DEFECT MEASURES OF EIGENFUNCTIONS WITH MAXIMAL L^∞ GROWTH

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Abstract. We characterize the defect measures of sequences of Laplace eigenfunctions with maximal L^∞ growth. As a consequence, we obtain new proofs of results on the geometry of manifolds with maximal eigenfunction growth obtained by Sogge–Toth–Zelditch [STZ11], and generalize those of Sogge–Zelditch [SZ16a] to the smooth setting. We also obtain explicit geometric dependence on the constant in Hörmander's L^∞ bound for high energy eigenfunctions, improving on estimates of Donnelly [Don01].

1. Introduction

Let (M, g) be a C^∞ compact manifold of dimension n without boundary. Consider the solutions to

$$(-\Delta_g - \lambda_j^2)u_{\lambda_j} = 0, \qquad \|u_{\lambda_j}\|_{L^2} = 1 \tag{1.1}$$

as $\lambda_j \to \infty$. It is well known [Ava56, Lev52, Hör68] (see also [Zwo12, Chapter 7]) that solutions to (1.1) satisfy

$$\|u_{\lambda_j}\|_{L^\infty(M)} \le C\lambda_j^{\frac{n-1}{2}} \tag{1.2}$$

and that this bound is saturated e.g. on the sphere. It is natural to consider situations which produce sharp examples for (1.2). Previous works [Bér77, IS95, TZ02, SZ02, TZ03, STZ11, SZ16a, SZ16b] have studied the connections between growth of L^∞ norms of eigenfunctions and the global geometry of the manifold M.

In this article, we study the relationship between L^∞ growth and L^2 concentration of eigenfunctions (this direction of inquiry was initiated in [GT17]). We measure L^2 concentration of eigenfunctions using *defect measures* - a sequence $\{u_{h_j}\}$ has defect measure μ if for any $a \in C_c^\infty(T^*M)$,

$$\langle a(x, h_jD)u_{h_j}, u_{h_j}\rangle \to \int_{T^*M} a(x, \xi)d\mu. \tag{1.3}$$

We write $a(x, hD)$ for a semiclassical pseudodifferential operator given by the quantization of the symbol $a(x, \xi)$ (see [Zwo12, Chapters 4, 14]) and let $h_j = \lambda_j^{-1}$ when considering the solutions to (1.1).

By an elementary compactness/diagonalization argument it follows that any L^2 bounded sequence u_h possesses a further subsequence that has a defect measure in the sense of (1.3) [Zwo12, Theorem 5.2]. Moreover, a standard commutator argument shows that if

$$p(x, hD)u = o_{L^2}(h),$$

for $p \in S^k(T^*M)$ real valued with

$$|p| \ge c\langle\xi\rangle^k \text{ on } |\xi| \ge R,$$

then μ is supported on $\Sigma := \{p = 0\}$ and is invariant under the bicharacteristic flow of p; that is, if $G_t = \exp(tH_p) : \Sigma \to \Sigma$ is the bicharacteristic flow, $(G_t)_*\mu = \mu$, $\forall t \in \mathbb{R}$ [Zwo12, Theorems 5.3, 5.4].

Rather than studying only eigenfunctions of the Laplacian, we replace $-\Delta_g - \lambda_j^2$ by a general semiclassical pseudodifferential operator and replace eigenfunctions with quasimodes. To this end, we say that u is compactly microlocalized if there exists $\chi \in C_c^\infty(\mathbb{R})$ with

$$(1-\chi(|hD|))u = O_{\mathcal{S}}(h^\infty).$$

For $P \in \Psi^m(M)$ an h-pseudodifferential operator, we say that u is a *quasimode for P* if

$$Pu = o_{L^2}(h), \qquad \|u\|_{L^2} = 1.$$

Let $\Sigma_x := \Sigma \cap T_x^*M$ and define respectively the *flow out of* Σ_x and *time T flowout of* Σ_x by

$$\Lambda_x := \bigcup_{T=0}^{\infty} \Lambda_{x,T}, \qquad \Lambda_{x,T} := \bigcup_{t=-T}^{T} G_t(\Sigma_x).$$

Let $\mathcal{H}^r$ denote the Hausdorff-r measure with respect to the Sasaki metric on T^*M (see for example [Bla10, Chapter 9] for a treatment of the Sasaki metric). For a Borel measure ρ on T^*M, let $\rho_x := \rho|_{\Lambda_x}$ i.e. $\rho_x(A) := \rho(A \cap \Lambda_x)$. Recall that two Borel measures on a set Ω, μ and ρ, are *mutually singular* (written $\mu \perp \rho$) if there exist disjoint sets $N, P \subset \Omega$ so that $\Omega = N \cup P$ and $\mu(N) = \rho(P) = 0$.

The main theorem characterizes the defect measures of quasimodes with maximal growth.

THEOREM **1.** *Let $P \in \Psi^m(M)$ be an h-pseudodifferential operator with real principal symbol p satisfying*

$$\partial_\xi p \neq 0 \text{ on } \{p = 0\}. \tag{1.4}$$

Suppose u is a compactly microlocalized quasimode for P with

$$\limsup_{h\to 0} h^{\frac{n-1}{2}} \|u\|_{L^\infty} > 0 \tag{1.5}$$

and defect measure μ. Then there exists $x \in M$ and $x(h) \to x$ so that

$$\limsup_{h\to 0} h^{\frac{n-1}{2}} |u(x(h))| > 0, \qquad \mu_x = \rho + f d\mathcal{H}_x^n, \tag{1.6}$$

where $0 \neq f \in L^1(\Lambda_x, \mathcal{H}_x^n)$, $\rho \perp \mathcal{H}_x^n$, and both $f d\mathcal{H}_x^n$, and ρ are invariant under G_t.

One way of interpreting Theorem 1 is that a quasimode with maximal L^∞ growth near x must have energy on a positive measure set of directions entering T_x^*M. That is, it must have concentration comparable to that of the zonal harmonic. (See [GT17, Section 4] for a description of the defect measure of the zonal harmonic.)

Theorem 1 is an easy consequence of the following theorem (see section 2 for the proof that Theorem 2 implies Theorem 1).

THEOREM **2.** *Let $x \in M$ and $P \in \Psi^m(M)$ be an h-pseudodifferential operator with real principal symbol p satisfying*

$$\partial_\xi p \neq 0 \text{ on } \{p = 0\}.$$

There exists a constant C_n depending only on n with the following property: Suppose that u is compactly microlocalized quasimode for P and has defect measure μ. Define $\rho \perp \mathcal{H}_x^n$ and $f \in L^1(\Lambda_x; \mathcal{H}_x^n)$ by

$$\mu_x =: \rho + f d\mathcal{H}_x^n.$$

Then for all $r(h) = o(1)$,

$$\limsup_{h\to 0} h^{\frac{n-1}{2}} \|u\|_{L^\infty(B(x,r(h)))} \leq C_n \int_{\Sigma_x} \sqrt{f} \sqrt{\frac{|\nu(H_p)|}{|\partial_\xi p|_g}} d\mathrm{Vol}_{\Sigma_x}$$

where ν is a unit (with respect to the Sasaki metric) conormal to Σ_x in Λ_x, Vol_{Σ_x} is the measure induced by the Euclidean metric on T^*_xM, and $|\partial_\xi p|_g = |\partial_\xi p\cdot\partial_x|_g$. Furthermore, $fd\mathcal{H}^n_x$ is G_t invariant.

In particular, if $\mu_x\perp\mathcal{H}^n_x$, then

$$\|u\|_{L^\infty(B(x,r(h))}=o(h^{\frac{1-n}{2}}).$$

To see that Theorem 2 applies to solutions of (1.1), let $h_j=\lambda_j^{-1}$. Writing $u=u_{\lambda_j}$ and $h=h_j$,

$$(-h^2\Delta_g-1)u=0.$$

Then, $(-h^2\Delta_g-1)=p(x,hD)$ with $p=|\xi|^2_g-1+hr$ and therefore, the elliptic parametrix construction shows that u is compactly microlocalized. Since $\partial_{\xi_j}p=2g^{ij}\xi_i$, $\partial_\xi p\neq 0$ on $p=0$ and Theorem 2 applies. In Section 2, we use Theorem 2 with $P=-h^2\Delta_g-1$ to give explicit bounds on the constant C in (1.2) in terms of the *injectivity radius of* M, $\mathrm{inj}(M)$, thereby improving on the bounds of [Don01] at high energies.

Corollary **1.1.** *There exists $\tilde{C}_n>0$ depending only on n so that for all (M,g) compact, boundaryless Riemannian manifolds of dimension n and all $\varepsilon>0$, there exists $\lambda_0=\lambda_0(\varepsilon,M,g)>0$ so that for $\lambda_j>\lambda_0$ and u_{λ_j} solving (1.1)*

$$\|u_{\lambda_j}\|_{L^\infty}\leq\Big(\frac{\tilde{C}_n}{\mathrm{inj}(M)^{1/2}}+\varepsilon\Big)\lambda_j^{\frac{n-1}{2}}.$$

Theorem 2 is sharp in the following sense. Let $P=-h^2\Delta_g-1$ and G_t as above.

Theorem **3.** *Suppose there exists $z\in M$, $T>0$ so that $G_T(z,\xi)=(z,\xi)$ for all $(z,\xi)\in S^*_zM$. Let $\rho\perp\mathcal{H}^n_z$ be a Radon measure on Λ_z invariant under G_t and $0\leq f\in L^1(\Lambda_z,\mathcal{H}^n_z)$ be invariant under G_t so that*

$$\|f\|_{L^1(\Lambda_z,\mathcal{H}^n_z)}+\rho(\Lambda_z)=1.$$

Then there exist $h_j\to 0$ and $\{u_{h_j}\}_{j=1}^\infty$ solving

$$(-h_j^2\Delta_g^2-1)u_{h_j}=o(h_j),\qquad \|u_{h_j}\|_{L^2}=1,\qquad \limsup_{j\to\infty}h_j^{\frac{n-1}{2}}\|u_{h_j}\|_{L^\infty}\geq(2\pi)^{\frac{1-n}{2}}\int_{\Sigma_z}\sqrt{f}d\mathrm{Vol}_{\Sigma_z}$$

and having defect measure $\mu=\rho+fd\mathrm{Vol}_{\Lambda_z}$.

Notice that we do not claim the existence of exact eigenfunctions having prescribed defect measures in Theorem 3, instead constructing only quasimodes.

1.1. Relation with previous results.

As far as the author is aware, the only previous work giving conditions on the defect measures of eigenfunctions with maximal L^∞ growth is [GT17]. Theorem 2 improves on the conditions given in [GT17, Theorem 3]; replacing $\mathcal{H}^n_x(\mathrm{supp}\,\mu_x)=0$ with the sharp condition $\mu_x\perp\mathcal{H}^n_x$. To see an example of how these conditions differ, fix $x\in M$ such that every geodesic through x is closed and let $\{\xi_k\}_{k=1}^\infty\subset S^*_xM$ be a countable dense subset. Suppose that the defect measure of $\{u_{\lambda_j}\}$ is given by

$$\mu=\sum_k a_k\delta_{\gamma_k},\qquad a_k>0$$

where γ_k is the geodesic emanation from (x,ξ_k). Then $\mathrm{supp}\,\mu_x=\Lambda_x$, but $\mu_x\perp\Lambda_x$, so Theorem 2 applies to this sequence but the results of [GT17] do not. Furthermore, Theorem 2 gives quantitative estimates on the growth rates of quasimodes in terms of their defect measures.

We are able to draw substantial conclusions about the global geometry of a manifold M having quasimodes with maximal L^∞ growth from Theorem 2. The results of [STZ11, Theorems 1(1), 2] and hence also [SZ02, Theorem 1.1] are corollaries of Thoerem 2. For $x \in M$, define the map $T_x : \Sigma_x \to \mathbb{R} \sqcup \{\infty\}$ by

$$T_x(\xi) := \inf\{t > 0 \mid G_t(x,\xi) \in \Sigma_x\}. \tag{1.7}$$

Then, define the *loop set* by

$$\mathcal{L}_x := \{\xi \in \Sigma_x \mid T_x(\xi) < \infty\},$$

and the *first return map* $\eta_x : \mathcal{L}_x \to \Sigma_x$ by

$$G_{T_x(\xi)}(x,\xi) = (x, \eta_x(\xi)).$$

Finally, define the set of *recurrent points* by

$$\mathcal{R}_x := \left\{ \xi \in \Sigma_x \mid \xi \in \left(\bigcap_{T>0} \overline{\bigcup_{t>T} G_t(x,\xi) \cap \Sigma_x} \right) \bigcap \left(\bigcap_{T>0} \overline{\bigcup_{t>T} G_{-t}(x,\xi) \cap \Sigma_x} \right) \right\}, \tag{1.8}$$

where the closure is with respect to the subspace topology on Σ_x.

