[-\Delta, \chi] = 0$. Therefore, taking such a $\tilde{\chi}$ with $\tilde{\chi} \equiv 1$ in a neighborhood, U of Γ shows that $\chi T_r \phi$ is a quasimode.

To prove (2.18), we let $A = I$. Then, by the bounds (2.6) in Theorem 2.8 together with Lemmas 2.14 and 2.15, for $s \geq 0$,

$$\|\chi T_r \phi\|_{\overline{H}^s(\Gamma)} \leq C_s \langle r\rangle^s \|\chi T_r \phi\|_{L^2(\widetilde{\Gamma})} \leq C_s \langle r\rangle^{\frac{1}{4}+s} \|\chi T_r \phi\|_{L^2(\mathbb{R}^d)} \leq C_s \langle r\rangle^{\frac{1}{4}+s} \|\phi\|_{L^2(S_r^{d-1})}, \tag{2.27}$$

and if Γ is curved then

$$\|\chi T_r \phi\|_{\overline{H}^s(\Gamma)} \leq C \langle r\rangle^{\frac{1}{6}+s} \|\phi\|_{L^2(S_r^{d-1})}. \tag{2.28}$$

To prove (2.17), we take $A = L$. Observe that

$$\gamma_{\widetilde{\Gamma}} \chi L T_r \phi = \gamma_{\widetilde{\Gamma}} L \chi T_r \phi.$$

Hence, using the fact that $L = \partial_\nu$ on Γ together with the bound (2.7) in Theorem 2.8, we can estimate $L T_r \phi$.

$$\|\chi L T_r \phi\|_{L^2(\widetilde{\Gamma})} = \|L \chi T_r \phi\|_{L^2(\widetilde{\Gamma})} \leq C \langle r\rangle \|\chi T_r \phi\|_{L^2(\mathbb{R}^d)}. \tag{2.29}$$

In particular, for $s \geq 0$,

$$\|\chi L T_r \phi\|_{\overline{H}^s(\Gamma)} \leq C_s \langle r\rangle^{s+1} \|\phi\|_{L^2(S_r^{d-1})}.$$

Applying Cauchy-Schwarz together with (2.26), (2.27), (2.28) and (2.29) completes the proof of Lemma 2.13, since we have shown that

$$|\langle \widehat{f\delta_\Gamma}(\xi), \phi(\xi)\rangle_{L^2(S_r^{d-1})}| \leq C_s \langle r\rangle^{\frac{1}{4}+s} \|f\|_{\dot{H}^{-s}(\Gamma)} \|\phi\|_{L^2(S_r^{d-1})},$$
$$|\langle \widehat{L^*(f\delta_\Gamma)}(\xi), \phi(\xi)\rangle_{L^2(S_r^{d-1})}| \leq C_s \langle r\rangle^{1+s} \|f\|_{\dot{H}^{-s}(\Gamma)} \|\phi\|_{L^2(S_r^{d-1})},$$

and if Γ is curved,

$$|\langle \widehat{f\delta_\Gamma}(\xi), \phi(\xi)\rangle_{L^2(S_r^{d-1})}| \leq C_s \langle r\rangle^{\frac{1}{6}+s} \|f\|_{\dot{H}^{-s}(\Gamma)} \|\phi\|_{L^2(S_r^{d-1})}.$$

■

Lemma 2.16 (Low-frequency estimates) *Let s_2 be either 0 or 1. If $\partial\Omega$ is piecewise smooth and Lipschitz, then*

$$\|\gamma^\pm R_0(k)\psi(k^{-1}D)\gamma^* f\|_{H^{s_2}(\partial\Omega)} \leq C\langle k\rangle^{2\beta-1+s_2} \log\langle k\rangle \|f\|_{L^2(\partial\Omega)} \tag{2.30}$$
$$\|\gamma R_0(k)\psi(k^{-1}D) L_1^* \gamma^* f\|_{H^{s_2}(\partial\Omega)} \leq C\langle k\rangle^{\beta+s_2} \log\langle k\rangle \|f\|_{L^2(\partial\Omega)} \tag{2.31}$$
$$\|\gamma^\pm L_2 R_0(k)\psi(k^{-1}D)\gamma^* f\|_{H^{s_2}(\partial\Omega)} \leq C\langle k\rangle^{\beta+s_2} \log\langle k\rangle \|f\|_{L^2(\partial\Omega)}. \tag{2.32}$$

with $\beta = 1/4$. If $\partial\Omega$ is piecewise curved and Lipschitz then (2.30)-(2.32) hold with $\beta = 1/6$.

Proof of Lemma 2.16. By the duality property of $\overline{H^s}(\Gamma)$ and $\dot{H}^{-s}(\Gamma)$ (discussed after Definition 2.5), Lemma 2.13 and the estimates (2.19) and (2.20) imply for $s_1, s_2 \geq 0$ that there exists $C > 0$ independent of $k > 1$ so that

$$\|\gamma_{\Gamma_2} R_0(k)\psi(k^{-1}D)\gamma^*_{\Gamma_1} f\|_{\overline{H}^{s_2}(\Gamma_2)} \leq C\langle k\rangle^{2\beta-1+s_1+s_2}\log\langle k\rangle \|f\|_{\dot{H}^{-s_1}(\Gamma_1)}, \tag{2.33}$$

$$\|\gamma_{\Gamma_2} R_0(k)\psi(k^{-1}D)L_1^*\gamma^*_{\Gamma_1} f\|_{\overline{H}^{s_2}(\Gamma_2)} \leq C\langle k\rangle^{\beta+s_1+s_2}\log\langle k\rangle \|f\|_{\dot{H}^{-s_1}(\Gamma_1)}, \tag{2.34}$$

$$\|\gamma_{\Gamma_2} L_2 R_0(k)\psi(k^{-1}D)\gamma^*_{\Gamma_1} f\|_{\overline{H}^{s_2}(\Gamma_2)} \leq C\langle k\rangle^{\beta+s_1+s_2}\log\langle k\rangle \|f\|_{\dot{H}^{-s_1}(\Gamma_1)}. \tag{2.35}$$

Since $\partial\Omega$ is piecewise-smooth, $\partial\Omega = \sum_{i=1}^N \Gamma_i$ with Γ_i smooth hypersurfaces. Since $\psi(k^{-1}D)$ is a smoothing operator on $\mathcal{S}'$, by elliptic regularity $R_0(k)\psi(k^{-1}D)$ is smoothing and hence its restriction to $\partial\Omega$ maps compactly supported distributions into $H^1(\partial\Omega)$. Applying (2.33)-(2.35) with $s_1 = 0$, $\Gamma = \Gamma_i$, and then summing over i, we find that, for $0 \leq s_2 \leq 1$,

$$\|\gamma R_0(k)\psi(k^{-1}D)\gamma^* f\|_{\overline{H}^{s_2}(\partial\Omega)} \leq C\langle k\rangle^{2\beta-1+s_2}\log\langle k\rangle \|f\|_{L^2(\partial\Omega)} \tag{2.36}$$

$$\|\gamma^{\pm} R_0(k)\psi(k^{-1}D)L_1^*\gamma^* f\|_{\overline{H}^{s_2}(\partial\Omega)} \leq C\langle k\rangle^{\beta+s_2}\log\langle k\rangle \|f\|_{L^2(\partial\Omega)} \tag{2.37}$$

$$\|\gamma^{\pm} L_2 R_0(k)\psi(k^{-1}D)\gamma^* f\|_{\overline{H}^{s_2}(\partial\Omega)} \leq C\langle k\rangle^{\beta+s_2}\log\langle k\rangle \|f\|_{L^2(\partial\Omega)}. \tag{2.38}$$

Applying (2.36)-(2.38) with $s_2 = 1$ (using the norm bound (2.3)) and $s_2 = 0$, we obtain the estimates (2.30)-(2.32). ∎

High frequency estimates. Next, we obtain an estimate on the high frequency ($\geq kM$) components of S_k, D_k, and D'_k. We start by analyzing the high frequency components of the free resolvent, proving two lemmata on the structure of the free resolvent there.