Corollary **1.2.** *Let (M,g) be a compact boundaryless Riemannian manifold and P satisfy (1.4). Suppose that* $\mathrm{Vol}_{\Sigma_x}(\mathcal{R}_x) = 0$. *Than for any $r(h) = o(1)$ and u a compactly microlocalized quasimode for P,*

$$\|u\|_{L^\infty(B(x,r(h))} = o(h^{\frac{1-n}{2}}).$$

Moreover, the forward direction of [SZ16a, Theorem 1.1] with the analyticity assumption removed is an easy corollary of Theorem 2. To state the theorem let $d\mathrm{Vol}_{\Sigma_x}$ denote the measure induced on Σ_x from the Euclidean metric on T^*_xM. We define the unitary Perron–Frobenius operator $U_x : L^2(\mathcal{R}_x, \sqrt{|\nu(H_p)|}d\mathrm{Vol}_{\Sigma_x}) \to L^2(\mathcal{R}_x, \sqrt{|\nu(H_p)|}d\mathrm{Vol}_{\Sigma_x})$ by

$$U_x(f)(\xi) := \sqrt{J_x(\xi)}f(\eta_x(\xi)), \tag{1.9}$$

where, writing

$$G_t(x,\xi) = (x_t(x,\xi), \eta_t(x,\xi)),$$

we have that

$$J_x(\xi) = \left|\det D_\xi \eta_t|_{t=T_x(\xi)}\right| \cdot \left| \frac{\nu(H_p)(\eta_x(\xi))}{\nu(H_p)(\xi)} \right| \tag{1.10}$$

is the Jacobian factor so that for $f \in L^1(\Sigma_x)$ supported on $\mathcal{L}_x$,

$$\int \eta_x^* f J_x(\xi) |\nu(H_p)|(\xi) d\mathrm{Vol}_{\Sigma_x} = \int f(\xi)|\nu(H_p)|(\xi) d\mathrm{Vol}_{\Sigma_x}.$$

See [Saf88, Section 4] for a more detailed discussion of U_x. We say that x is *dissipative* if

$$\left\{ f \in L^2\left(\mathcal{R}_x, \sqrt{|\nu(H_p)|}d\mathrm{Vol}_{\Sigma_x}\right) \middle| U_x(f) = f \right\} = \{0\}. \tag{1.11}$$

Corollary **1.3.** *Let (M,g) be a compact boundaryless Riemannian manifold and P satisfy (1.4). Suppose that x is dissipative. Then for $r(h) = o(1)$ and u a compactly microlocalized quasimode for P,*

$$\|u\|_{L^\infty(B(x,r(h)))} = o(h^{\frac{1-n}{2}}).$$

The dynamical arguments in [SZ16b] show that if (M,g) is a real analytic surface and $P = -h^2\Delta_g - 1$, then x being non-dissipative implies that x is a periodic point for the geodesic flow, i.e. a point so that there is a $T > 0$ so that every geodesic starting from $(x,\xi) \in S^*_xM$ smoothly closes at time T.

1.2. **Comments on the proof.** While the assumption $Pu = o_{L^2}(h)$ implies a global assumption on u, similar to that in [GT17], the analysis here is entirely local. The global consequences in Corollaries 1.2 and 1.3 follow from dynamical arguments using invariance of defect measures.

We take a different approach from that in [GT17] choosing to base our method on the Koch–Tataru–Zworski method [KTZ07] rather than explicit knowledge of the spectral projector. This approach gives a more explicit explanation for the L^∞ improvements from defect measures. In Section 7 we sketch the proof of Theorem 2 in the case that $\mu_x \perp \mathcal{H}^n_x$ using the spectral projector.

The idea behind our proof is to estimate the absolute value of u at x in terms of the degree to which energy concentrates along any bicharacteristics passing through Σ_x. Either too much localization or too little localization will yield an improvement over the naive bound. By covering Λ_x with appropriate cutoffs to tubes around bicharacteristics we are then able to give $o(h^{\frac{1-n}{2}})$ bounds whenever $\mu_x \perp \mathcal{H}^n_x$. The proof relies, roughly, on the fact that if a compactly microlocalized function u on $\mathbb{R}^m$ has defect measure supported at (x_0, ξ_0), then $\|u\|_{L^\infty} = o(h^{-m/2})$ rather than the standard estimate $O(h^{-m/2})$.

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2. Consequences of Theorem 2

We first formulate a local result matching those in [SZ02, STZ11] more closely.

COROLLARY **2.1.** *Let $x \in M$ and $P \in \Psi^m(M)$ satisfying the assumption of Theorem 2. Then there exists a constant C_n depending only on n with the following property. Suppose that u is a compactly microlocalized quasimode for P, and has defect measure μ. Define $\rho \perp \mathcal{H}^n_x$ and $f \in L^1(\Lambda_x; \mathcal{H}^n_x)$ by*

$$\mu_x =: \rho + f d\mathcal{H}^n_x.$$

Then for all $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}(\varepsilon)$ of x and $h_0(\varepsilon)$ such that for $0 < h < h_0(\varepsilon)$,

$$\|u\|_{L^\infty(\mathcal{N}(\varepsilon))} \le h^{-\frac{n-1}{2}} \left(C_n \int_{\Sigma_x} \sqrt{f} \sqrt{\frac{|\nu(H_p)|}{|\partial_\xi p|_g}} d\mathrm{Vol}_{\Sigma_x} + \varepsilon \right).$$

Proof that Theorem 2 implies Corollary 2.1. Let

$$\tilde{A}_x := C_n \int_{\Sigma_x} \sqrt{f} \sqrt{\frac{\nu(H_p)}{|\partial_\xi p|_g}} d\mathrm{Vol}_{\Sigma_x}$$

and suppose that there exists $\varepsilon > 0$ such that for all $r > 0$,

$$\limsup_{h\to 0} h^{\frac{1-n}{2}} \|u_h\|_{L^\infty(B(x,r))} > \tilde{A}_x + \varepsilon. \tag{2.1}$$

Fix $r_0 > 0$. Then by (2.1) there exists $x_0 \in B(x, r_0)$, $h_0 > 0$ so that

$$|u_{h_0}(x_0)| h_0^{\frac{n-1}{2}} \ge \tilde{A}_x + \frac{\varepsilon}{2}.$$

Assume that there exist $\{h_j\}_{j=0}^N$ and $\{x_j\}_{j=0}^N$ so that

$$h_j \le \frac{h_{j-1}}{2}, \qquad x_j \in B(x, r_0 2^{-j}), \qquad h_j^{\frac{n-1}{2}} |u(x_j)| \ge \tilde{A}_x + \frac{\varepsilon}{2}.$$

By (2.1), there exists $h_k \downarrow 0$ and $x_k \in B(x, r_0 2^{-N-1})$ such that

$$h_k^{\frac{1-n}{2}} |u_{h_k}(x_k)| \ge \tilde{A}_x + \frac{\varepsilon}{2}.$$

Therefore, we can choose k_0 large enough so that $h_{k_0} \leq \frac{h_N}{2}$ and let $(h_{N+1}, x_{N+1}) = (h_{k_0}, x_{k_0})$, Hence, by induction, there exists $h_j \downarrow 0$, $x_j \to x$ such that

$$h_j^{\frac{n-1}{2}} |u_{h_j}(x_j)| \geq \tilde{A}_x + \frac{\varepsilon}{2},$$

contradicting Theorem 2. □

Proof that Theorem 2 implies Theorem 1. Compactness of M together with Corollary 2.1 with $f \equiv 0$ implies the contrapositive of Theorem 1, in particular, if $\mu_x \perp \mathcal{H}^n_x$ for all x, then $\|u\|_{L^\infty} = o(h^{\frac{1-n}{2}})$. □

2.1. Proof of Corollaries 1.2 and 1.3 from Theorem 2.

Lemma **2.2.** *Fix $x \in M$ and suppose that u is compactly microlocalized with $Pu = o_{L^2}(h)$. Define $\rho \perp \mathcal{H}^n_x$ and $f \in L^1(\Lambda_x; \mathcal{H}^n_x)$ by*

$$\mu_x = \rho + f d\mathcal{H}^n_x.$$

Then $\operatorname{supp} f|_{\Sigma_x} \subset \mathcal{R}_x$.

Proof. For $\xi \in \Sigma_x$ and $\varepsilon > 0$ let $B(\xi, \varepsilon) \subset \Sigma_x$ be the open ball of radius ε and

$$V := \bigcup_{-2\delta < t < 2\delta} G_t(B(\xi, \varepsilon)).$$

Observe that by Theorem 2 the triple $(\Lambda_x, f d\mathcal{H}^n_x, G_t)$ forms a measure preserving dynamical system. The Poincaré recurrence theorem [BS02, Proposition 4.2.1, 4.2.2] implies that for $f d\mathcal{H}^n_x$ a.e. $(x_0, \xi_0) \in V$ there exists $t_n^{\pm} \to \pm\infty$ so that $G_{t_n^{\pm}}(x_0, \xi_0) \in V$. By the definition of V, there exists $s_n^{\pm}$ with $|s_n^{\pm} - t_n^{\pm}| < 2\delta$ such that $G_{s_n^{\pm}}(x_0, \xi_0) \in B(\xi, \varepsilon)$. In particular, for $f d\mathcal{H}^n_x$ a.e. $(x_0, \xi_0) \in V$,

$$\left(\bigcap_{T>0} \bigcup_{t \geq T} G_t(x_0, \xi_0) \cap B(\xi, \varepsilon) \right) \bigcup \left(\bigcap_{T>0} \bigcup_{t \geq T} G_{-t}(x_0, \xi_0) \cap B(\xi, \varepsilon) \right) \neq \emptyset. \tag{2.2}$$

Let

$$\mu_{\Sigma_x} := f|_{\Sigma_x} |\nu(H_p)||_{\Sigma_x} d\mathrm{Vol}_{\Sigma_x}.$$

We next show that (2.2) holds for μ_{Σ_x} a.e. point in $B(\xi, \varepsilon)$. To do so, suppose the opposite. Then there exists $A \subset B(\xi, \varepsilon)$ with $\mu_{\Sigma_x}(A) > 0$ so that for each $(x_0, \xi_0) \in A$, there exists $T > 0$ with

$$\left(\left[\bigcup_{t \geq T} G_t(x_0, \xi_0) \right] \bigcup \left[\bigcup_{t \geq T} G_{-t}(x_0, \xi_0) \right] \right) \bigcap B(\xi, \varepsilon) = \emptyset. \tag{2.3}$$

Let

$$A_\delta := \bigcup_{t=-\delta}^{\delta} G_t(A).$$

Then $A_\delta \subset V$ and for all $(x_0, \xi_0) \in A_\delta$, there exists $T > 0$ so that (2.3) holds. Moreover, invariance of $f d\mathcal{H}^n_x$ under G_t together with Lemma 3.4 implies that

$$(f d\mathcal{H}^n_x)(A_\delta) = 2\delta \mu_{\Sigma_x}(A) > 0$$

which contradicts (2.2). Thus (2.2) holds for μ_{Σ_x} a.e. point in $B(\xi, \varepsilon)$.

Let $\{B(\xi_i, \varepsilon_i)\}$ be a countable basis for the topology on Σ_x. Then for each i, there is a subset of full measure, $\tilde{B}_i \subset B(\xi_i, \varepsilon_i)$ so that for every point of $\tilde{B}_i$ (2.2) holds with $\xi = \xi_i$, $\varepsilon = \varepsilon_i$. Noting that $X_i = \tilde{B}_i \cup (\Sigma_x \setminus B(\xi_i, \varepsilon_i))$ has full measure, we conclude that $\tilde{\Sigma}_x = \cap_i X_i \subset \mathcal{R}_x$ has full measure and thus, $\mu_{\Sigma_x}(\mathcal{R}_x) = \mu_{\Sigma_x}(\Sigma_x)$, finishing the proof of the lemma. □

Proof of Corollary 1.2. Let u solve $Pu = o_{L^2}(h)$. Then we can extract a subsequence with a defect measure μ. By Lemma 2.2, $\mu_x = \rho + f d\mathcal{H}^n_x$ with $\rho \perp \mathcal{H}^n_x$ and $\operatorname{supp} f|_{\Sigma_x} \subset \mathcal{R}_x$. Now, if $\mathrm{Vol}_{\Sigma_x}(\mathcal{R}_x) = 0$,

$$\int_{\Sigma_x} \sqrt{f}\sqrt{\frac{|\nu(H_p)|}{|\partial_\xi p|_g}} d\mathrm{Vol}_{\Sigma_x} = 0.$$

Plugging this into Theorem 2 proves the corollary ☐

Proof of Corollary 1.3. Let u solve $Pu = o_{L^2}(h)$. Then we can extract a subsequence with a defect measure μ. By Lemma 2.2 and Theorem 2, $\mu_x = \rho + f d\mathcal{H}^n_x$ where $\rho \perp \mathcal{H}^n_x$, $\operatorname{supp} f|_{\Sigma_x} \subset \mathcal{R}_x$, and $f d\mathcal{H}^n_x$ is G_t invariant.

Let T_x be as in (1.7). Fix $T < \infty$ and suppose

$$A \subset \Omega_T := \{\eta \in \Sigma_x \mid T_x(\eta) \leq T\}.$$

Write $(0, T] = \bigsqcup_{i=1}^{N(\varepsilon)} (T_i - \varepsilon, T_i + \varepsilon]$ and

$$\Omega_T = \bigsqcup_{i=1}^{N(\varepsilon)} T_x^{-1}((T_i - \varepsilon, T_i + \varepsilon]).$$

Then, by Lemma 3.4, for any $0 < \delta$ small enough

$$\begin{aligned}
\int 1_A f |\nu(H_p)| d\mathrm{Vol}_{\Sigma_x} &= \frac{1}{2\delta}\int 1_{\bigcup_{-\delta}^{\delta} G_t(A)} f d\mathcal{H}^n_x \\
&= \sum_i \frac{1}{2\delta}\int 1_{\bigcup_{t_i-\delta}^{t_i+\delta} G_t(A\cap\Omega_i)} f d\mathcal{H}^n_x \\
&= \sum_i \frac{1}{2\delta}\int 1_{\bigcup_{-\delta+O(\varepsilon)}^{\delta+O(\varepsilon)} G_t(\eta_x(A\cap\Omega_i))} f d\mathcal{H}^n_x \\
&= \frac{1}{2\delta}\int 1_{\bigcup_{-\delta+O(\varepsilon)}^{\delta+O(\varepsilon)} G_t(\eta_x(A))} f d\mathcal{H}^n_x
\end{aligned}$$

So, sending $\varepsilon \to 0$, applying the dominated convergence theorem and then $\delta \to 0$ gives

$$\int 1_A f |\nu(H_p)| d\mathrm{Vol}_{\Sigma_x} = \int 1_{\eta_x(A)} f |\nu(H_p)| d\mathrm{Vol}_{\Sigma_x}$$

for all $A \subset \Omega_T$ measurable. Taking $T \to \infty$ then proves this for all $A \subset \mathcal{L}_x$ measurable. In particular, changing variables, and using that $\operatorname{supp} f \subset \mathcal{R}_x \subset \mathcal{L}_x$, and writing $J_x(\xi)$ as in (1.10)

$$f(\xi) \cdot |\nu(H_p)|(\xi) d\mathrm{Vol}_{\Sigma_x}(\xi) = f(\eta_x(\xi)) \cdot J_x(\xi) \cdot |\nu(H_p)|(\xi) d\mathrm{Vol}_{\Sigma_x}(\xi)$$

which implies $U_x\sqrt{f} = \sqrt{f}$ where U_x is defined in (1.9). Observe that since x is dissipative and $\sqrt{f} \in L^2(\mathcal{R}_x, \sqrt{|\nu(H_p)|} d\mathrm{Vol}_{\Sigma_x})$, (1.11) implies $\sqrt{f} = 0$. Theorem 2 then completes the proof. ☐

2.2. **Spectral cluster estimates for $-\Delta_g$.** Let (M, g) be a smooth, compact, boundaryless Riemannian manifold of dimension n, $p = |\xi|_g^2 - 1$, $G_t = \exp(tH_p)$ and

$$A_x := \frac{C_n}{2}\left(\frac{\mathrm{Vol}_{\Sigma_x}(\mathcal{R}_x)}{\inf_{\xi\in\mathcal{R}_x} T_x(\xi)}\right)^{1/2}$$

where T_x is as in (1.7) and C_n is the constant in Theorem 2. We consider an orthonormal basis $\{u_{\lambda_j}\}_{j=1}^\infty$ of eigenfunctions of $-\Delta_g$ (i.e. solving (1.1)) and let

$$\Pi_{[\lambda,\lambda+\delta]} := 1_{[\lambda,\lambda+\delta]}(\sqrt{-\Delta_g}).$$

Corollary **2.3.** *For all $\varepsilon > 0$, $x \in M$, there exists $\delta = \delta(x,\varepsilon) > 0$, a neighborhood $\mathcal{N}(x,\varepsilon)$ of x, and $\lambda_0 = \lambda_0(x,\varepsilon) > 0$ so that for $\lambda > \lambda_0$,*

$$\|\Pi_{[\lambda,\lambda+\delta]}\|^2_{L^2\to L^\infty(\mathcal{N}(x,\varepsilon))} = \sup_{y\in\mathcal{N}(x,\varepsilon)} \sum_{\lambda_j\in[\lambda,\lambda+\delta]} |u_{\lambda_j}(y)|^2 \leq (A_x^2+\varepsilon)\lambda^{n-1}. \tag{2.4}$$

Note that since $G_t|_{S^*M}$ parametrizes the speed 2 geodesic flow and therefore

$$\inf_{\xi\in\mathcal{R}_x} T_x(\xi) \geq \frac{1}{2}L(x,M) \geq \operatorname{inj}(M),$$

$$L(x,M) := \inf\{t>0 \mid \text{ there exists a geodesic of length } t \text{ starting and ending at } x\},$$

and $\operatorname{inj}(M)$ denotes the injectivity radius of M. Therefore, we could replace A_x in (2.4) by either of

$$A_x' = C_n\left(\frac{\operatorname{Vol}_{\Sigma_x}(\mathcal{R}_x)}{2\cdot L(x,M)}\right)^{1/2}, \qquad A_x'' = C_n\left(\frac{\operatorname{Vol}_{\Sigma_x}(\mathcal{R}_x)}{4\cdot \operatorname{inj}(M)}\right)^{1/2}.$$

to obtain a weaker, but more easily understood statement. Corollary 2.3 is closely related to the work of Donnelly [Don01] and gives explicit dependence of the constant in the Hörmander bound in terms of geometric quantities.

Proof. For $U \subset M$

$$\|\Pi_{[\lambda,\lambda+\delta]}\|^2_{L^2(M)\to L^\infty(U)} = \sup_{x\in U} \sum_{\lambda_j\in[\lambda,\lambda+\delta]} |u_{\lambda_j}(x)|^2. \tag{2.5}$$

For $w \in L^2(M)$,

$$\|(-\Delta_g-\lambda^2)\Pi_{[\lambda,\lambda+\delta]}w\|_{L^2} \leq 2\lambda\delta\|\Pi_{[\lambda,\lambda+\delta]}w\|_{L^2}. \tag{2.6}$$

Suppose that for some $\varepsilon > 0$ no δ, $\mathcal{N}(x)$, and λ_0 exist so that (2.4) holds. Then for all $\delta > 0$, $r > 0$,

$$\limsup_{\lambda\to\infty} \lambda^{\frac{1-n}{2}}\|\Pi_{[\lambda,\lambda+\delta]}\|_{L^2(M)\to L^\infty(B(x,r))} > A_x+\varepsilon.$$

Therefore, for all $\delta > 0$, there exists $\lambda_{k,\delta}\uparrow\infty$ so that

$$\lambda_{k,\delta}^{\frac{1-n}{2}}\|\Pi_{[\lambda_{k,\delta},\lambda_{k,\delta}+\delta]}\|_{L^2(M)\to L^\infty(B(x,r))} > A_x+\varepsilon.$$

Moreover, we may assume that for $\delta_1 < \delta_2$, $\lambda_{k,\delta_1} > \lambda_{k,\delta_2}$. So, since for $\delta_1 \leq \delta_2$,

$$\|\Pi_{[\lambda,\lambda+\delta_1]}\|_{L^2(M)\to L^\infty(B(x,r))} \leq \|\Pi_{[\lambda,\lambda+\delta_2]}\|_{L^2(M)\to L^\infty(B(x,r))},$$

letting $\lambda_l = \lambda_{l,l^{-1}}$, $\lambda_l\to\infty$ and

$$\lambda_l^{\frac{1-n}{2}}\|\Pi_{[\lambda_l,\lambda_l+l^{-1}]}\|_{L^2(M)\to L^\infty(B(x,r))} > A_x+\varepsilon.$$

By (2.6) for $w \in L^2(M)$

$$\|(-\lambda_l^{-2}\Delta_g-1)\Pi_{[\lambda_l,\lambda_l+l^{-1}]}w\|_{L^2\to L^2} = o(\lambda_l^{-1})\|\Pi_{[\lambda_l,\lambda_l+l^{-1}]}w\|_{L^2\to L^2}.$$

Fix $w_l \in L^2(M)$ with $\|w_l\|_{L^2} = 1$, so that

$$\lambda_l^{\frac{1-n}{2}}\|v_l\|_{L^\infty(B(x,r))} > A_x+\varepsilon, \qquad v_l := \Pi_{[\lambda_l,\lambda_l+l^{-1}]}w_l.$$

Then extracting a further subsequence, subsequence if necessary, we may assume that v_l has defect measure μ with $\mu_x = \rho + f d\mathcal{H}^n_x$ and hence that Corollary 2.1 applies to v_l. Furthermore, since $\|v_l\|_{L^2} \leq \|w_l\|_{L^2} = 1$,

$$\int_{\Lambda_x} f d\mathcal{H}^n_x \leq 1. \tag{2.7}$$

By computing in normal geodesic coordinates at x, observe that for $p = |\xi|_g^2 - 1$, $|\nu(H_p)| = |\partial_\xi p|_g = 2$. Thus, Corollary 2.1, implies the existence of $r > 0$ small enough so that

$$A_x + \varepsilon \leq \limsup_{l\to\infty} \lambda_l^{\frac{1-n}{2}} \|v_l\|_{L^\infty(B(x,r))} \leq C_n \int_{\Sigma_x} \sqrt{f} d\mathrm{Vol}_{\Sigma_x} \tag{2.8}$$

Finally, by Lemma 2.2 and (2.7), $\operatorname{supp} f \subset \mathcal{R}_x$ and $\|f\|_{L^1(\Lambda_x, \mathcal{H}^n_x)} \leq 1$. Therefore,

$$\begin{aligned} C_n \int_{\Sigma_x} \sqrt{f} d\mathrm{Vol}_{\Sigma_x} &\leq C_n \left(\frac{1}{2} \int_{\Sigma_x} f |\nu(H_p)| d\mathrm{Vol}_{\Sigma_x} \right)^{1/2} \left(\mathrm{Vol}_{\Sigma_x}(\mathcal{R}_x) \right)^{1/2} \\ &= C_n \left(\frac{1}{4 \cdot \inf_{\xi \in \mathcal{R}_x} (T_x(\xi))} \int_{\Lambda_{x, \inf_{\mathcal{R}_x} T_x(\xi)}} f d\mathcal{H}^n_x \right)^{1/2} \left(\mathrm{Vol}_{\Sigma_x}(\mathcal{R}_x) \right)^{1/2} \\ &\leq \frac{C_n}{2} \left(\frac{|\mathcal{R}_x|}{\inf_{\xi \in \mathcal{R}_x} (T_x(\xi))} \right)^{1/2} = A_x, \end{aligned}$$

contradicting (2.8). □

Compactness of M, the fact that $\mathrm{Vol}_{\Sigma_x}(\mathcal{R}_x) \leq \mathrm{Vol}(S^{n-1})$, and Corollary 2.3 imply Corollary 1.1.

3. Dynamical and measure theoretic preliminaries

3.1. **Dynamical preliminaries.** The following lemma gives an estimate on how much spreading the geodesic flow has near a point.