Lemma 2.17 *Suppose that $z \in [-E, E]$. Let $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $[-2E^2, 2E^2]$. Then for $\chi \in C_c^\infty(\mathbb{R}^d)$.*

$$\chi R_0(zk)\chi(1 - \psi(|k^{-1}D|)) = B_1, \qquad (1 - \psi(|k^{-1}D|))\chi R_0(zk)\chi = B_2$$

where $B_i \in k^{-2}\Psi^{-2}(\mathbb{R}^d)$ with

$$\sigma(B_i) = \frac{\chi^2 k^{-2}(1 - \psi(|\xi|))}{|\xi|^2 - z^2}.$$

Proof of Lemma 2.17. Let $\chi_0 = \chi \in C_c^\infty(\mathbb{R}^d)$ and $\chi_n \in C_c^\infty(\mathbb{R}^d)$ have $\chi_n \equiv 1$ on $\operatorname{supp}\chi_{n-1}$ for $n \geq 1$. Let $\psi_0 = \psi \in C_c^\infty(\mathbb{R})$ have $\psi \equiv 1$ on $[-2E^2, 2E^2]$, let $\psi_n \in C_c^\infty(\mathbb{R})$ have $\psi_n \equiv 1$ on $[-3E^2/2, 3E^2/2]$ and $\operatorname{supp}\psi_n \subset \{\psi_{n-1} \equiv 1\}$ for $n \geq 1$. Finally, let $\varphi_n = (1 - \psi_n)$. Then,

$$k^2\chi R_0(zk)\chi(-k^{-2}\Delta - z^2) = (\chi^2 - \chi k^2\chi_1 R_0(zk)\chi_1[\chi, k^{-2}\Delta]). \tag{2.39}$$

Now, by Lemma 2.3 there exists $A_0 \in k^{-2}\Psi^{-2}(\mathbb{R}^d)$ with

$$A_0 = k^{-2}a_0(x, k^{-1}D) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}), \qquad \operatorname{supp} a_0 \subset \{\operatorname{supp}\varphi_0\} \tag{2.40}$$

such that

$$k^2(-k^{-2}\Delta - z^2)A_0 = \varphi(|k^{-1}D|) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}) \tag{2.41}$$

and A_0 has

$$\sigma(A_0) = \frac{k^{-2}\varphi(|\xi|)}{|\xi|^2 - z^2}. \tag{2.42}$$

(Indeed, since we are working on $\mathbb{R}^d$,

$$k^2(-k^{-2}\Delta - z^2)\frac{k^{-2}\varphi(|k^{-1}D|)}{|k^{-1}D|^2 - z^2} = \varphi(|k^{-1}D|)$$

with no remainder.)

Composing (2.39) on the right with A_0, we have

$$\begin{aligned}\chi R_0\chi\varphi(|k^{-1}D|) &= \chi^2 A_0 - k^2\chi\chi_1 R_0\chi_1\varphi_1(|k^{-1}D|)[\chi, k^{-2}\Delta]A_0 + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}),\\ &= \chi^2 A_0 - \chi\chi_1 R_0\chi_1\varphi_1(|k^{-1}D|)\mathcal{O}_{\Psi^{-1}}(k^{-1}) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).\end{aligned}$$

Now, applying the same arguments, there exists $A_n \in k^{-2}\Psi^{-2}(\mathbb{R}^d)$ such that

$$\chi_n R_0\chi_n\varphi_n(|k^{-1}D|) = \chi_n^2 A_n + \chi_{n+1}R_0\chi_{n+1}\varphi_{n+1}(|k^{-1}D|)\mathcal{O}_{\Psi^{-1}}(k^{-1}) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).$$

Hence, by induction

$$\chi R_0\chi\varphi(|k^{-1}D|) = B_1 \in k^{-2}\Psi^{-2}(\mathbb{R}^d),$$

with

$$\sigma(B_1) = \frac{k^{-2}\chi^2(1-\psi(|\xi|))}{|\xi|^2 - z^2}$$

as desired. The proof of the statement for B_2 is identical. ∎

Next, we prove an estimate on the difference between the resolvent at high energy and that at fixed energy.

Lemma 2.18 *Suppose that $z \in [0,E]$. Let $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $[-2E^2, 2E^2]$. Then for $\chi \in C_c^\infty(\mathbb{R}^d)$,*

$$\chi(R_0(zk) - R_0(1))\chi(1-\psi(|k^{-1}D|)) \in k^{-2}\Psi^{-4}(\mathbb{R}^d).$$

Proof of Lemma 2.18. We proceed as in the proof of Lemma 2.17. Let $\chi_0 = \chi \in C_c^\infty(\mathbb{R}^d)$ and $\chi_n \in C_c^\infty(\mathbb{R}^d)$ have $\chi_n \equiv 1$ on $\operatorname{supp}\chi_{n-1}$ for $n \geq 1$. Let $\psi_0 = \psi \in C_c^\infty(\mathbb{R})$ have $\psi \equiv 1$ on $[-2E^2, 2E^2]$, let $\psi_n \in C_c^\infty(\mathbb{R})$ have $\psi_n \equiv 1$ on $[-3E^2/2, 3E^2/2]$ and $\operatorname{supp}\psi_n \subset \{\psi_{n-1} \equiv 1\}$ for $n \geq 1$. Finally, let $\varphi_n = (1-\psi_n)$. Then,

$$\begin{aligned}&k^2\chi(R_0(zk) - R_0(1))\chi(-k^{-2}\Delta - z^2)\\ &\qquad= \chi R_0(1)\left(z^2k^2 - 1\right)\chi - \chi k^2\chi_1(R_0(zk) - R_0(1))\chi_1[\chi, k^{-2}\Delta]). \end{aligned}\tag{2.43}$$

Now, by Lemma 2.3 there exists $A_0 \in k^{-2}\Psi^{-2}(\mathbb{R}^d)$ such that (2.40), (2.41), and (2.42) hold. Composing (2.43) on the right with $k^{-2}A_0$, we have

$$\begin{aligned}&\chi(R_0(zk) - R_0(1))\chi\varphi(|k^{-1}D|)\\ &= k^2\chi R_0(1)\chi(z^2 - k^{-2})A_0 - k^2\chi\chi_1(R_0(zk) - R_0(1))\chi_1\varphi_1(|k^{-1}D|)[\chi, k^{-2}\Delta]A_0 + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).\end{aligned}\tag{2.44}$$

In particular, iterating using the same argument to write

$$\begin{aligned}&\chi_1(R_0(zk) - R_0(1))\chi_1\varphi_1(|k^{-1}D|)\\ &= k^2\chi_1 R_0(1)\chi_1(z^2 - k^{-2})A_1 - k^2\chi_1\chi_2(R_0(zk) - R_0(1))\chi_2\varphi_2(|k^{-1}D|)[\chi_1, k^{-2}\Delta]A_1 + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}),\end{aligned}$$

we see that the right hand side of (2.44) is in $k^{-2}\Psi^{-4}(\mathbb{R}^d)$. ∎

With Lemma 2.17 and 2.18 in hand, we obtain the high-frequency estimates of the boundary-integral operators by comparing them to those at fixed frequency.

Lemma 2.19 (High-frequency estimates) *Let $M > 1$ and $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ for $|\xi| < M$. Suppose that $\partial\Omega$ is a piecewise-smooth hypersurface (in the sense of Definition 1.5). Then for $k > 1$ and $\chi \in C_c^\infty(\mathbb{R}^d)$*

$$\gamma R_0(k)\chi(1-\psi(|k^{-1}D|))\gamma^* = \mathcal{O}_{L^2(\partial\Omega)\to H_k^1(\partial\Omega)}(k^{-1}(\log k)^{1/2}).\tag{2.45}$$

If, in addition, $\partial\Omega$ is $C^{2,\alpha}$ for some $\alpha > 0$, then

$$\mp\frac{1}{2}I + \gamma^\pm R_0(k)\chi(1-\psi(|k^{-1}D|))L^*\gamma^* = \mathcal{O}_{L^2(\partial\Omega)\to H_k^1(\partial\Omega)}(\log k)\tag{2.46}$$

$$\pm\frac{1}{2}I + \gamma^\pm L R_0(k)\chi(1-\psi(|k^{-1}D|))\gamma^* = \mathcal{O}_{L^2(\partial\Omega)\to H_k^1(\partial\Omega)}(\log k).\tag{2.47}$$

Remark 2.20 *The factors of* $\log k$ *in the bounds of Lemma 2.19 are likely artifacts of our proof, but since they do not affect our final results, we do not attempt to remove them here. In fact, if* $\partial\Omega$ *is smooth (rather than piecewise smooth), then one can show that the logarithmic factors can be removed from the bounds in Lemma 2.19 using the analysis in [38, Section 4.4].*

Proof of Lemma 2.19. By Lemma 2.18,

$$A_k := \chi(R_0(k) - R_0(1))\chi(1 - \psi(k^{-1}D)) \in k^{-2}\Psi^{-4}.$$

Note that for $s > 1/2$,

$$\gamma = \mathcal{O}_{H_k^s(\mathbb{R}^d)\to H_k^{s-1/2}(\partial\Omega)}(k^{1/2}); \tag{2.48}$$

this bound follows from repeating the proof of the trace estimate in [61, Lemma 3.35] but working in semiclassically rescaled spaces.

Let $B_k := \gamma A_k\gamma^*$, $C_k := \gamma A_k L^*\gamma^*$, $C_k' := \gamma L A_k\gamma^*$. Then, using (2.48) and the fact that $L, L^* = \mathcal{O}_{H_k^s\to H_k^{s-1}}(k)$, we have that $B_k = \mathcal{O}_{L^2\to H_k^1}(k^{-1})$ and C_k, $C_k' = \mathcal{O}_{L^2\to H_k^1}(1)$.

Recalling the notation for S_k (2.11), D_k, and D_k' (2.16), and the mapping properties recapped in §2.4.1, we have

$$\gamma R_0(1)\chi\gamma^* : L^2(\partial\Omega) \to H^1(\partial\Omega)$$

when $\partial\Omega$ is Lipschitz, and

$$\pm\frac{1}{2}I + \gamma^\pm R_0(1)\chi L^*\gamma^* : L^2(\partial\Omega) \to H^1(\partial\Omega)$$
$$\mp\frac{1}{2}I + \gamma^\pm L R_0(1)\chi\gamma^* : L^2(\partial\Omega) \to H^1(\partial\Omega)$$

when $\partial\Omega$ is $C^{2,\alpha}$.