Lemma **3.1.** *Fix $x \in M$. Then there exists $\delta > 0$ small enough so that uniformly for $t \in [0, \delta]$,*

$$\frac{1}{2} d(\xi_1, \xi_2) + O(d(\xi_1, \xi_2)^2) \leq d(G_t(x, \xi_2), G_t(x, \xi_1)) \leq 2 d(\xi_1, \xi_2) + O(d(\xi_1, \xi_2)^2). \tag{3.1}$$

Furthermore if $G_t(x, \xi_i) = (x_i(t), \xi_i(t))$,

$$d(x_1(t), x_2(t)) = O(d(\xi_1, \xi_2)\delta). \tag{3.2}$$

Proof. By Taylor's theorem

$$G_t(x, \xi_1) - G_t(x, \xi_2) = d_\xi G_t(x, \xi_2))(\xi_1 - \xi_2) + O_{C^\infty}(\sup_{q \in \Sigma} |d_\xi^2 G_t(q)| (\xi_1 - \xi_2)^2)$$

Now,

$$G_t(x, \xi) = (x, \xi) + (\partial_\xi p(x, \xi) t, -\partial_x p(x, \xi) t) + O(t^2)$$

so

$$d_\xi G_t(x, \xi) = (0, I) + t(\partial_\xi^2 p, -\partial_{\xi x}^2 p) + O(t^2)$$

In particular,

$$G_t(x, \xi_1) - G_t(x, \xi_2) = ((0, I) + O(t))(\xi_1 - \xi_2) + O((\xi_1 - \xi_2)^2)$$

and choosing $\delta > 0$ small enough gives the result. □

3.2. **Measure theoretic preliminaries.** We will need a few measure theoretic lemmas to prove our main theorem.

Lemma **3.2.** *Suppose that $\mu_x = \rho_x + f d\mathcal{H}^n_x$ is a finite Borel measure invariant under G_t and $\rho_x \perp \mathcal{H}^n_x$. Then ρ_x and $f d\mathcal{H}^n_x$ are invariant under G_t.*

Proof. Since $\rho_x \perp \mathcal{H}^n_x$, there exist disjoint N, P such that $\rho_x(P) = \mathcal{H}^n_x(N) = 0$ and $\Lambda_x = N \cup P$. Suppose A is Borel. Then the invariance of μ_x implies

$$\int (1_A \circ G_{-t} - 1_A) d\rho_x = \int (1_A - 1_A \circ G_{-t}) f d\mathcal{H}^n_x. \tag{3.3}$$

Now, if $A \subset N$ then the fact that G_t is a diffeomorphism implies $\mathcal{H}^n_x(A) = \mathcal{H}^n_x(G_t(A)) = 0$. Therefore,

$$\rho_x(A) = \rho_x(G_t(A)), \qquad A \subset N \tag{3.4}$$

In particular,

$$\rho_x(N) = \rho_x(G_t(N)) = \rho_x(\Lambda_x).$$

Using again that for $t \in \mathbb{R}$, $G_t : \Sigma \to \Sigma$ is a diffeomorphism, we have

$$\rho_x(G_t(P)) = \rho_x(\Lambda_x \setminus G_t(N)) = \rho_x(\Lambda_x) - \rho_x(G_t(N)) = 0.$$

So, in particular,

$$\rho_x(G_t(A)) = 0, \qquad A \subset P. \tag{3.5}$$

Combining (3.4) with (3.5) proves that ρ_x is G_t invariant and hence (3.3) proves the lemma. $\square$

Let $B(\xi, r) \subset \Sigma_x$ be the geodesic ball (with respect tot he Sasaki metric) of radius r around ξ and define

$$T(\xi, r) := \bigcup_{t=-\infty}^{\infty} G_t(\{(x, \xi) \mid \xi \in B(\xi, r)\}). \tag{3.6}$$

Lemma **3.3.** *Suppose ρ_x is a finite measure invariant under G_t and $\rho_x \perp \mathcal{H}^n_x$. Then for all $\varepsilon > 0$, there exist $\xi_j \in \Sigma_x$ and $r_j > 0$, $j = 1, \dots$ so that*

$$\sum r_j^{n-1} < \varepsilon, \qquad \rho_x\left(\bigcup_j T(\xi_j, r_j)\right) = \rho_x(\Lambda_x). \tag{3.7}$$

Proof. Fix $\delta > 0$ so that

$$[-\delta, \delta] \times \Sigma_x \ni (t, \xi) \mapsto G_t(x, \xi) \in \Lambda_{x,\delta}$$

is a diffeomorphism and use (t, ξ) as coordinates on $\Lambda_{x,\delta}$.

We integrate ρ_x over $\Lambda_{x,\delta}$ to obtain a measure on Σ_x. In particular, for $A \subset \Sigma_x$ Borel, define the measure

$$\tilde{\rho}_x(A) := \frac{1}{\delta} \rho_x \left(\bigcup_{t=0}^{\delta} G_t(A) \right). \tag{3.8}$$

Then, the invariance of ρ_x implies that $\rho_x \ll dt \times d\tilde{\rho}_x$ and in particular, there exists $f \in L^1(dt \times d\tilde{\rho}_x)$ so that

$$\rho_x = f(t, \xi) dt \times d\tilde{\rho}_x(\xi).$$

Moreover, since ρ_x is invariant under G_t, $f(t, \xi) = f(\xi)$. Finally (3.8) implies f is a constant and in particular,

$$\rho_x = dt \times d\tilde{\rho}_x. \tag{3.9}$$

Now, notice that $\mathcal{H}^n_x = g(t, \xi) dt \times d\mathrm{Vol}_{\Sigma_x}$ where $0 < c < g \in C^\infty$. In particular, since

$$dt \times d\tilde{\rho}_x \perp dt \times d\mathrm{Vol}_{\Sigma_x}$$

we have that $\tilde{\rho}_x \perp d\mathrm{Vol}_{\Sigma_x}$.

Thus, there exists $N, P \subset \Sigma_x$ so that $\tilde{\rho}_x(P) = \mathrm{Vol}_{\Sigma_x}(N) = 0$ and $\Sigma_x = N \sqcup P$. Hence for any $\varepsilon > 0$, there exist $\xi_j \in \Sigma_x$ and $r_j > 0$ so that

$$\sum_j r_j^{n-1} < \varepsilon, \quad \tilde{\rho}_x\left(\bigcup_j B(\xi_j, r_j)\right) = \tilde{\rho}_x(\Sigma_x).$$

The lemma then follows from (3.9) and invariance of ρ_x. □

LEMMA **3.4.** *Suppose that $0 \leq f \in L^1(\Lambda_x, \mathcal{H}^n_x)$ with $f d\mathcal{H}^n_x$ invariant under G_t. Then for $\delta_0 > 0$ small enough, write*

$$[-\delta_0, \delta_0] \times \Sigma_x \ni (t, q) \mapsto G_t(q) \in \Lambda_x$$

for coordinates on Λ_{x,δ_0}. We have

$$f 1_{\Lambda_{x,\delta_0}} d\mathcal{H}^n_x = \tilde{f}(q) 1_{[-\delta_0,\delta_0]}(t) dt \times d\mathrm{Vol}_{\Sigma_x}$$

where

$$\tilde{f}(q) = f(0, q)|\nu(H_p)|(0, q)$$

and ν is a unit normal to $\Sigma_x \Subset \Lambda_{x,\delta_0}$ with respect to the Sasaki metric.

Proof. Observe that $1_{\Lambda_{x,\delta_0}} d\mathcal{H}^n_x$ is the volume measure on Λ_{x,δ_0}. Therefore, $1_{\Lambda_{x,\delta_0}} d\mathcal{H}^n_x \ll 1_{[-\delta_0,\delta_0]}(t) dt \times d\mathrm{Vol}_{\Sigma_x}$ and in particular,

$$f 1_{\Lambda_{x,\delta_0}} d\mathcal{H}^n_x = f(t, q) \frac{d(d\mathcal{H}^n_x)}{d(dt \times d\mathrm{Vol}_{\Sigma_x})}(t, q) 1_{[-\delta_0,\delta_0]}(t) dt \times d\mathrm{Vol}_{\Sigma_x}.$$

But, since $f d\mathcal{H}^n_x$ is invariant under G_t. That is, under translation in t,

$$f(t, q) \frac{d(d\mathcal{H}^n_x)}{d(dt \times d\mathrm{Vol}_{\Sigma_x})}(t, q) = \tilde{f}(q)$$

is constant in time.

To compute $\tilde{f}(q)$, we need only compute

$$\frac{d(d\mathcal{H}^n_x)}{d(dt \times d\mathrm{Vol}_{\Sigma_x})}(0, q).$$

For this, observe that $1_{\Lambda_{x,\delta}} \mathcal{H}^n_x$ is the volume measure on $\Lambda_{x,\delta}$ with respect to the Sasaki metric. Therefore, we have $d\mathrm{Vol}_{\Sigma_x} = N \lrcorner d\mathrm{Vol}_{\Lambda_{x,\delta_0}}$ where N is a unit normal to Σ_x. More precisely, if $r \in C^\infty(\Lambda_{x_0,\delta_0})$ has $dr|_{\Sigma_x}(V) = \langle N, V \rangle_{g_s}$ where g_s denotes the Sasaki metric and $V \in T_{\Sigma_x} \Lambda_{x,\delta_0}$, then $\nu = dr|_{\Sigma_x}$ is a unit conormal to Σ_x and

$$\frac{d(d\mathcal{H}^n_x)}{d(dt \times d\mathrm{Vol}_{\Sigma_x})}(0, q) = |\partial_t (r \circ G_t)|_{t=0}|(q) = |\nu(H_p)|(q).$$

□

4. L^∞ ESTIMATES MICROLOCALIZED TO Λ_x

For the next two sections, we assume that u is compactly microlocalized and $Pu = o_{L^2}(h)$ where P is as in Theorem 2.

LEMMA **4.1.** *Suppose that P is as in Theorem 2, u is compactly microlocalized, and $Pu = o_{L^2}(h)$. Then for $q, a \in S^\infty(T^*M)$*

$$\|a(x, hD) q(x, hD) u\|^2_{L^2} = \int |a|^2 |q|^2 d\mu + o(1),$$

$$\|a(x, hD) P q(x, hD) u\|^2_{L^2} = h^2 \int |a|^2 |H_p q|^2 d\mu + o(h^2).$$

Proof. First observe that since u is compactly microlocalized, there exists $\chi \in C_c^\infty(T^*M)$ so that

$$u = \chi(x, hD)u + O_{\mathcal{S}}(h^\infty).$$

Therefore, we may assume $q, a \in C_c^\infty(T^*M)$. The first equality then follows from the definition of the defect measure and the fact that $[a(x, hD)]^* = \bar{a}(x, hD) + O_{L^2\to L^2}(h)$. For the second, note that

$$\begin{aligned} Pq(x, hD)u &= q(x, hD)Pu + [P, q(x, hD)]u \\ &= q(x, hD)Pu + \frac{h}{i}\{p, q\}(x, hD)u + O_{L^2}(h^2). \end{aligned}$$

The lemma follows since $Pu = o_{L^2}(h)$. □

At this point, following the argument in Koch–Tataru–Zworski [KTZ07], we work h-microlocally. The first step is to reduce the $L^2 \to L^\infty$ bounds to a neighbourhood of $\Sigma = \{p = 0\}$.

Lemma **4.2.** *Suppose that u is compactly microlocalized and $Pu = o_{L^2}(h)$. Then for $\chi_\Sigma \in C_c^\infty(T^*M)$ with $\chi_\Sigma \equiv 1$ in a neighborhood of $\Sigma = \{p = 0\}$,*

$$\|(1 - \chi_\Sigma(x, hD))u\|_{L^\infty} = o(h^{\frac{2-n}{2}}). \tag{4.1}$$

Proof. Since u is compactly microlocalized, there exists $\chi \in C_c^\infty(T^*M)$ so that

$$u = \chi(x, hD)u + O_{\mathcal{S}}(h^\infty).$$

For $\chi_\Sigma \in C_c^\infty(T^*M)$ with $\chi_\Sigma \equiv 1$ in a neighborhood of Σ, $|p| \geq c > 0$ on $\operatorname{supp}(1 - \chi_\Sigma)\chi$. Therefore, by the elliptic parametrix construction, for any $q \in S^\infty(T^*M)$, there exists $e \in C_c^\infty(T^*M)$ so that

$$e(x, hD)P = (1 - \chi_\Sigma)(x, hD)q(x, hD)\chi(x, hD) + O_{\mathcal{D}'\to\mathcal{S}}(h^\infty)$$

and in particular,

$$(1 - \chi_\Sigma)(x, hD)q(x, hD)u = o_{L^2}(h). \tag{4.2}$$

Tthe compact microlocalization of u together with (4.2) and the Sobolev estimate [Zwo12, Lemma 7.10] implies

$$\|(1 - \chi_\Sigma(x, hD))u\|_{L^\infty} = o(h^{\frac{2-n}{2}}).$$

□

To simplify the writing somewhat, we introduce the notation $u_\Sigma := \chi_\Sigma(x, hD)u$.

4.1. Microlocal L^∞ bounds near Σ.

In view of (4.1), it suffices to consider points in an arbitrarily small tubular neighborhood of $\Sigma = \{p = 0\}$. More precisely, we cover $\operatorname{supp}\chi_\Sigma$ by a union $\cup_{j=0}^N B_j$ of open balls B_j centered at points $(x_j, \xi_j) \in \Sigma \subset \{p = 0\}$. We let $\chi_j \in C_0^\infty(B_j)$ be a corresponding partition of unity with

$$u_\Sigma = \sum_{j=0}^N \chi_j(x, hD)u_\Sigma + O_{\mathcal{S}}(h^\infty)$$

By possible refinement, the supports of χ_j can be chosen arbitrarily small.

Since the argument here is entirely local, it suffices to h-microlocalize to $\operatorname{supp}\chi_0 \subset B_0$ where B_0 has center $(x_0, \xi_0) \in \{p = 0\}$. Since we have assumed $\partial_\xi p \neq 0$ in $\{p = 0\}$, we may assume that $\partial_{\xi_1} p(x_0, \xi_0) \neq 0$ and $\partial_{\xi'} p(x_0, \xi_0) = 0$. Therefore, choosing $\operatorname{supp}\chi$ supported sufficiently close to (x_0, ξ_0), it follows from the implicit function theorem that

$$p\chi = e(x, \xi)(\xi_1 - a(x, \xi'))$$

with $e(x,\xi)$ elliptic on $\operatorname{supp}\chi_0$ provided the latter support is chosen small enough. Thus,

$$P\chi_0 = E(x,hD)(hD_{x_1} - a(x,hD_{x'}))\chi_0(x,hD) + hR\chi_0(x,hD).$$

Therefore,

$$(hD_{x_1} - a(x_1,x',hD_{x'}))\chi_0 q(x,hD)u = E^{-1}(x,hD)P\chi_0 q(x,hD)u + hR_1\chi_0(x,hD)q(x,hD)u.$$

In particular, from the standard energy estimate (see for example [KTZ07, Lemma 3.1]) with $(x_1,x')\in\mathbb{R}^n$,

$$\begin{aligned}\|\chi_0 q(x,hD)u_\Sigma(x_1=s,\cdot)\|_{L^2_{x'}(\mathbb{R}^{n-1})} &\leq \|\chi_0 q(x,hD)u_\Sigma(x_1=t,\cdot)\|_{L^2_{x'}(\mathbb{R}^{n-1})}\\ &+ Ch^{-1}|s-t|^{1/2}(\|P\chi_0 q(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)} + h\|R_1\chi_0 q(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)}).\end{aligned} \tag{4.3}$$

4.2. **Microlocalization to the flowout.** Our next goal will be to insert microlocal cutoffs restricting to a neighborhood of $\Lambda_{x_0,\delta}$ for some $\delta>0$ into the right hand side of (4.3).

Let $\varepsilon\ll\delta$, $\chi_{\varepsilon,x_0}\in C_c^\infty(M;[0,1]))$ with

$$\chi_{\varepsilon,x_0}\equiv 1 \text{ on } B(x_0,\varepsilon), \qquad \operatorname{supp}\chi_{\varepsilon,x_0}\subset B(x_0,2\varepsilon).$$

Let $b_{\varepsilon,x_0}\in C_c^\infty(T^*M;[0,1])$ with

$$\begin{gathered}\operatorname{supp} b_{\varepsilon,x_0}\cap\{p=0\}\subset\bigcup_{x\in B(x_0,3\varepsilon)}\Lambda_{x,3\delta}, \qquad \operatorname{supp} b_{\varepsilon,x_0}\subset\{|p|<2\varepsilon\},\\ b_{\varepsilon,x_0}\equiv 1 \text{ on } \bigcup_{t=-2\delta}^{2\delta} G_t\left\{(x,\xi)\mid |p(x,\xi)|<\varepsilon, d(x,x_0)<2\varepsilon\right\}.\end{gathered} \tag{4.4}$$

Lemma **4.3.** *There exists $C>0,\delta_0>0$ so that for all $\chi_j\in C_c^\infty(T^*M)$ supported sufficiently close to (x_0,ξ_0), $0<\varepsilon\ll\delta<\delta_0$, χ_{ε,x_0}, b_{ε,x_0} as above, $q\in S^\infty(T^*M)$, and $y_1\in\mathbb{R}$*

$$\begin{aligned}\|(q\chi_{\varepsilon,x_0}\chi_j)(x,hD)u_\Sigma|_{x_1=y_1}\|_{L^2_{x'}(\mathbb{R}^{n-1})} &\leq 2\delta_0^{-1/2}\|b_{\varepsilon,x_0}(x,hD)q(x,hD)\chi_j(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)}\\ &+ C\delta_0^{\frac12}h^{-1}\|b_{\varepsilon,x_0}(x,hD)Pq(x,hD)\chi_j(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)} + o_{\varepsilon,\delta}(1)\end{aligned} \tag{4.5}$$

where $\delta_0:=\delta|\partial_\xi p(x_0,\xi_0)|_g$ and $|\partial_\xi p|_g:=|\partial_\xi p\cdot\partial_x|_g$.

Remark 4.4: In (4.5), the local defining functions x_1 depend on j, but we will abuse notation somewhat and suppress the dependence on the index.

Proof. Let

$$A(x_1,y_1,x',hD_{x'}) := -\int_{y_1}^{x_1} a(s,x',hD_{x'})ds$$

and $w=\chi_0 q(x,hD)u_\Sigma$. Then

$$w(y_1,x') = e^{-\frac{i}{h}A(t,y_1,x',hD_{x'})}w|_{x_1=t} - \frac{i}{h}\int_{y_1}^t e^{-\frac{i}{h}A(s,y_1,x',hD_{x'})}f(s,x')ds$$

where

$$f(x) := E^{-1}(x,hD)P\chi_0 q(x,hD)u_\Sigma + hR_1\chi_0(x,hD)q(x,hD)u_\Sigma. \tag{4.6}$$

Let $\delta_0:=\delta|\partial_\xi p(x_0,\xi_0)|_g$ and $\psi\in C_c^\infty(\mathbb{R};[0,1])$ with $\operatorname{supp}\psi\subset[0,\delta_0]$ and $\int\psi=1$. Then, integrating in x_1,

$$w(y_1,x') = \int\psi(t)e^{-\frac{i}{h}A(t,y_1,x',hD_{x'})}w|_{x_1=t}dt - \frac{i}{h}\int\psi(t)\int_{y_1}^t e^{-\frac{i}{h}A(s,y_1,x',hD_{x'})}f(s,x')dsdt$$

Applying propagation of singularities,

$$\chi_{\varepsilon,x_0}w(y_1,x') = \int \psi(t)\chi_{\varepsilon,x_0}e^{-\frac{i}{h}A(t,y_1,x',hD_{x'})}(b_{\varepsilon,x_0}(x,hD)w)|_{x_1=t}dt \\ - \frac{i}{h}\chi_{\varepsilon,x_0}\int \psi(t)\int_{y_1}^{t} e^{-\frac{i}{h}A(s,y_1,x',hD_{x'})}(b_{\varepsilon,x_0}(x,hD)f)(s,x')dsdt + o_{\varepsilon,\delta}(1)_{L^\infty_{y_1}L^2_{x'}} \tag{4.7}$$

More precisely, for $q_1 \in S^0(T^*M)$, $s \in [0,\delta_0]$, we show that

$$\chi_{\varepsilon,x_0}(y_1,x')e^{-\frac{i}{h}A(s,y_1,x',hD_{x'})}(1-b_{\varepsilon,x_0}(x,hD))q_1(x,hD)u = o_\varepsilon(h)_{L^2_x}.$$

Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \equiv 1$ on $[-1,1]$. By (4.2)

$$\chi_{\varepsilon,x_0}(y_1,x')e^{-\frac{i}{h}A(x_1,x',hD_{x'})}(1-b_{\varepsilon,x_0}(x,hD))q_1(x,hD)(1-\varphi(\varepsilon^{-2}p(x,hD)))u = o_\varepsilon(h)_{L^2_x}.$$

Therefore, we need only estimate

$$\chi_{\varepsilon,x_0}(y_1,x')e^{-\frac{i}{h}A(s,y_1,x',hD_{x'})}(1-b_{\varepsilon,x_0}(x,hD))q_1(x,hD)\varphi(\varepsilon^{-2}p(x,hD))u. \tag{4.8}$$

Let $\tilde{G}_t$ denote the Hamiltonian flow of $\xi_1 - a(x,\xi')$. Then, for $(x,\xi) \in \{|p| \le C\varepsilon^2\}$ and $|t| \le 1$, $d(G_t(x,\xi),\tilde{G}_t(x,\xi)) \le C\varepsilon^2$. By (4.4), b_{ε,x_0} is identically 1 in an ε neighborhood of

$$\bigcup_{x\in \operatorname{supp}\chi_{\varepsilon,x_0}} \Lambda_{x_0,2\delta}$$

and thus for $\varepsilon > 0$ small enough on

$$\bigcup_{t=-2\delta}^{2\delta} \tilde{G}_t\left\{(x,\xi) \mid x \in \operatorname{supp}\chi_{x_0,\varepsilon},\ |p| \le C\varepsilon^2\right\}.$$

In particular, since we assume that $\partial_{\xi'}p(x_0,\xi_0) = 0$, and $\operatorname{supp}\psi \subset [0,\delta_0]$,

$$\psi(s)\chi_{\varepsilon,x_0}(y_1,x')e^{-\frac{i}{h}A(s,y_1,x',hD_{x'})}(1-b_{\varepsilon,x_0}(x,hD))a(x,hD)\varphi(\varepsilon^{-2}p(x,hD))u = O_\varepsilon(h^\infty)_{L^2_x}. \tag{4.9}$$

Together (4.8) and (4.9) give (4.7) which implies

$$\|\chi_{\varepsilon,x_0}w(y_1,\cdot)\|_{L^2_{x'}(\mathbb{R}^{n-1})} \le \delta_0^{-1/2}\|b_{\varepsilon,x_0}(x,hD)w\|_{L^2_x(\mathbb{R}^n)} + C\delta_0^{\frac{1}{2}}h^{-1}\|b_{\varepsilon,x_0}(x,hD)f\|_{L^2_x(\mathbb{R}^n)} + o_{\varepsilon,\delta}(1).$$

Now,

$$q(x,hD)\chi_{\varepsilon,x_0}\chi_0(x,hD)u_\Sigma = \chi_{\varepsilon,x_0}\chi_0(x,hD)q(x,hD)u_\Sigma + [q(x,hD),\chi_{\varepsilon,x_0}\chi_0(x,hD)]u_\Sigma.$$

Therefore, since

$$\|[q(x,hD),\chi_{\varepsilon,x_0}\chi_0(x,hD)]u_\Sigma(x_1,\cdot)\|_{L^2_{x'}(\mathbb{R}^{n-1})} = O_\varepsilon(h^{1/2}),$$

we have the following L^2 bound along the section $x_1 = y_1$ of $\operatorname{supp}\chi_0 \subset \operatorname{supp}\chi_\Sigma$.

$$\|q(x,hD)\chi_{\varepsilon,x_0}\chi_0(x,hD)u_\Sigma(y_1,\cdot)\|_{L^2_{x'}(\mathbb{R}^{n-1})} \le \\ \delta_0^{-1/2}\|b_{\varepsilon,x_0}(x,hD)w\|_{L^2_x(\mathbb{R}^n)} + C\delta_0^{1/2}h^{-1}\|b_{\varepsilon,x_0}(x,hD)f\|_{L^2_x(\mathbb{R}^n)} + o_{\varepsilon,\delta}(1). \tag{4.10}$$

Since the proof of (4.10) is local, by refining the supports of $\chi_j; j = 1,...,N$ if necessary and using the definition of f, (4.6), (4.5) follows for all $j = 1,...,N$, $x_0 \in M$, χ_{ε,x_0} supported in an ε neighborhood of x_0. □