Now, note that for $\tilde{\Gamma}$ a precompact smooth hypersurface, and $\psi \in C_c^\infty(\mathbb{R})$,

$$\|\psi(|k^{-1}D|)\gamma_{\tilde{\Gamma}}^*\|_{L^2(\tilde{\Gamma})\to H^s(\mathbb{R}^d)} + \|\gamma_{\tilde{\Gamma}}\psi(|k^{-1}D|)\|_{H^{-s}(\mathbb{R}^d)\to\bar{H}^{-s-1/2}(\tilde{\Gamma})} \le C\begin{cases}1 & s < -1/2\\ (\log k)^{1/2} & s = -1/2\\ k^{(s+1/2)} & s > -1/2.\end{cases}$$

Thus, since $\psi(|k^{-1}D|) : \mathcal{S}'(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ and in particular, $\gamma\psi(|k^{-1}D|) : \mathcal{S}'(\mathbb{R}^d) \to \overline{H^1}(\Gamma)$,

$$\|\psi(|k^{-1}D|)\gamma^*\|_{L^2(\Gamma)\to H^s(\mathbb{R}^d)} + \|\gamma_\Gamma\psi(|k^{-1}D|)\|_{H^{-s}(\mathbb{R}^d)\to\bar{H}^{-s-1/2}(\Gamma)} \le C\begin{cases}1 & s < -1/2\\ (\log k)^{1/2} & s = -1/2\\ k^{(s+1/2)} & s > -1/2.\end{cases} \tag{2.49}$$

Furthermore, notice that by Lemma 2.17, if $\psi_1 \in C_c^\infty(\mathbb{R})$ has $\psi_1 \equiv 1$ on $\operatorname{supp}\psi$, then

$$\chi R_0(1)\chi\psi(|k^{-1}D|) = \psi_1(|k^{-1}D|)\chi R_0(1)\chi\psi(|k^{-1}D|) + \mathcal{O}_{\Psi^{-\infty}}(k^{-\infty}).$$

In particular, using this estimate together with (2.49) and that $\chi R_0(1)\chi : H^s(\mathbb{R}^d) \to H^{s+2}(\mathbb{R}^d)$,

$$\gamma^\pm R_0(1)\chi\psi(|k^{-1}D|)\gamma^* = \begin{cases}\mathcal{O}_{L^2(\Gamma)\to\overline{H}^1(\Gamma)}((\log k)^{1/2})\\ \mathcal{O}_{L^2(\Gamma)\to L^2(\Gamma)}(1),\end{cases}$$

$$\gamma^\pm R_0(1)\chi\psi(|k^{-1}D|)L^*\gamma^* = \begin{cases}\mathcal{O}_{L^2(\Gamma)\to\overline{H}^1(\Gamma)}(k)\\ \mathcal{O}_{L^2(\Gamma)\to L^2(\Gamma)}(\log k),\end{cases}$$

$$\gamma^\pm L R_0(1)\chi\psi(|k^{-1}D|)\gamma^* = \begin{cases}\mathcal{O}_{L^2(\Gamma)\to\overline{H}^1(\Gamma)}(k)\\ \mathcal{O}_{L^2(\Gamma)\to L^2(\Gamma)}(\log k),\end{cases}$$

Hence,

$$\gamma R_0(k)\chi(1 - \psi(|k^{-1}D|))\gamma^* = \gamma R_0(1)\chi(1 - \psi(|k^{-1}D|))\gamma^* + B_k = \mathcal{O}_{L^2\to\bar{H}^1}((\log k)^{1/2}).$$

Furthermore, since $R_0(k)\chi(1-\psi(|k^{-1}D|)) \in k^{-2}\Psi^{-2}(\mathbb{R}^d)$, and we have (2.48),

$$\gamma R_0(k)\chi(1-\psi(|k^{-1}D|))\gamma^* = \mathcal{O}_{L^2\to L^2}(k^{-1}). \tag{2.50}$$

Next, observe that

$$\mp\frac{1}{2} + \gamma^\pm R_0(k)\chi(1-\psi(|k^{-1}D|))L^*\gamma^* = \mp\frac{1}{2} + \gamma^\pm R_0(1)\chi(1-\psi(|k^{-1}D|))L^*\gamma^* + C_k$$
$$= \begin{cases} \mathcal{O}_{L^2\to\bar{H}^1}(k) \\ \mathcal{O}_{L^2\to L^2}(\log k), \end{cases} \tag{2.51}$$

$$\pm\frac{1}{2} + \gamma^\pm L R_0(k)\chi(1-\psi(|k^{-1}D|))\gamma^* = \pm\frac{1}{2} + \gamma^\pm L R_0(1)\chi(1-\psi(|k^{-1}D|))\gamma^* + C_k'$$
$$= \begin{cases} \mathcal{O}_{L^2\to\bar{H}^1}(k) \\ \mathcal{O}_{L^2\to L^2}(\log k). \end{cases} \tag{2.52}$$

Since $\partial\Omega$ is piecewise-smooth, $\partial\Omega = \sum_{i=1}^N \Gamma_i$. Applying (2.50)-(2.52) with $\Gamma = \Gamma_i$, summing over i, and then using the result (2.3) we obtain (2.45)- (2.47) ∎

Proof of Parts (a) and (b) of Theorem 2.10. This follows from combining the low-frequency estimates (2.30)-(2.32) in Lemma 2.16 with the high-frequency estimates (2.45)-(2.47) in Lemma 2.19, recalling the decompositions (2.11) and (2.16). ∎

2.5 Proof of Part (c) of Theorem 2.10

Proof of Part (c) of Theorem 2.10. Observe that [38, Theorems 4.29, 4.32] imply that for $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ on $[-2, 2]$,

$$\psi(|k^{-1}D'|)S_k = \mathcal{O}_{L^2\to H_k^1}(k^{-2/3}), \qquad \psi(|k^{-1}D'|)D_k = \mathcal{O}_{L^2\to H_k^1}(1).$$

Then [38, Lemma 4.25] shows that $(1-\psi(|k^{-1}D'|))S_k \in k^{-1}\Psi^{-1}(\partial\Omega)$ and $(1-\psi(|k^{-1}D'|))D_k \in k^{-1}\Psi^{-1}(\partial\Omega)$ and hence

$$(1-\psi(|k^{-1}D'|))S_k = \mathcal{O}_{L^2\to H_k^1}(k^{-1}), \qquad (1-\psi(|k^{-1}D'|))D_k = \mathcal{O}_{L^2\to H_k^1}(k^{-1}).$$

An identical analysis shows that

$$D_k' = \mathcal{O}_{L^2\to H_k^1}(1).$$

∎

2.6 Proof of Corollary 1.14

This follows in exactly same way as [42, Proof of Corollary 1.2, page 193]. The two ideas are that (i) the relationships

$$\int_\Gamma \phi\, S_k\psi \,\mathrm{d}s = \int_\Gamma \psi\, S_k\phi \,\mathrm{d}s, \text{ and } \int_\Gamma \phi\, D_k\psi \,\mathrm{d}s = \int_\Gamma \psi\, D_k'\phi \,\mathrm{d}s, \tag{2.53}$$

for $\phi, \psi \in L^2(\partial\Omega)$ (see, e.g., [18, Equation 2.37]), and the duality of $H^1(\partial\Omega)$ and $H^{-1}(\partial\Omega)$ (see, e.g., [61, Page 98]) allow us to convert bounds on S_k, D_k, and D_k' as mappings from $L^2(\partial\Omega) \to H^1(\partial\Omega)$ into bounds on these operators as mappings from $H^{-1}(\partial\Omega) \to L^2(\partial\Omega)$; and
(ii) interpolation then allows us to obtain bounds from $H^{s-1/2}(\partial\Omega) \to H^{s+1/2}(\partial\Omega)$ for $|s| \leq 1/2$.

3 Sharpness of the bounds in Theorem 1.10

We now prove that the powers of k in the $\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$ bounds in Theorem 2.10 are optimal. The analysis in [46, §A.3] proves that the powers of k in the $\|D_k\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}$ bounds are optimal, but can be adapted in a similar way to below to prove the sharpness of the $\|D_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$ bounds.

In this section we write $x \in \mathbb{R}^d$ as $x = (x', x_d)$ for $x' \in \mathbb{R}^{d-1}$, and $x' = (x_1, x'')$ (in the case $d = 2$, the x'' variable is superfluous).

Lemma 3.1 (Lower bound on $\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$ when $\partial\Omega$ contains a line segment) *If $\partial\Omega$ contains the set*

$$\big\{(x_1, 0) : |x_1| < \delta\big\}$$

for some $\delta > 0$ and is C^2 in a neighborhood thereof (i.e. $\partial\Omega$ contains a line segment), then there exists $k_0 > 0$ and $C > 0$ (independent of k), such that, for all $k \geq k_0$,

$$\|S_k\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \geq Ck^{-1/2} \quad \textit{and} \quad \|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \geq Ck^{1/2}.$$

This result shows that the bound (1.14), when $\partial\Omega$ is piecewise smooth, is sharp up to a factor of $\log k$.