Lemma **4.5.** *Suppose that for some $\delta > 0$, $q \in S^0(T^*M)$ has $q \equiv 0$ on $\Lambda_{x_0,3\delta}$. Then for $r(h) = o(1)$.*

$$\limsup_{h\to 0} h^{\frac{n-1}{2}}\|q(x,hD)u_\Sigma\|_{L^\infty(B(x_0,r(h)))} = 0.$$

Proof. Observe that Lemma 4.3 gives for each $j = 1, \dots N$,

$$\|(q\chi_{\varepsilon,x_0}\chi_j)(x,hD)u_\Sigma|_{x_1=y_1}\|_{L^2_{x'}(\mathbb{R}^{n-1})} \leq 2\delta_0^{-1/2}\|b_{\varepsilon,x_0}(x,hD)q(x,hD)\chi_j(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)} + C\delta_0^{\frac{1}{2}}h^{-1}\|b_{\varepsilon,x_0}(x,hD)Pq(x,hD)\chi_j(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)} + o_{\varepsilon,\delta}(1).$$

Applying the Sobolev estimate [Zwo12, Lemma 7.10] and Lemma 4.1 gives

$$\limsup_{h\to 0} h^{n-1}\|(q\chi_j)(x,hD)u_\Sigma\|^2_{L^\infty(B(x_0,r(h)))} \leq 2\delta_0^{-1}\int b^2_{\varepsilon,x_0}(x,hD)q^2(x,hD)\chi_j^2 d\mu + C\delta_0\int b^2_{\varepsilon,x_0}(x,hD)|H_p(q(x,hD)\chi_j)|^2 d\mu.$$

Sending $\varepsilon \to 0$ and using the dominated convergence thoerem proves the lemma since $\mu(T^*M) = 1 < \infty$, $\lim_{\varepsilon\to 0} b^2_{\varepsilon,x_0} \leq 1_{\Lambda_{x_0,3\delta}}$, and q vanishes identically on $\Lambda_{x,3\delta}$. □

5. Decomposition into wave packets

We now choose a convenient partition χ_j and functions $q_{j,i}$, $i = 2, \dots n$ to prove the main theorem. The χ_j localize to individual bicharacteristics, and $\sum_i q_{j,i}$ will measure concentration in neighborhoods of each bicharacteristic. We then show that understanding the mass localization to finer and finer neighborhoods of geodesics yields the structure of the defect measure.

5.1. L^∞ contributions near geodesics.

We need the following version of the L^∞ Sobolev embedding.

Lemma **5.1.** *Suppose $v \in H^l(\mathbb{R}^{n-1})$ with $l > (n-1)/2$. Then for all $\varepsilon > 0$*

$$\|v\|^2_{L^\infty} \leq C_{n,l}h^{-n+1}\Big(\varepsilon^{n-1}\|v\|^2_{L^2} + \varepsilon^{n-2l-1}\sum_{i=1}^{n-1}\|(hD_{x_i})^l v\|^2_{L^2}\Big).$$

In particular this holds if v is compactly microlocalized.

Proof. Let $\zeta \in C_c^\infty([-2,2])$ with $\zeta \equiv 1$ on $[-1,1]$ and $\zeta_\varepsilon(x) = \zeta(\varepsilon^{-1}x)$.

Then

$$v(x) = (2\pi h)^{-n+1}\int e^{i\langle x,\xi\rangle/h}[\zeta_\varepsilon(|\xi|) + (1-\zeta_\varepsilon(|\xi|))]\mathcal{F}_h(v)(\xi)d\xi$$

Applying the triangle inequality and Cauchy–Schwarz, and letting $w_l(\xi) = \sqrt{\sum_{i=1}^{n-1}\xi_i^{2l}}$

$$\|v\|^2_{L^\infty} \leq h^{-2(n-1)}(\varepsilon^{n-1}\|\zeta\|^2_{L^2}\|\mathcal{F}_h v\|^2_{L^2} + \|(1-\zeta_\varepsilon)w_l^{-1}\|^2_{L^2}\|w_l\mathcal{F}_h v\|^2_{L^2}) \tag{5.1}$$

Now,

$$\|(1-\zeta_\varepsilon)w_l^{-1}\|^2_{L^2} = \varepsilon^{n-2l-1}\|(1-\zeta)w_l^{-1}\|^2_{L^2}$$

$$\|w_l\mathcal{F}_h v\|^2_{L^2} = \int\sum_{i=1}^{n-1}\xi_i^{2l}|\mathcal{F}_h v(\xi)|^2 d\xi = \sum_{i=1}^{n-1}\|\mathcal{F}_h(hD^l_{x_i}v)\|^2_{L^2}.$$

Using this in (5.1) proves the Lemma. □

Lemma **5.2.** *There exists $C_n > 0$ depending only on n, $\delta_1 > 0$ so that for $0 < \delta < \delta_1$ there exists $r_0 > 0$ so that if $(x_0,\xi) \in \Sigma_{x_0}$, $0 < r < r_0$ and $\chi_j \in C_c^\infty(T^*M)$ with*

$$\operatorname{supp}\chi_j \cap \Lambda_x \subset T(\xi,r), \qquad H_p\chi_j \equiv 0, \text{ on } \Lambda_{x_0,3\delta}$$

where $T(\xi,r)$ is as in (3.6). Then

$$\limsup_{h\to 0} h^{n-1}\|\chi_j u_\Sigma\|^2_{L^\infty(B(x_0,r(h)))}\leq C_n\delta^{-1}|\partial_\xi p(x,\xi)|_g^{-1}\int_{\Lambda_{x_0,3\delta}}\chi_j^2 r^{n-1}d\mu. \tag{5.2}$$

Proof. Let $a_{j,i}(x_1)$, $i=2,\dots n$ so that $\xi_i-a_{j,i}(x_1)$ vanishes on the bicharacteristic emanating from (x,ξ_j). This is possible since we have chosen coordinates so that $\partial_{\xi_1}p(x_0,\xi_j)\neq 0$ and hence a bicharacteristic may be written locally as

$$\gamma=\{(x,\xi)\mid x_1\in(-3\delta,3\delta),\ x'=x'(x_1),\ \xi=a(x_1)\}.$$

Let $2l>n-1$ and $q_{j,i}=(\xi_i-a_i(x_1))^l$. Then, using $q=q_{j,i}$ in (4.5) gives

$$\begin{aligned}\|(hD_{x_i}-a_i(x_1))^l\chi_{\varepsilon,x_0}\chi_j(x,hD)u_\Sigma(x_1,\cdot)\|_{L^2_{x'}(\mathbb{R}^{n-1})}&\leq 2\delta_0^{-1/2}\|b_{\varepsilon,x_0}(x,hD)q_{j,i}(x,hD)\chi_j(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)}\\&+C\delta_0^{1/2}h^{-1}\|b_{\varepsilon,x_0}(x,hD)Pq_{j,i}(x,hD)\chi_j(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)}+o_{\varepsilon,\delta}(1)\end{aligned}$$

where $|\partial_\xi p|_g=|\partial_\xi p\cdot\partial_x|_g$. Next, $q=1$ in (4.5) gives

$$\|\chi_{\varepsilon,x_0}\chi_j u_\Sigma\|_{L^2_{x'}}\leq 2\delta_0^{-1/2}\|b_{\varepsilon,x_0}(x,hD)\chi_j(x,hD)u_\Sigma\|_{L^2_x(\mathbb{R}^n)}+C\delta_0^{1/2}h^{-1}\|b_{\varepsilon,x_0}(x,hD)P\chi_j u_\Sigma\|_{L^2_x(\mathbb{R}^n)}+o_{\varepsilon,\delta}(1).$$

Therefore, letting $w=e^{-i\langle x',a_j(x_1)\rangle/h}\chi_{\varepsilon,x_0}\chi_j u$ with $a_j(x_1)=(a_{j,2}(x_1),\dots,a_{j,n}(x_1))$ we see that

$$\|(hD_{x_i})^l w\|_{L^2_{x'}}\leq 2\delta_0^{-1/2}\|b_{\varepsilon,x_0}q_{j,i}\chi_j u_\Sigma\|_{L^2_x(\mathbb{R}^n)}+C\delta_0^{1/2}h^{-1}\|b_{\varepsilon,x_0}Pq_{j,i}\chi_j u_\Sigma\|_{L^2_x(\mathbb{R}^n)}+o_{\varepsilon,\delta}(1)$$

and

$$\|w\|_{L^2_{x'}}\leq 2\delta_0^{-1/2}\|b_{\varepsilon,x_0}\chi_j u_\Sigma\|_{L^2_x(\mathbb{R}^n)}+C\delta_0^{1/2}h^{-1}\|b_{\varepsilon,x_0}P\chi_j u_\Sigma\|_{L^2_x(\mathbb{R}^n)}+o_{\varepsilon,\delta}(1).$$

Applying Lemma 5.1 to w (with $\varepsilon=\alpha$) and using the fact that $\|w\|_{L^\infty}=\|\chi_{\varepsilon,x_0}\chi_j u_\Sigma\|_{L^\infty}$ gives for any $\alpha>0$ and $r(h)=o(1)$

$$\begin{aligned}\limsup_{h\to 0}h^{n-1}\|\chi_j u_\Sigma\|^2_{L^\infty(B(x_0,r(h))}&\leq C_{n,l}\alpha^{n-1}\left(\limsup_{h\to 0}\left[\delta_0^{-1}\|b_{\varepsilon,x_0}\chi_j u_\Sigma\|^2_{L^2_x}+C\delta_0h^{-2}\|b_{\varepsilon,x_0}P\chi_j u_\Sigma\|^2_{L^2_x}\right]\right)\\&+C_{n,l}\alpha^{n-2l-1}\left(\sum_{i=2}^n\limsup_{h\to 0}\left[\delta_0^{-1}\|b_{\varepsilon,x_0}q_{j,i}\chi_j u_\Sigma\|^2_{L^2_x}+C\delta_0h^{-2}\|b_{\varepsilon,x_0}Pq_{j,i}\chi_j u_\Sigma\|^2_{L^2_x}\right]\right)\end{aligned}$$

In particular, applying Lemma 4.1,

$$\begin{aligned}\limsup_{h\to 0}h^{n-1}\|\chi_j u_\Sigma\|^2_{L^\infty(B(x_0,r(h)))}&\leq C_{n,l}\alpha^{n-1}\int b^2_{\varepsilon,x_0}(\delta_0^{-1}\chi_j^2+C\delta_0|H_p\chi_j|^2)d\mu\\&+C_{n,l}\alpha^{n-2l-1}\sum_{i=2}^n\int b^2_{\varepsilon,x_0}(\delta_0^{-1}\chi_j^2q_{j,i}^2+C\delta_0|H_p\chi_jq_{i,j}|^2)d\mu.\end{aligned}$$

Observe that by (4.4), $0\leq b^2_{\varepsilon,x_0}\leq 1$ and

$$\lim_{\varepsilon\to 0}b^2_{\varepsilon,x_0}\leq 1_{\Lambda_{x_0,3\delta}}.$$

Sending $\varepsilon\to 0$ and using $H_p\chi_j=0$ on $\Lambda_{x_0,3\delta}$ (together with $\mu(T^*M)=1$ to apply the dominated convergence theorem) we have

$$\begin{aligned}\limsup_{h\to 0}h^{n-1}\|\chi_j u_\Sigma\|^2_{L^\infty(B(x_0,r(h)))}&\leq C_{n,l}\delta_0^{-1}\alpha^{n-1}\int_{\Lambda_{x_0,3\delta}}\chi_j^2d\mu\\&+C_{n,l}\alpha^{n-2l-1}\sum_{i=2}^n\int_{\Lambda_{x_0,3\delta}}\chi_j^2(\delta_0^{-1}q_{j,i}^2+C\delta_0|H_pq_{i,j}|^2)d\mu\end{aligned}\tag{5.3}$$

Now, χ_j is supported on $T(\xi,r)$ (see (3.6)). Letting γ be the bicharacteristic through (x,ξ), we have by (3.1)

$$\sup\{d((x,\xi_1),\gamma)\mid (x,\xi_1)\in T(\xi,r)\cap\Lambda_{x_0,3\delta}\}\leq 3r.$$

Hence,

$$\sup_{T(\xi,r)\cap\Lambda_{x_0,3\delta}}|H_pq_{j,i}|\leq Cr^l.$$

Furthermore, by (3.2)

$$\sup_{T(\xi,r)\cap\Lambda_{x_0,3\delta}}|q_{j,i}|\leq r^l(1+C\delta)^l+O(r^{2l})$$

Thus, choosing δ small enough we obtain from (5.3) that

$$\limsup_{h\to 0}h^{n-1}\|\chi_j u_\Sigma\|^2_{L^\infty(B(x_0,r(h)))}\leq C_{n,l}\delta_0^{-1}\int_{\Lambda_{x_0,3\delta}}\chi_j^2(\alpha^{n-1}+\alpha^{n-2l-1}r^{2l})d\mu.$$

Optimizing in α and fixing $l=n$ gives (5.2). □

We now find an appropriate cover of Λ_{x_0} that is adapted to μ_x.

5.2. **Decomposition of Λ_{x_0}.** We start by constructing a convenient partition of unity to which Lemma 5.2 applies.

Lemma **5.3.** *Fix $(x_0,\xi_j)\in\Sigma_{x_0}$ and $r_j>0$, $j=1,\dots K<\infty$, $\delta>0$. Then there exist $\chi_j\in C_c^\infty(T^*M;[0,1])$, $j=1\dots K$ so that*

$$\operatorname{supp}\chi_j\cap\Lambda_x\subset T(\xi_j,2r_j)\cap\Lambda_{x,4\delta},\qquad H_p\chi_j\equiv 0 \text{ on } \Lambda_{x_0,3\delta}$$

$$\sum_j\chi_j\equiv 1 \text{ on } \bigcup_{j=1}^K T(\xi_j,r_j)\cap\Lambda_{x_0,3\delta},\qquad 0\leq\sum_j\chi_j\leq 1, \text{ on } \Lambda_x \tag{5.4}$$

Furthermore, if

$$\bigcup_{j=1}^K T(\xi_j,2r_j)\supset\Lambda_{x_0,3\delta}, \tag{5.5}$$

there exists χ_j with (5.4) and

$$\sum_j\chi_j\equiv 1 \text{ on } \Lambda_{x_0,3\delta}. \tag{5.6}$$

Proof. Let $\tilde\chi_j\in C_c^\infty(\Sigma_{x_0};[0,1])$ have

$$\sum_j\tilde\chi_j\equiv 1 \text{ on } \bigcup_{j=1}^K B(\xi_j,r_j),\qquad \operatorname{supp}\tilde\chi_j\subset B(\xi_j,2r_j)\cap\Sigma_{x_0},\qquad 0\leq\sum_j\tilde\chi_j\leq 1.$$

Next, let $\psi\in C_c^\infty(\mathbb{R};[0,1])$ with $\psi\equiv 1$ on $[-3\delta,3\delta]$ and $\operatorname{supp}\psi\subset(-4\delta,4\delta)$. For $\delta>0$ small enough, $G_t:[-4\delta,4\delta]\times\Sigma_{x_0}\to\Lambda_{x_0,4\delta}$ is a diffeomorphism and so we can define $\chi_j\in C_c^\infty(\Lambda_{x_0,4\delta};[0,1])$ by

$$\chi_j(G_t(x,\xi))=\psi(t)\tilde\chi_j(x,\xi)$$

so that $H_p\chi_j\equiv 0$ on $\Lambda_{x_0,3\delta}$. Finally, extend χ_j from $\Lambda_{x_0,4\delta}$ to a compactly supported function on T^*M arbitrarily. Then χ_j $j=1,\dots K$ satisfy (5.4).

If (5.5) holds, then we may take $\tilde\chi_j$ a partition of unity on Σ_{x_0} subordinate to $B(\xi_j,2r_j)$ and hence obtain (5.6) by the same construction. □

Proof of Theorem 2. Recall that

$$\mu_{x_0} = \rho_{x_0} + f d\mathcal{H}^n_{x_0}$$

where $\rho_{x_0} \perp \mathcal{H}^n_{x_0}$ and μ_{x_0} is invariant under G_t. Therefore, by Lemma 3.2, ρ_{x_0} and $f d\mathcal{H}^n_{x_0}$ are invariant under G_t.

Fix $0 < \varepsilon \ll \delta$ arbitrary. By Lemma 3.3, there exist $((x_0, \xi_j), r_j) \in \Sigma_{x_0} \times \mathbb{R}_+$ satisfying (3.7). Let K be large enough so that

$$\rho_{x_0}\left(\Lambda_{x_0} \setminus \bigcup_{j=1}^{K} T(\xi_j, r_j)\right) < \varepsilon. \tag{5.7}$$

Let $\chi_j \in C_c^\infty(T^*M; [0,1])$ satisfy (5.4) for $((x_0, \xi_j), r_j)$ $j = 1, \dots K$.

Define $\psi = 1 - \sum \chi_j$. Applying Lemma 5.2 (with $\xi = \xi_j$, $r = r_j$, $\chi = \chi_j$), summing and using the triangle inequality, we have

$$\begin{aligned}\limsup_{h\to 0} h^{\frac{n-1}{2}} \|(1 - \psi(x, hD))u_\Sigma\|_{L^\infty(B(x_0, r(h)))} &\leq C_{n,\delta} \sum_{j=1}^{K} r_j^{(n-1)/2} \left(\int_{\Lambda_{x_0}} \chi_j^2 d\mu\right)^{1/2} \\ &\leq C_{n,\delta} \left(\sum_j r_j^{n-1}\right)^{1/2} \left(\int_{\Lambda_{x_0}} \sum_j \chi_j^2 d\mu\right)^{1/2} \\ &\leq C_{n,\delta} \varepsilon^{1/2} \mu(\Lambda_{x_0})\end{aligned} \tag{5.8}$$

where in the last line we use $0 \leq \chi_j \leq 1$ and $0 \leq \sum \chi_j \leq 1$.

Next we estimate $\psi(x, hD)u_\Sigma$. By the Besicovitch–Federer Covering Lemma [Hei01, Theorem 1.14, Example (c)], there exists a constant C_n depending only on n and $\gamma_0 = \gamma_0(\Sigma_{x_0})$ so that for all $0 < \gamma < \gamma_0$, there exists $\xi_1, \dots \xi_{N(\gamma)}$ with $N(\gamma) \leq C\gamma^{1-n}$ so that

$$\Sigma_{x_0} \subset \bigcup_{j=1}^{N(\gamma)} B(\xi_k, \gamma)$$

and each point in Σ_{x_0} lies in at most C_n balls $B(\xi_k, \gamma)$. Let ψ_k, $k = 1, \dots N(\gamma)$ satisfy (5.4), (5.6) (with $\xi_j = \xi_k$, $2r_j = \gamma$, and $K = N(\gamma)$). Observe that applying Lemma 5.2 (with $\xi = \xi_k$, $r = \gamma$, and $\chi_j = \psi_k$),

$$\limsup_{h\to 0} h^{n-1} \|\psi(x, hD)\psi_k(x, hD)u_\Sigma\|^2_{L^\infty(B(x_0, r(h)))} \leq C_n \delta^{-1} |\partial_\xi p(x_0, \xi_k)|_g^{-1} \int_{\Lambda_{x_0, 3\delta}} \psi_k^2 \psi^2 \gamma^{n-1} d\mu$$

Notice that

$$\sum_k \psi\psi_k \equiv 1 \text{ on } \Lambda_{x_0, 3\delta}$$

and therefore Lemma 4.5 implies

$$\limsup_{h\to 0} h^{\frac{n-1}{2}} \left\| \psi(x, hD)\Big[1 - \sum_k \psi_k(x, hD)\Big] u_\Sigma \right\|_{L^\infty(B(x_0, r(h)))} = 0.$$

So, applying the triangle inequality,

$$\begin{aligned}&\limsup_{h\to 0} h^{\frac{n-1}{2}}\left\|\psi(x,hD)u_\Sigma\right\|_{L^\infty(B(x_0,r(h)))}\\ &\quad\le C_{n,\delta}\sum_k\left(\int_{\Lambda_{x_0,3\delta}}\psi_k^2\psi^2\gamma^{n-1}d\rho_{x_0}\right)^{1/2}+C_n\delta^{-1/2}\sum_k\left(\int_{\Lambda_{x_0,3\delta}}|\partial_\xi p(x_0,\xi_k)|_g^{-1}\psi_k^2\psi^2\gamma^{n-1}fd\mathcal{H}^n_{x_0}\right)^{1/2}\\ &\quad=: C_{n,\delta}I+II\end{aligned}$$

Use (5.7) to estimate

$$I\le C\gamma^{\frac{n-1}{2}}N(\gamma)^{1/2}\left(\int_{\Lambda_{x_0,3\delta}}\sum_k\psi_k^2\psi^2d\rho_{x_0}\right)^{1/2}\le C\rho_{x_0}\left(\Lambda_{x_0}\setminus\bigcup_{j=1}^K T(\xi_j,r_j)\right)^{1/2}\le C\varepsilon^{1/2}.$$

Since for γ small enough, $C_n^{-1}\gamma^{n-1}\le \mathrm{Vol}_{\Sigma_x}(B(\xi_k,\gamma))\le C_n\gamma^{n-1}$, where C_n depends only on n,

$$\begin{aligned}II&\le C_n\delta^{-1/2}\int_{\Sigma_{x_0}}\sum_k 1_{B(\xi_k,\gamma)}\left(\frac{1}{|\partial_\xi p(x_0,\xi_k)|_g\mathrm{Vol}_{\Sigma_{x_0}}(B(\xi_k,\gamma))}\int_{T(\xi_k,\gamma)\cap\Lambda_{x_0,3\delta}}fd\mathcal{H}^n_{x_0}\right)^{1/2}d\mathrm{Vol}_{\Sigma_{x_0}}\\ &\le C_n\int_{\Sigma_{x_0}}\sum_k 1_{B(\xi_k,\gamma)}\left(\frac{1}{|\partial_\xi p(x_0,\xi_k)|_g\mathrm{Vol}_{\Sigma_{x_0}}(B(\xi_k,\gamma))}\int_{B(\xi_k,\gamma)}f(0,q)|\nu(H_p)|(0,q)d\mathrm{Vol}_{\Sigma_{x_0}}\right)^{1/2}d\mathrm{Vol}_{\Sigma_{x_0}}\end{aligned}$$

where in the last line we use that $fd\mathcal{H}^n_x$ is G_t invariant and apply Lemma 3.4. The Lebesgue differentiation theorem [Fol99, Theorem 3.21] then shows that

$$\begin{aligned}&\limsup_{\gamma\to 0} C_n\sum_k 1_{B(\xi_k,\gamma)}\left(\frac{1}{|\partial_\xi p(x_0,\xi_k)|_g\mathrm{Vol}_{\Sigma_{x_0}}(B(\xi_k,\gamma))}\int_{B(\xi_k,\gamma)}f(0,q)|\nu(H_p)|(0,q)d\mathrm{Vol}_{\Sigma_{x_0}}\right)^{1/2}\\ &\qquad\qquad\le C_n\sqrt{\frac{f|\nu(H_p)|}{|\partial_\xi p|_g}}\quad\text{a.e.}\end{aligned}$$

Furthermore, the weak type 1-1 boundedness of the Hardy–Littlewood maximal function [Fol99, Theorem 3.17] implies

$$\mathrm{Vol}_{\Sigma_{x_0}}\left(\xi\in\Sigma_{x_0}\mid\left\{\sup_{\gamma>0}\left(\frac{1}{\mathrm{Vol}_{\Sigma_{x_0}}(B(\xi_k,\gamma))}\int_{B(\xi_k,\gamma)}f|\nu(H_p)|d\mathrm{Vol}_{\Sigma_{x_0}}\right)^{1/2}\ge\alpha\right\}\right)\le C\alpha^{-2}$$

and hence by the dominated convergence theorem,

$$\lim_{\gamma\to 0}\limsup_{h\to 0} h^{\frac{n-1}{2}}\|\psi(x,hD)u_\Sigma\|_{L^\infty(B(x_0,r(h)))}\le C_n\int_{\Sigma_{x_0}}\sqrt{\frac{f|\nu(H_p)|}{|\partial_\xi p|_g}}d\mathrm{Vol}_{\Sigma_{x_0}}+C\varepsilon^{1/2}. \tag{5.9}$$

Sending $h\to 0$, then $\varepsilon\to 0$, then $\gamma\to 0$ and using (5.8), (5.9) then proves the theorem. □

6. Construction of Modes - Proof of Theorem 3

Proof of Theorem 3. We apply the construction in [STZ11, Lemma 7]. Let $p=\frac{1}{2}(|\xi|_g^2-1)$ and $G_t=\exp(tH_p)$ so that $G_t|_{S^*M}$ is the unit speed geodesic flow. Let $g_1\in L^2(S^*_zM)$ have $|g_1|^2=f|_{S^*_zM}$ and $g_{1,\varepsilon}\in C^\infty(S^*_zM)$ have $\|g_{1,\varepsilon}-g_1\|_{L^2(S^*_zM)}<\varepsilon$. For $A\subset S^*_zM$ Borel, define the measure

$$\tilde\rho(A)=\frac{1}{2\delta}\rho\left(\bigcup_{t=-\delta}^{\delta}G_t(A)\right).$$

Let $g_{2,\varepsilon} \in C^\infty(S^*_zM)$ have $|g_{2,\varepsilon}|^2 dS_\phi \to \tilde{\rho}$ as a measure where S_ϕ is the surface measure on S^{n-1}. Finally, define $g_\varepsilon = g_{1,\varepsilon} + g_{2,\varepsilon}$.