Lemma 3.2 (General lower bound on $\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$) *If $\partial\Omega$ is C^2 in a neighborhood of a point then there exists $k_0 > 0$ and $C > 0$ (independent of k), such that, for all $k \geq k_0$,*

$$\|S_k\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \geq Ck^{-2/3} \quad \textit{and} \quad \|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \geq Ck^{1/3}.$$

This result shows that the bound (1.15), when $\partial\Omega$ is piecewise curved, is sharp up to a factor of $\log k$ and that the bound (1.16), when when $\partial\Omega$ is smooth and curved, is sharp.

Remark 3.3 *The lower bound $\|S_k\|_{\dot{H}^s(\Gamma)\to H^{s+1}(\Gamma)} \geq Ck^{1/2}$ when Γ is a flat screen (i.e. a bounded and relatively open subset of $\{x \in \mathbb{R}^d : x_d = 0\}$) and $s \in \mathbb{R}$ is proved in [19, Remark 4.2] (recall that $\dot{H}^s(\Gamma)$ is defined in Definition 2.5).*

Proof of Lemma 3.1. By assumption $\Gamma \subset \Omega$, where

$$\Gamma := \Big\{(x_1, x'', \gamma(x')) : |x'| < \delta\Big\}$$

for some $\gamma(x')$ with $\gamma(x_1, 0) = 0$ for $|x_1| < \delta$ (since the line segment $\{(x_1, 0) : |x_1| < \delta\} \subset \Gamma$).

By the definition of the operator norm, it is sufficient to prove that there exists $u_k \in L^2(\partial\Omega)$ with $\operatorname{supp} u_k \subset \Gamma$, $k_0 > 0$, and $C > 0$ (independent of k), such that, for all $k \geq k_0$,

$$\|S_k u_k\|_{L^2(\Gamma)} \geq Ck^{-1/2}\|u\|_{L^2(\Gamma)} \quad \text{and} \quad \|\partial_{x_1} S_k u_k\|_{L^2(\Gamma)} \geq Ck^{1/2}\|u\|_{L^2(\Gamma)}. \tag{3.1}$$

We begin by observing that the definition of $\Phi_k(x, y)$ (1.6) and the asymptotics of Hankel functions for large argument and fixed order (see, e.g., [69, §10.17]) imply that

$$\Phi_k(x, y) = C_d k^{d-2} e^{ik|x-y|}\Big((k|x-y|)^{-(d-1)/2} + \mathcal{O}(k|x-y|)^{-(d+1)/2}\Big), \tag{3.2}$$

$$\langle V, \partial_x\rangle\Phi_k(x, y) = C'_d k^{d-1}\frac{\langle V, x-y\rangle}{|x-y|} e^{ik|x-y|}\Big((k|x-y|)^{-(d-1)/2} + \mathcal{O}(k|x-y|)^{-(d+1)/2}\Big). \tag{3.3}$$

Let $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset [-2, 2]$, $\chi(0) \equiv 1$ on $[-1, 1]$ and define

$$\chi_{\epsilon,\gamma_1,\gamma_2}(x') = \chi\big(\epsilon^{-1}k^{\gamma_1}x_1\big)\chi\big(\epsilon^{-1}k^{\gamma_2}|x''|\big), \tag{3.4}$$

In what follows, we suppress the dependence of u on k for convenience. Let $u(x', \gamma(x')) := e^{ikx_1}\chi_{\epsilon,0,1/2}(x')$. The definition of χ implies that

$$\operatorname{supp} u = \Big\{(x', \gamma(x')) : |x_1| \leq 2\epsilon,\ |x''| \leq 2\epsilon k^{-1/2}\Big\},$$

and thus $\operatorname{supp} u \subset \Gamma$ for ϵ sufficiently small and k sufficiently large (say $\epsilon < (2\sqrt{2})^{-1}\delta$ and $k > 1$); for the rest of the proof we assume that ϵ and k are such that this is the case. Observe also that

$$\|u\|_{L^2(\Gamma)} \sim C_\epsilon k^{-(d-2)/4}. \tag{3.5}$$

Let

$$U := \Big\{(x', \gamma(x')) : M\epsilon \le x_1 \le 2M\epsilon,\ |x''| \le \epsilon k^{-1/2}, \quad M \gg 1\Big\};$$

the motivation for this choice comes from the analysis in Remark 4.5 below. Indeed, we know that S_k is largest microlocally near points that are glancing in both the incoming and outgoing variables. Since u concentrates microlocally at $x = 0$, $\xi = (1, 0)$ up to scale $k^{-1/2}$, the billiard trajectory emanating from this point is $\{t(1, 0) : t > 0\}$. This ray is always glancing since Γ is flat. Therefore, we choose U to contain this ray up to scale $k^{-1/2}$.

Then for $x \in U$, $y \in \operatorname{supp} u$,

$$|(x', \gamma(x')) - (y', \gamma(y'))|^2 = (x_1 - y_1)^2 + |x'' - y''|^2 + |\gamma(x') - \gamma(y')|^2,$$

Then, observe that by Taylor's formula

$$\gamma(x')-\gamma(y') = \gamma(x_1, 0)-\gamma(y_1, 0)+\partial_{x''}\gamma(x_1, 0)(x''-y'')+y''(\partial_{x''}\gamma(x_1, 0)-\partial_{x''}\gamma(y_1, 0))+\mathcal{O}(|x''|^2+|y''|^2).$$

Since $\gamma(x_1, 0) = 0$ for $|x_1| < \delta$,

$$|\gamma(x') - \gamma(y')|^2 = \mathcal{O}\big(|x'' - y''|^2\big) + \mathcal{O}(|x''|^2 + |y''|^2).$$

In particular,

$$\begin{aligned} |(x', \gamma(x')) - (y', \gamma(y'))| &= (x_1 - y_1) + \mathcal{O}\Big(\big[|x'' - y''|^2 + |x''|^2 + |y''|^2\big]|x_1 - y_1|^{-1}\Big) \\ &= x_1 - y_1 + \mathcal{O}(k^{-1}M^{-1}\epsilon), \qquad (3.6) \\ &= x_1\big(1 + \mathcal{O}(M^{-1}) + \mathcal{O}(k^{-1}M^{-2})\big). \qquad (3.7) \end{aligned}$$

We have from the Hankel-function asymptotics (3.2) and the definition of u that, for $x \in U$,

$$S_k u(x) = C_d k^{d-2} \int_\Gamma e^{ik|x-y|+iky_1} \bigg(k^{-(d-1)/2}|x - y|^{-(d-1)/2} + \mathcal{O}((k|x - y|)^{-(d+1)/2}\bigg)\chi_{\epsilon,0,1.2}(y')\mathrm{d}s(y),$$

and then using the asymptotics (3.6) in the exponent of the integrand and the asymptotics (3.7) in the rest of the integrand, we have, for $x \in U$,

$$S_k u(x) = C_d k^{d-2} \frac{e^{ikx_1}}{k^{(d-1)/2}|x_1|^{(d-1)/2}} \int_\Gamma \big(1 + \mathcal{O}(M^{-1}\epsilon)\big) \bigg(1 + \mathcal{O}(M^{-1}) + \mathcal{O}_{\epsilon,M}(k^{-1})\bigg)\chi_{\epsilon,0,1/2}(y')\mathrm{d}s(y).$$

Therefore, with M large enough, ϵ small enough, and then k_0 large enough, the contribution from the integral over Γ is determined by the cutoff $\chi_{\epsilon,0,1/2}$, yielding $k^{-(d-2)/2}$, and thus

$$|S_k u(x')| \ge C k^{(d-2)/2} \frac{1}{k^{(d-1)/2}|x_1|^{(d-1)/2}}, \quad x' \in U,\ k \ge k_0. \tag{3.8}$$

In the step of taking ϵ sufficiently small, we can also take ϵ small enough to ensure that $U \subset \Gamma$ for all $k \ge 1$. Using (3.8), along with the fact that the measure of $U \sim k^{-(d-2)/2}$, we have that

$$\|S_k u\|_{L^2(U)} \ge C k^{-1/2-(d-2)/4}. \tag{3.9}$$

Since we have ensured that $U \subset \Gamma$, (3.9) and (3.5) imply that the first bound in (3.1) holds. It easy to see that if we repeat the argument above but with (3.3) instead of (3.2), then we obtain the second bound in (3.1). ∎

Proof of Lemma 3.2. Let $x_0 \in \partial\Omega$ be a point so that $\partial\Omega$ is C^2 in a neighborhood of x_0 and let x' be coordinates near x_0 so that

$$\Gamma := \Big\{(x', \gamma(x')) : |x'| < \delta\Big\} \subset \partial\Omega, \qquad \text{with } \gamma \in C^2,\ \gamma(0) = \partial\gamma(0) = 0.$$

Similar to the proof of Lemma 3.1, it is sufficient to prove that there exists $u_k \in L^2(\partial\Omega)$ with $\operatorname{supp} u_k \subset \Gamma$, $k_0 > 0$, and $C > 0$ (independent of k), such that

$$\|S_k u_k\|_{L^2(\Gamma)} \geq Ck^{-1/2}\|u_k\|_{L^2(\Gamma)} \quad \text{and} \quad \|\partial_{x_1} S_k u\|_{L^2(\Gamma)} \geq Ck^{1/2}\|u\|_{L^2(\Gamma)} \tag{3.10}$$

for all $k \geq k_0$.

The idea in the curved case is the same as in the flat case: choose u concentrating as close as possible to a glancing point and measure near the point given by the billiard map. More practically, this amounts to ensuring that $|x - y|$ looks like $x_1 - y_1$ modulo terms that are much smaller than k^{-1}. The fact that Γ may be curved will force us to choose u differently and cause our estimates to be worse than in the flat case (leading to the weaker - but still sharp - lower bound).

With $\chi_{\epsilon,\gamma_1,\gamma_2}$ defined by (3.4), let $u(x', \gamma(x')) := \mathrm{e}^{\mathrm{i}kx_1}\chi_{\epsilon,1/3,2/3}(x')$ where, as in the proof of Lemma 3.1, we have $x' = (x_1, x'')$ and as in Lemma 3.1, $\operatorname{supp} u \subset \Gamma$ for ϵ sufficiently small and k sufficiently large, and for the rest of the proof we assume that this is the case. Then

$$\|u\|_{L^2(\Gamma)} \leq C_\epsilon k^{-1/6} k^{-(d-2)/3}. \tag{3.11}$$

Define

$$U := \Big\{(x', \gamma(x')) : M\epsilon k^{-1/3} \leq x_1 \leq 2M\epsilon k^{-1/3},\ |x''| \leq \epsilon k^{-2/3}, \quad M \gg 1\Big\}.$$

Then, for $y \in \operatorname{supp} u$ and $x \in U$,

$$\begin{aligned} |(x', \gamma(x')) - (y', \gamma(x'))| &= (x_1 - y_1) + \mathcal{O}\big((|x'|^2 + |y'|^2)^2|x_1 - y_1|^{-1}\big) + \mathcal{O}\big(|x'' - y''|^2|x_1 - y_1|^{-1}\big) \\ &= x_1 - y_1 + \mathcal{O}(k^{-1}M^3\epsilon^3) + \mathcal{O}(\epsilon k^{-1}M^{-1}) \qquad (3.12) \\ &= x_1\big(1 + \mathcal{O}(M^{-1}) + \mathcal{O}(k^{-2/3}M^2\epsilon^2) + \mathcal{O}(k^{-2/3}M^{-2})\big). \qquad (3.13) \end{aligned}$$

From (3.2) and the definition of u, we have for $x' \in U$,

$$S_k u(x) = C_d k^{d-2} \int_\Gamma \mathrm{e}^{\mathrm{i}k|x-y| + \mathrm{i}ky_1}\Big(k^{-(d-1)/2}|x-y|^{-(d-1)/2} + \mathcal{O}((k|x-y|)^{-(d+1)/2}\Big)\chi_{\epsilon,1/3,2/3}(y')\mathrm{d}s(y),$$

and then, using (3.12) in the exponent of the integrand and (3.13) in the rest, we have, for $x' \in U$,

$$\begin{aligned} S_k u(x) = C_d k^{d-2} &\frac{\mathrm{e}^{\mathrm{i}kx_1}}{k^{(d-1)/2}|x_1|^{(d-1)/2}} \\ &\int_\Gamma \Big(1 + \mathcal{O}(M^3\epsilon^3) + \mathcal{O}(M^{-1}\epsilon)\Big)\Big(1 + \mathcal{O}(M^{-1}) + \mathcal{O}_{\epsilon,M}(k^{-2/3})\Big)\chi_{\epsilon,1/3,2/3}(y')\mathrm{d}s(y). \end{aligned}$$

Thus, fixing M large enough, then ϵ small enough, then k_0 large enough, we have

$$|S_k u(x')| \geq Ck^{(d-2)/3}\frac{1}{k^{(d-1)/2}|x_1|^{(d-1)/2}}k^{-1/3}, \quad x' \in U,\ k \geq k_0 \tag{3.14}$$

In the step of taking ϵ sufficiently small, we can also take ϵ small enough so that when $x' \in U$, $|x'| < \delta$, and thus $x' \in \Gamma$. Using the lower bound (3.14), and the fact that the measure of U $\sim k^{-1/3}k^{-2(d-2)/3}$, we have that

$$\|S_k u\|_{L^2(\Gamma)} \geq \|S_k u\|_{L^2(U)} \geq Ck^{-2/3-1/6-(d-2)/3}$$

and so using (3.11) we obtain the first bound in (3.10). Similar to before, if we repeat this argument with (3.3) instead of (3.2), we find the second bound in (3.10). ∎

4 Proofs of Theorems 1.15, 1.16 (the results concerning Q1)

4.1 Proof of Theorem 1.15

The heart of the proof of Theorem 1.15 is the following lemma.

Lemma 4.1 *There exists a $\tilde{C} > 0$ such that under the condition*

$$h \left\| D'_k - \mathrm{i}\eta S_k \right\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \leq \widetilde{C} \tag{4.1}$$

the Galerkin equations (1.11) *have a unique solution satisfying* (1.20).

The presence of $\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}$ in (4.1) means that before proving Theorem 1.15 using Lemma 4.1 we need to recall the following bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}$.

Theorem 4.2 ([23, Theorem 4.3], [9, Theorem 1.13]) *If $|\eta| \sim k$ and* either *Ω is star-shaped with respect to a ball and C^2 in a neighbourhood of almost every point on Γ* or *Ω is nontrapping, then, given $k_0 > 0$, $\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \lesssim 1$ for all $k \geq k_0$.*

Proof of Theorem 1.15 using Lemma 4.1. Using the triangle inequality, a sufficient condition for (4.1) to hold is

$$h \Big(\|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + |\eta| \, \|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \Big) \|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \leq \widetilde{C}; \tag{4.2}$$

we show in Remark 4.5 below that we do not lose anything by doing this, i.e., (4.2) is no less sharp than (4.1) in terms of k-dependence.

The mesh thresholds (1.19), (1.21), (1.22) then follow from using the bound $\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \lesssim 1$ from Theorem 4.2 and the different bounds on $\|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$ and $\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$ in Theorem 1.10 (apart from when $d = 2$ when we use the bound on S_k (1.17) instead of (1.14)). ■

To prove Theorem 1.15 we therefore only need to prove Lemma 4.1. This was proved in [42, Corollary 4.1], but since the proof is short we repeat it here for completeness.

We first introduce some notation: let P_h denote the orthogonal projection from $L^2(\partial\Omega)$ onto $\mathcal{V}_h$ (see, e.g, [6, §3.1.2]); then the Galerkin equations (1.11) are equivalent to the operator equation

$$P_h A'_{k,\eta} v_h = P_h f_{k,\eta}. \tag{4.3}$$

The proof requires us to treat $A'_{k,\eta}$ as a (compact) perturbation of the identity, and thus we let $L_{k,\eta} := D'_k - \mathrm{i}\eta S_k$. Furthermore, to make the notation more concise, we let $\lambda = 1/2$. Therefore, the left-hand side of (4.3) becomes $(\lambda I + P_h L_{k,\eta})v_h$, and the question of existence of a solution to (4.3) boils down to the invertibility of $(\lambda I + P_h L_{k,\eta})$. Note also that, with the P_h-notation, the best approximation error on the right-hand side of (1.20) is $\|(I - P_h)v\|_{L^2(\partial\Omega)}$.

The heart of the proof of Lemma 4.1 is the following lemma.

Lemma 4.3 *If*

$$\|(I - P_h)L_{k,\eta}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \leq \frac{\delta}{1+\delta} \tag{4.4}$$

for some $\delta > 0$, then the Galerkin equations have a unique solution, v_h, which satisfies the quasi-optimal error estimate

$$\|v - v_h\|_{L^2(\partial\Omega)} \leq \lambda(1+\delta)\|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}\|(I - P_h)v\|_{L^2(\partial\Omega)}. \tag{4.5}$$

Proof of Lemma 4.1 using Lemma 4.3. By the polynomial-approximation result (1.12),

$$\|(I - P_h)L_{k,\eta}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \lesssim h \, \|L_{k,\eta}\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} .$$

Therefore, choosing, say, $\delta = 1$, we find that there exists a $\tilde{C} > 0$ such that (4.1) implies that (4.4) holds. ■

Thus, to prove Theorem 1.15, we only need to prove Lemma 4.3.

Proof of Lemma 4.3. Since

$$\lambda I + P_h L_{k,\eta} = \lambda I + L_{k,\eta} - (I - P_h)L_{k,\eta} = (\lambda I + L_{k,\eta}) \left(I - (\lambda I + L_{k,\eta})^{-1} (I - P_h)L_{k,\eta} \right),$$

if

$$\left\| (\lambda I + L_{k,\eta})^{-1} (I - P_h)L_{k,\eta} \right\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} < 1,$$

then $(\lambda I + P_h L_{k,\eta})$ is invertible using the classical result that $I - A$ is invertible if $\|A\| < 1$. In this abstract setting $\|(I - A)^{-1}\| \leq (1 - \|A\|)^{-1}$, and thus if (4.4) holds we have

$$\begin{aligned} \left\|(\lambda I + P_h L_{k,\eta})^{-1}\right\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} &\leq \left\| (\lambda I + L_{k,\eta})^{-1} \right\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \frac{1}{1 - \delta/(1+\delta)}, \\ &= (1+\delta) \left\|(\lambda I + L_{k,\eta})^{-1}\right\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}. \end{aligned} \tag{4.6}$$

Writing the direct equation as $(\lambda I + L_{k,\eta})v = f$ and the Galerkin equation as $(\lambda I + P_h L_{k,\eta})v_h = P_h f$, we have

$$\begin{aligned} v - v_h = v - (\lambda I + P_h L_{k,\eta})^{-1} P_h f &= (\lambda I + P_h L_{k,\eta})^{-1}(\lambda v - P_h(f - L_k v)) \\ &= \lambda \left(\lambda I + P_h L_{k,\eta}\right)^{-1} (I - P_h)v, . \end{aligned} \tag{4.7}$$

and the result (4.5) follows from using the bound (4.6) in (4.7). ■

Remark 4.4 (Is there a better choice of η than $|\eta| \sim k$?) *Theorem 1.15 is proved under the assumption that $|\eta| \sim k$. This choice of η is widely recommended from studies of the condition number of $A'_{k,\eta}$; see [18, Chapter 5] for an overview of these. From (4.2) we see that the best choice of η, from the point of view of obtaining the least-restrictive threshold for k-independent quasi-optimality, will minimise the k-dependence of*

$$\left(\|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + |\eta| \, \|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}\right) \|(A'_{k,\eta})^{-1}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}.$$

There does not yet exist a rigorous proof that $|\eta| \sim k$ minimises this quantity, but [9, §7.1] outlines exactly the necessary results still to prove.

Remark 4.5 (Using the triangle inequality on $\|D'_k - i\eta S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$) *We now show that we do not lose anything, from the point of view of k-dependence, by using the triangle inequality $\|D'_k - i\eta S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \leq \|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + |\eta|\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$. First, recall that D'_k and S_k have wavefront set relation given by the billiard ball relation (see for example [38, Chapter 4]). Denote the relation by $C_\beta \subset \overline{B^*\partial\Omega} \times \overline{B^*\partial\Omega}$ i.e.*

$$C_\beta = \left\{(x, \xi, y, \eta) : (x, \xi) = \beta(y, \eta)\right\}$$

where β is the billiard ball map (see Figure 1). To see that the optimal bound in terms of powers of k for $\|D'_k - i\eta S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$ is equal to that for $\|D_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + |\eta|\|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$, observe that the largest norm for S_k corresponds microlocally to points $(q_1, q_2) \in C_\beta \cap (S^*\partial\Omega \times S^*\partial\Omega)$ (i.e. "glancing" to "glancing"). On the other hand, these points are damped (relative to the worst bounds) for D'_k. In particular, microlocally near such points, one expects that

$$\|D'_k f_{q_2}\|_{H^1(\partial\Omega)} \leq Ck, \qquad \|S_k f_{q_2}\|_{H^1(\partial\Omega)} \geq \begin{cases} Ck^{1/2}, & \partial\Omega \text{ flat}, \\ Ck^{1/3}, & \partial\Omega \text{ curved}, \end{cases}$$

where $\|f_{q_2}\|_{L^2(\partial\Omega)} = 1$ and f_{q_2} is microlocalized near q_2.

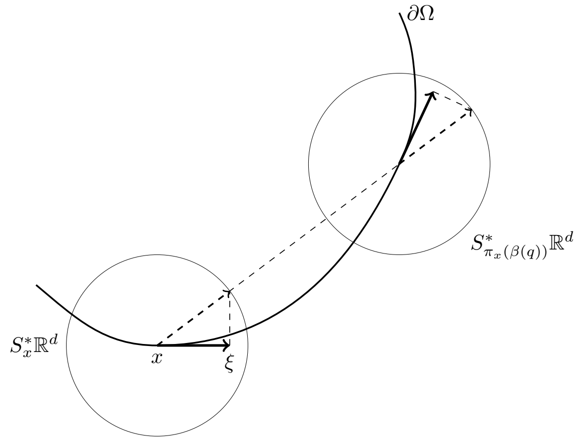

Figure 1: A recap of the billiard ball map. Let $q = (x, \xi) \in B^*\partial\Omega$ (the unit ball in the cotangent bundle of $\partial\Omega$). The solid black arrow on the left denotes the covector $\xi \in B^*_x\partial\Omega$, with the dashed arrow denoting the unique inward-pointing unit vector whose tangential component is ξ. The dashed arrow on the right is the continuation of the dashed arrow on the left, and the solid black arrow on the right is $\xi(\beta(q)) \in B^*_{\pi_x(\beta(q))}\partial\Omega$. The center of the left circle is x and that of the right is $\pi_x(\beta(q))$. If this process is repeated, then the dashed arrow on the right is reflected in the tangent plane at $\pi_x(\beta(q))$: the standard "angle of incidence equals angle of reflection" rule.

The norm for D'_k is maximized microlocally near $(p_1, p_2) \in C_\beta \cap (S^*\partial\Omega \times B^*\partial\Omega)$ (i.e. "transversal" to "glancing"), but near these points, the norm of S_k is damped relative to its worst bound. In particular, microlocally near (p_1, p_2), one expects

$$\|D'_k f_{p_2}\|_{H^1(\partial\Omega)} \geq \begin{cases} Ck^{5/4}, & \partial\Omega \text{ flat}, \\ Ck^{7/6}, & \partial\Omega \text{ curved}, \end{cases} \qquad \|S_k f_{p_2}\|_{H^1(\partial\Omega)} \leq \begin{cases} Ck^{1/4}, & \partial\Omega \text{ flat}, \\ Ck^{1/6}, & \partial\Omega \text{ curved}, \end{cases}$$

where $\|f_{p_2}\|_{L^2(\partial\Omega)} = 1$ and f_{p_2} is microlocalized near p_2. Therefore, even if $|\eta|$ is chosen so that $\|D_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \sim |\eta| \|S_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$, this analysis shows that there cannot be cancellation since the worst norms occur at different points of phase space.

4.2 Proof of Theorem 1.16

Proof of Theorem 1.16. By the polynomial-approximation result (1.12), we only need to prove that the bound (1.24) hold with the different functions $A(k)$. The idea is to take the H^1 norm of the integral equation (1.2) and then use the $L^2(\partial\Omega) \to L^2(\partial\Omega)$ and $L^2(\partial\Omega) \to H^1(\partial\Omega)$ bounds from Theorems 2.10 and 1.10 respectively.

Taking the H^1 norm of (1.2) and using the notation that $A'_{k,\eta} = \frac{1}{2}I + L_{k,\eta}$ and $v := \partial_n^+ u$ as in the proof of Theorem 1.15 above, we have

$$\|v\|_{H^1(\partial\Omega)} \lesssim \|L_{k,\eta}\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} + \|f_{k,\eta}\|_{H^1(\partial\Omega)}.$$

In this inequality, η is just a parameter that appears in $L_{k,\eta}$ and $f_{k,\eta}$, with the equation holding for all values of η; in other words, the unknown $v(= \partial_n^+ u)$ does not depend on the value of η. We now seek to minimise the k-dependence of $\|L_{k,\eta}\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$. Looking at the k-dependence of the $L^2(\partial\Omega) \to H^1(\partial\Omega)$-bounds on S_k and D'_k in Theorem 1.10, we see that, under each of the different geometric set-ups, the best choice is $\eta = 0$, and thus

$$\|v\|_{H^1(\partial\Omega)} \lesssim \|D'_k\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} + k^2 \tag{4.8}$$

where we have explicitly worked out the k-dependence of $\|f_{k,\eta}\|_{H^1(\partial\Omega)}$ using the definition (1.10).

Taking the L^2 norm of (1.2) (with $\eta = 0$), and noting that $\|f_{k,\eta}\|_{L^2(\partial\Omega)} \sim k$, we have that

$$\left(1 + \|D'_k\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}\right) \|v\|_{L^2(\partial\Omega)} \gtrsim k. \tag{4.9}$$

Using (4.9) in (4.8), we have

$$\|v\|_{H^1(\partial\Omega)} \lesssim \Big(\|D_k'\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)} + k\big(1+\|D_k'\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}\big)\Big)\,\|v\|_{L^2(\partial\Omega)}\,. \tag{4.10}$$

Since the bounds on the $L^2(\partial\Omega) \to H^1(\partial\Omega)$-norm of D_k' in Theorem 1.10 are one power of k higher that the $L^2(\partial\Omega) \to L^2(\partial\Omega)$-bounds in Theorem 2.10, using these norm bounds in (4.10) results in the bound $\|v\|_{H^1(\partial\Omega)} \lesssim A(k)\|v\|_{L^2(\partial\Omega)}$ with the functions of $A(k)$ as in the statement of theorem (and equal to the right-hand sides of the bounds on $\|D_k'\|_{L^2(\partial\Omega)\to H^1(\partial\Omega)}$ in Theorem 1.10). ■

5 Proofs of Theorem 1.21 (the result concerning Q2)

To prove Theorem 1.21 we need to recall (i) the result about coercivity of $A'_{k,\eta}$ when Ω is convex, C^3, piecewise analytic, and curved from [77], and (ii) the refinement of the Elman estimate in [11].

Theorem 5.1 (Coercivity of $A'_{k,\eta}$ for Ω convex, C^3, piecewise analytic, and curved [77]) *Let Ω be a convex domain in either 2- or 3-d whose boundary, $\partial\Omega$, is curved and is both C^3 and piecewise analytic. Then there exist constants $\eta_0 > 0$, $k_0 > 0$ (with $\eta_0 = 1$ when Ω is a ball) and a function of k, $\alpha_k > 0$, such that for $k \geq k_0$ and $\eta \geq \eta_0 k$,*

$$\left|\big(A'_{k,\eta}\phi,\phi\big)_{L^2(\partial\Omega)}\right| \geq \alpha_k \|\phi\|^2_{L^2(\partial\Omega)} \qquad \text{for all } \phi \in L^2(\partial\Omega), \tag{5.1}$$

where

$$\alpha_k = \frac{1}{2} - \mathcal{O}(k^{-2/3}\log k) \qquad \text{as } k \to \infty. \tag{5.2}$$

In stating this result we have used the bound (2.9) on S_k in [77, Remark 3.3] to get the asymptotics (5.2). The fact that $\eta_0 = 1$ when Ω is a ball follows from [76, Corollary 4.8].

Theorem 5.2 (Refinement of the Elman estimate [11]) *Let $\mathbf{A}$ be a matrix with $0 \notin W(\mathbf{A})$, where $W(\mathbf{A}) := \big\{(\mathbf{A}\mathbf{v},\mathbf{v}) : \mathbf{v} \in \mathbb{C}^N, \|\mathbf{v}\|_2 = 1\big\}$ is the* numerical range *of $\mathbf{A}$. Let $\beta \in [0,\pi/2)$ be defined such that*

$$\cos\beta = \frac{\operatorname{dist}\big(0,W(\mathbf{A})\big)}{\|\mathbf{A}\|_2},$$

and let γ_β be defined by

$$\gamma_\beta := 2\sin\left(\frac{\beta}{4-2\beta/\pi}\right). \tag{5.3}$$

Suppose the matrix equation $\mathbf{A}\mathbf{v} = \mathbf{f}$ is solved using GMRES, and let $\mathbf{r}_m := \mathbf{A}\mathbf{v}_m - \mathbf{f}$ be the m-th GMRES residual. Then

$$\frac{\|\mathbf{r}_m\|_2}{\|\mathbf{r}_0\|_2} \leq \left(2 + \frac{2}{\sqrt{3}}\right)(2+\gamma_\beta)\,\gamma_\beta^m. \tag{5.4}$$

When we apply the estimate (5.4) to $\mathbf{A}$, we find that $\beta = \pi/2 - \delta$, where $\delta = \delta(k)$ is such that $\delta \to 0$ as $k \to \infty$. We therefore specialise the result (5.4) to this particular situation in the following corollary.

Corollary 5.3 *If $\beta = \pi/2 - \delta$ with $0 < \delta < \delta_0$, then there exists $C_1 > 0$ and $\delta_1 > 0$ (both independent of δ) such that, for $0 < \varepsilon < 1$,*

$$\text{if} \quad m \geq \frac{C_1}{\delta}\log\left(\frac{12}{\varepsilon}\right) \qquad \text{then} \qquad \frac{\|\mathbf{r}_m\|_D}{\|\mathbf{r}_0\|_D} \leq \varepsilon \tag{5.5}$$

for all $0 < \delta < \delta_1$.

That is, choosing $m \gtrsim \delta^{-1}$ is sufficient for GMRES to converge in an δ-independent way as $\delta \to 0$.

Proof of Corollary 5.3. If $\beta = \pi/2 - \delta$, with $\delta \to 0$, then $\cos\beta = \sin\delta = \delta + \mathcal{O}(\delta^3)$ as $\delta \to 0$. From the definition of the convergence factor γ_β, (5.3), we have

$$\gamma_\beta := 2\sin\left(\frac{\beta}{4-2\beta/\pi}\right) = 2\sin\left(\frac{\pi}{6} - \frac{4\delta}{9} + \mathcal{O}(\delta^2)\right) = 1 - \frac{4\delta}{3\sqrt{3}} + \mathcal{O}(\delta^2) \quad \text{as} \quad \delta \to 0, \tag{5.6}$$

and then

$$\log\gamma_\beta = -\frac{4\delta}{3\sqrt{3}} + \mathcal{O}(\delta^2) \quad \text{as} \quad \delta \to 0,$$

and so there exist $C_2 > 0$ and $\delta_1 > 0$ such that

$$\gamma_\beta^m = \mathrm{e}^{m\log\gamma_\beta} \le \mathrm{e}^{-m\delta/C_2} \quad \text{for all } 0 < \delta \le \delta_1,$$

and the bound (5.5) follows since $(2 + 2/\sqrt{3})(2 + \gamma_\beta) < 3(2 + 2/\sqrt{3}) < 12$. ■

Remark 5.4 (Comparison of (5.4) with the original Elman estimate) *The estimate*

$$\frac{\|\mathbf{r}_m\|_2}{\|\mathbf{r}_0\|_2} \le \sin^m\beta \tag{5.7}$$

was essentially proved in [37, 36] (see also the review [73, §6] and the references therein). When $\beta = \pi/2 - \delta$, the convergence factor in (5.7) is

$$\sin\beta = \cos\delta = 1 - \frac{\delta^2}{2} + \mathcal{O}(\delta^4);$$

by comparing this to (5.6) we can see that (5.7) is indeed a weaker bound.

Proof of Theorem 1.21. The set up of the Galerkin method in §1.1 implies that, for any $v_N, w_N \in \mathcal{V}_N$, $(A'_{k,\eta}v_N, w_N)_{L^2(\partial\Omega)} = (\mathbf{A}\mathbf{v}, \mathbf{w})_2$, where $(\cdot,\cdot)_2$ denotes the euclidean inner product on l^2. Therefore, the continuity of $A'_{k,\eta}$ and the norm equivalent (1.13) implies that

$$|(\mathbf{A}\mathbf{v}, \mathbf{w})_2| \lesssim \left\|A'_{k,\eta}\right\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} h^{d-1} \|\mathbf{v}\|_2 \|\mathbf{w}\|_2 \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{C}^N. \tag{5.8}$$

Furthermore, if $A'_{k,\eta}$ is coercive with coercivity constant $\alpha_{k,\eta}$, i.e., (5.1) holds, then

$$|(\mathbf{A}\mathbf{v}, \mathbf{v})_2| \gtrsim \alpha_{k,\eta} h^{d-1} \|\mathbf{v}\|_2^2 \quad \text{for all } \mathbf{v} \in \mathbb{C}^N. \tag{5.9}$$

The bounds (5.8) and (5.9) together imply that the ratio $\cos\beta$ in (5.7) satisfies

$$\cos\beta \gtrsim \frac{\alpha_{k,\eta}}{\|A'_{k,\eta}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}}.$$

Since Ω is C^∞ and curved, the bound $\|A'_{k,\eta}\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \lesssim k^{1/3}$ follows from the bounds in Theorem 2.10 (recalling that $\eta_0 k \le \eta \lesssim k$). Since $\partial\Omega$ is piecewise analytic, C^3, and curved, from Theorem 5.1 there exists a $k_0 > 0$ such that $\alpha_{k,\eta} \sim 1$ for all $k \ge k_1$. Combining these two bounds we have $\cos\beta \gtrsim k^{-1/3}$ for all $k \ge k_0$ and thus Corollary 5.3 holds with $\delta \sim k^{-1/3}$ for all $k \ge k_0$; the result (5.8) then follows from (5.5).

Note that the assumption in the theorem that $\partial\Omega$ is analytic comes from the fact that if $\partial\Omega$ is both piecewise analytic and C^∞, then $\partial\Omega$ must be analytic, where the notion of piecewise analyticity in Theorem 5.1 is inherited from [26, Definition 4.1]. ■

Remark 5.5 (The star-combined operator) *The bound on the number of iterations in Theorem 1.21 crucially depended on the coercivity result of Theorem 5.1. Although numerical experiments in [13] indicate that $A'_{k,\eta}$ is coercive, uniformly in k, for a wider class of obstacles that those in Theorem 5.1, this has yet to be proved. Nevertheless, there* does *exist an integral operator that (i) can be used to solve the sound-soft scattering problem, and (ii) is provable coercive for a wide class of obstacles. Indeed, the* star-combined operator *$\mathscr{A}_k$, introduced in [76] and defined by*

$$\mathscr{A}_k := (x\cdot n)\left(\frac{1}{2}I + D'_k\right) + x\cdot\nabla_{\partial\Omega}S - \mathrm{i}\eta S_k$$

(where $\nabla_{\partial\Omega}$ is the surface gradient operator on $\partial\Omega$; see, e.g., [18, Page 276]), has the following two properties: (i) if u solves the sound-soft scattering problem, then

$$\mathscr{A}_k \partial_n^+ u = x \cdot \gamma^+ (\nabla u^I) - \mathrm{i}\eta\gamma^+ u^I \tag{5.10}$$

[76, Lemma 4.1] (see also [18, Theorem 2.36]), and

(ii) if Ω is a 2- or 3-d Lipschitz obstacle that is star-shaped with respect to a ball and $\eta := k|x| + \mathrm{i}(d-1)/2$, then

$$\operatorname{Re}\left(\mathscr{A}_k\phi, \phi\right)_{L^2(\partial\Omega)} \geq \frac{1}{2} \operatorname*{ess\,inf}_{x\in\partial\Omega}(x \cdot n(x) > 0$$

for all $k > 0$ [76, Theorem 1.1].

The refinement of the Elman estimate in Theorem 5.2 can therefore be used to prove results about the number of iterations required when GMRES is applied to the Galerkin discretisation of (5.10). Since the coercivity constant of the star-combined operator is independent of k, the k-dependence of the analogue of the bound (1.25) for $\mathscr{A}_k$ rests on the bounds on $\|\mathscr{A}_k\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)}$.

For convex Ω with smooth and curved $\partial\Omega$, Theorems 2.10 and Theorem 1.10 imply that $\|\mathscr{A}_k\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \lesssim k^{1/3}$, and we therefore obtain the same bound on m as for $A'_{k,\eta}$ (i.e. (1.25)). For general piecewise-smooth Lipschitz obstacles that are star-shaped with respect to a ball, Theorems 2.10 and 1.10, along with the bound

$$\|S_k\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \lesssim k^{-1/2} \quad \text{for } d = 2,$$

([17, Theorem 3.3], [39, Theorem 6]) and the bound (1.17), show that $\|\mathscr{A}_k\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \lesssim k^{1/2}$ when $d = 2$ and $\lesssim k^{1/2}\log k$ when $d = 3$. Corollary 5.3 then implies that $m \gtrsim k^{1/2}$ for $d = 2$ and $m \gtrsim k^{1/2}\log k$ for $d = 3$. Recall that GMRES always converges in at most N steps (in exact arithmetic), and when $h \sim 1/k$ we have that $N \sim k^{d-1}$; these bounds on m are therefore nontrivial.

6 Numerical experiments concerning Q2

The main purpose of this section is to show that the $k^{1/3}$ growth in the number of iterations given by Theorem 1.21 is effectively sharp.

Details of the scattering problems considered We solve the sound-soft scattering problem of Definition 1.7 with $\widehat{a} = (1, 0, 0)$ (i.e the incident plane wave propagates in the x_1-direction), using the direct integral equation (1.2) and the Galerkin method (1.11). The subspace $\mathcal{V}_h$ is taken to be piecewise constants on a shape regular mesh, and the meshwidth h is taken to be $2\pi/(10k)$, i.e. we are choosing ten points per wavelength. We solve the resulting linear system with GMRES, with tolerance 1×10^{-5}. We consider two obstacles:

1. Ω the unit sphere, and
2. Ω the ellipsoid with semi-principal axes of lengths 3, 1, and 1 (in the x_1-, x_2-, and x_3-directions respectively.

The computations were carried out using version 3.0.3 of the BEM++ library [74] on one node of the "Balena" cluster at the University of Bath. The cluster consists of Intel Xeon E5-2650 v2 (Ivybridge, 2.60 GHz) CPUs and the used node had 512GB of main memory. BEM++ was compiled with version 5.2 of the GNU C compiler and the Python code was run under Anaconda 2.3.0.

Numerical results Tables 1 and 2 displays the number of degrees of freedom, number of iterations required for GMRES to converge, and time taken to converge, with $\eta = k$, and with Ω the sphere or ellipsoid. The difference between Tables 1 and 2 is that, in the first, k starts as 2 and then doubles until it equals 128, and in the second, k starts as 3 and then doubles until it equals 96; we performed the second set of experiments when the $k = 128$ run for the ellipsoid failed to complete. Figure 2 plots the iteration counts from both tables and compares them to the $k^{1/3}$ rate

	Sphere			Ellipsoid		
k	$\#_{\text{DOF}}$	$\#_{\text{iterations}}$	time (s)	$\#_{\text{DOF}}$	$\#_{\text{iterations}}$	time (s)
4	1304	13	3.10	3230	16	5.26
8	4998	15	7.42	12324	18	19.30
16	19560	18	40.30	48526	21	113.95
32	77224	22	271.42	190784	25	926.47
64	307454	28	2674.54	754236	31	10354.29
128	1225260	34	31024.43	*	*	*

Table 1: With Ω the sphere or ellipsoid and $\eta = k$, the number of degrees of freedom, number of iterations required for GMRES to converge (with tolerance 1×10^{-5}), and time taken to converge, when GMRES is applied to the Galerkin matrix corresponding to the direct integral equation (1.2), starting with $k = 4$ and then doubling until $k = 128$. * denotes that the run did not complete.

	Sphere			Ellipsoid		
k	$\#_{\text{DOF}}$	$\#_{\text{iterations}}$	time (s)	$\#_{\text{DOF}}$	$\#_{\text{iterations}}$	time (s)
3	846	13	1.12	1806	16	6.20
6	2880	15	3.85	6874	17	9.51
12	11054	17	18.56	26994	19	55.64
24	43688	20	107.18	107272	23	373.45
48	173264	26	928.61	426026	28	3985.63
96	689894	31	10753.95	1691328	34	43423.69

Table 2: Same as Table 1 but for a different range of k.

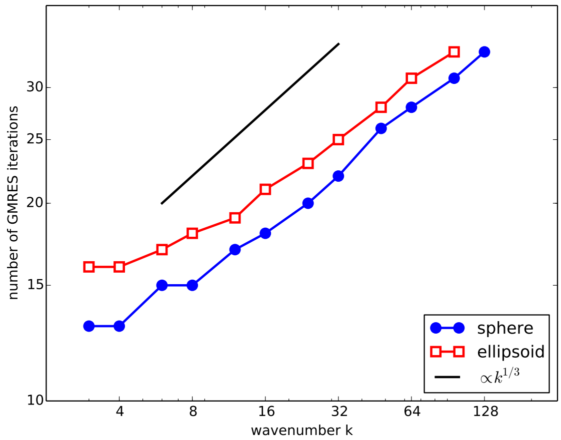

Figure 2: The number of iterations required for GMRES to converge (with tolerance 1×10^{-5}) when GMRES is applied to the Galerkin matrix corresponding to the direct integral equation (1.2) with $\eta = k$, and with Ω the sphere or ellipsoid, and the values of k from Tables 1 and 2. The $k^{1/3}$ rate is the upper bound on the rate guaranteed by Theorem 1.21.

from Theorem 1.21 (the graph is plotted on a log-log scale so that a dependence $\#_{\text{iterations}} \sim k^{\alpha}$ appears as a straight line with gradient α).

We see from Figure 2 that the $k^{1/3}$ growth predicted by Theorem 1.21 appears to be sharp. Indeed, the plot of the iterations for the ellipsoid becomes roughly linear from $k = 12$ onwards, and estimating the slope of this line using the numbers of iterations at $k = 12$ and $k = 96$ we have that the $\#_{\text{iterations}} \sim k^{0.28}$. Using the numbers of iterations at $k = 12$ and $k = 96$ to estimate the rate of growth for the sphere we have that $\#_{\text{iterations}} \sim k^{0.29}$.

Finally, Table 3 compares the iteration counts and times for the sphere when $\eta = k$ and when $\eta = -k$. We see that, for every value of k considered, the number of iterations when $\eta = -k$ is much greater than when $\eta = k$. Table 3 only goes up to $k = 32$, since the $k = 64$ run for the sphere with $\eta = -k$ did not complete.

We performed the experiment in Table 3 because, in the engineering acoustics literature, Marburg recently considered collocation discretisations of the direct integral equation for the Neumann problem (i.e. the Neumann-analogue of equation (1.2)) and showed that the analogue of the choice $\eta = k$ leads to much slower growth than the analogue of the choice $\eta = -k$ [57], [58].

A heuristic explanation for this dependence of the number of iterations on the sign of η is essentially contained in the work of Levadoux and Michielsen [54], [55], and Antoine and Darbas [3]; the understanding is that $i\eta$ should, in some sense, approximate the Dirichlet-to-Neumann map in Ω_+, and (at least for smooth convex obstacles) ik is a better approximation to the Dirichlet-to-Neumann map than $-ik$.

	$\eta = k$		$\eta = -k$	
k	$\#_{\text{iterations}}$	time (s)	$\#_{\text{iterations}}$	time (s)
4	13	3.10	44	3.46
8	15	7.42	88	9.04
16	18	40.30	405	75.38
32	22	271.42	11191	4502.05

Table 3: With Ω the sphere and $\eta = k$ or $\eta = -k$, the number of iterations required for GMRES to converge (with tolerance 1×10^{-5}) and time taken to converge, when GMRES is applied to the Galerin matrix corresponding to the direct integral equation (1.2).

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