Then, letting $\mathbb{T}_T = \mathbb{R}/T\mathbb{Z}$ and parametrizing Λ_z by $\mathbb{T}_T \times S^{n-1} \ni (t,\omega) \to G_t(z,\omega)$ there exists $\Phi_{\varepsilon,j} \in \mathcal{O}^{\frac{n-1}{2}}(M, \Lambda_z, \{h_j\})$ with symbol

$$\sigma(\Phi_{\varepsilon,j})(t,\omega) = g_\varepsilon(\omega)(2\pi)^{\frac{n-1}{2}} h_j^{\frac{1-n}{2}} |d\mu_\omega dt|^{1/2}$$

and having

$$\|(-h_j^2\Delta_g - 1)\Phi_{\varepsilon,j}\|_{L^2} = O_\varepsilon(h_j^2), \qquad C + O_\varepsilon(h_j) \geq \|\Phi_{\varepsilon,j}\|_{L^2} \geq c + O_\varepsilon(h_j).$$

Moreover, using normal geodesic coordinates at z, we have in a neighborhood thereof,

$$\Phi_{\varepsilon,j}(x) = (2\pi h_j)^{\frac{1-n}{2}} \int e^{i\left\langle x, \frac{\theta}{|\theta|}\right\rangle/h_j} g_\varepsilon\left(\frac{\theta}{|\theta|}\right)\chi_R(|\theta|)d\theta,$$

where $\chi_R \in C_c^\infty((0,\infty);[0,1])$ with $\chi_R \equiv 1$ on $[1,R]$, $\operatorname{supp}\chi_R \subset (0,2R)$ and

$$\int \chi_R(\alpha)\alpha^{n-1}d\alpha = 1. \tag{6.1}$$

Choose $\varepsilon_j \to 0$ so slowly that

$$\lim_{j\to\infty} \|(-h_j^2\Delta_g - 1^2)\Phi_{\varepsilon_j,j}\|_{L^2} h_j^{-1} \to 0, \qquad 2C \geq \limsup_{j\to\infty}\|\Phi_{\varepsilon_j,j}\|_{L^2} \geq \liminf_{j\to\infty}\|\Phi_{\varepsilon_j,j}\|_{L^2} > c/2.$$

Then,

$$\|(-h_j^2\Delta_g - 1)\Phi_{\varepsilon_j,j}\|_{L^2} = o(h_j\|\Phi_{\varepsilon_j,j}\|_{L^2}).$$

Fix $N > 0$ to be chosen large and $\varepsilon_j \to 0$ slowly enough so that

$$\sup_{|\alpha|\leq N}\sup_{S^*_xM} |\partial^{|\alpha|} g_{\varepsilon_j}| h_j \to 0. \tag{6.2}$$

Under this condition, we compute the defect measure of $\Phi_{\varepsilon_j,j}$. Let $b \in C_c^\infty(T^*M)$ supported in

$$A_\delta := \{x \mid \delta \leq |r(z,x)| \leq 2\delta\}.$$

Then, letting $\psi \in C_c^\infty(\mathbb{R}\setminus\{0\})$ have $\psi \equiv 1$ on $[\delta, 2\delta]$,

$$b(x,h_jD)\Phi_{\varepsilon_j,j} = (2\pi h_j)^{\frac{1-3n}{2}} \int e^{i\left(\langle x-y,\xi\rangle + \left\langle y, \frac{\theta}{|\theta|}\right\rangle\right)/h_j} b(x,\xi)\psi(|y|) g_{\varepsilon_j}\left(\frac{\theta}{|\theta|}\right)\chi_R(|\theta|)d\theta dy d\xi + O_{L^2}(h_j^\infty).$$

Performing stationary phase in the (y,ξ) variables gives

$$b(x,h_jD)\Phi_{\varepsilon_j,j} = (2\pi h_j)^{\frac{1-n}{2}} \int e^{i\left\langle x, \frac{\theta}{|\theta|}\right\rangle/h_j}\left[b\left(x, \frac{\theta}{|\theta|}\right) + h_j e(x,\theta)\right] g_{\varepsilon_j}\left(\frac{\theta}{|\theta|}\right)\chi_R(|\theta|)d\theta + O_{L^2}(h_j^\infty)$$

where $e \in C^\infty(\mathbb{R}^{2n})$ has $\operatorname{supp} r \subset \operatorname{supp} b$ and is independent of ε.

$$\langle b(x,h_jD)\Phi_{\varepsilon_j,j}, \Phi_{\varepsilon_j,j}\rangle =$$

$$(2\pi h_j)^{1-n}\int_{A_\delta}\int e^{i\frac{|x|}{h_j}\left\langle \frac{x}{|x|}, \frac{\theta}{|\theta|} - \frac{\omega}{|\omega|}\right\rangle} g_{\varepsilon_j}\left(\frac{\theta}{|\theta|}\right)\left[b\left(x,\frac{\theta}{|\theta|}\right) + h_j e(x,\theta)\right]\overline{g_{\varepsilon_j}\left(\frac{\omega}{|\omega|}\right)}\chi_R(|\theta|)\chi_R(|\omega|)d\theta d\omega dx + O(h_j^\infty).$$

We write the integral in polar coordinates $x = r\phi$, $\theta = \alpha\Theta$, and $\omega = \beta\Omega$. Since $|r| > \delta$ on A_δ, we perform stationary phase in Ω and Θ. Using (6.2) with $M > n+2$ together with the remainder estimate [Zwo12, Theorem 3.16] to control the error uniformly as $j \to \infty$, gives

$$\int_{S^{n-1}} \int_{\mathbb{R}^3_+} [|g_{\varepsilon_j}(\phi)|^2 b(r\phi, \phi) + |g_{\varepsilon_j}(-\phi)|^2 b(r\phi, -\phi) \\ + c_1 e^{2ir/h} g_{\varepsilon_j}(\phi)\overline{g_{\varepsilon_j}(-\phi)} b(r\phi, \phi) + c_2 e^{-2ir/h} g_{\varepsilon_j}(-\phi)\overline{g_{\varepsilon_j}(\phi)} b(r\phi, -\phi)] \alpha^{n-1}\beta^{n-1} \\ \chi_R(\alpha)\chi_R(\beta)\psi(r) d\alpha d\beta dr dS_\phi + o(1)$$

Integration by parts in r then shows that the second two terms are lower order and yields

$$\int_{S^{n-1}} \int_{\mathbb{R}^3_+} [|g_{\varepsilon_j}(\phi)|^2 b(r\phi, \phi) + |g_{\varepsilon_j}(-\phi)|^2 b(r\phi, -\phi)] \alpha^{n-1}\beta^{n-1} \chi_R(\alpha)\chi_R(\beta) d\alpha d\beta dr dS_\phi + o(1)$$

Sending $j \to \infty$ gives

$$\left(\int_0^\infty \chi_R(\alpha)\alpha^{n-1} d\alpha\right)^2 \int_{\mathbb{R}} \int_{S^{n-1}} b(r\phi, \phi)(d\tilde{\rho}(\phi) + |g_1|^2 dS_\phi) dr = \int_{\Lambda_z} b(x, \xi)(d\rho + f d\mathrm{Vol}_{\Lambda_z})$$

where we use (6.1).

Using that the defect measure of $\Phi_{\varepsilon_j, j}$ is invariant under G_t then shows that $\Phi_{\varepsilon_j . j}$ has defect measure

$$\mu = d\rho + f d\mathrm{Vol}_{\Lambda_z}.$$

and hence $\|\Phi_{\varepsilon_j, j}\|_{L^2} \to 1$. Moreover,

$$\Phi_{\varepsilon_j, j}(z) = (2\pi h_j)^{\frac{1-n}{2}} \int_{\mathbb{R}^n} g_{\varepsilon_j}\left(\frac{\theta}{|\theta|}\right) \chi_R(|\theta|) d\theta = (2\pi h_j)^{\frac{1-n}{2}} \int_{S^{n-1}} (g_{1,\varepsilon_j}(\phi) + g_{2,\varepsilon_j}(\phi)) dS_\phi.$$

Since $\tilde{\rho} \perp d\mathrm{Vol}_{\Sigma_x}$ and $|g_{2,\varepsilon_j}|^2 dS_\phi \to \tilde{\rho}$ as a measure, for any $\delta > 0$, there exists $A \subset S^{n-1}$ so that

$$\int_{A^c} |g_{2,\varepsilon_j}|^2 dS_\phi \to 0, \qquad \int_A dS_\phi < \delta.$$

Therefore,

$$\left|\int_{S^{n-1}} g_{2,\varepsilon_j}(\phi) dS_\phi\right| \leq C \left(\int_{A^c} |g_{2,\varepsilon_j}|^2 dS_\phi\right)^{1/2} + \left(\int_{S^{n-1}} |g_{2,\varepsilon_j}|^2 dS_\phi\right)^{1/2} \delta^{1/2}$$

so, for all $\delta > 0$,

$$\limsup_{j\to\infty} \left|\int_{S^{n-1}} g_{2,\varepsilon_j}(\phi) dS_\phi\right| \leq C\delta^{1/2}.$$

In particular,

$$\lim_{j\to\infty} \int_{S^{n-1}} g_{2,\varepsilon_j}(\phi) dS_\phi = 0.$$

Finally, using that $g_{1,\varepsilon_j} \to g_1$ in L^2 and hence also in L^1

$$\lim_{j\to\infty} u_j(z) h_j^{\frac{n-1}{2}} = (2\pi)^{\frac{1-n}{2}} \int_{S^{n-1}} g_1(\phi) dS_\phi.$$

Letting $u_j = \Phi_{\varepsilon_j, j} / \|\Phi_{\varepsilon_j, j}\|_{L^2}$ then proves the lemma. □

7. A proof of Theorem 2 for the Laplacian

One can use a strategy similar to that in [GT17] to prove Theorem 2 for eigenfunctions of the Laplacian (or Schrödinger operators). We sketch the proof in the case $\mu_x \perp \mathcal{H}^n_x$ for the convenience of the reader.

Sketch. Fix $\delta > 0$ and let $\rho \in S(\mathbb{R})$ with $\rho(0) = 1$ and $\operatorname{supp} \hat{\rho} \subset [\delta, 2\delta]$. Let

$$S^*M(\gamma) := \{(x, \xi); ||\xi|_x - 1| \leq \gamma\}$$

and $\chi(x, \xi) \in C_0^\infty(T^*M)$ be a cutoff near the cosphere S^*M with $\chi(x, \xi) = 1$ for $(x, \xi) \in S^*M(\gamma)$ and $\chi(x, \xi) = 0$ when $(x, \xi) \in T^*M \setminus S^*M(2\gamma)$.

Suppose that $(-h^2\Delta_g - 1)u_h = 0$, and u_h has defect measure μ with

$$\mu_x = \rho \perp \mathcal{H}^n_x.$$

Then

$$u_h = \rho(h^{-1}[-h^2\Delta - 1])u_h = \int_{\mathbb{R}} \hat{\rho}(t) e^{it[-h^2\Delta-1]/h} \chi(x, hD) u_h \, dt + O_\gamma(h^\infty). \tag{7.1}$$

Setting $V(t, x, y, h) := \Big(\hat{\rho}(t) e^{it[-h^2\Delta-1]/h} \chi(x, hD)\Big)(t, x, y)$, by propagation of singularities,

$$WF_h'(V(t, \cdot, \cdot, h)) \subset \{(x, \xi, y, \eta); (x, \xi) = G_t(y, \eta),\ ||\xi| - 1| \leq 2\gamma\,, t \in [\delta, 2\delta]\}.$$

Let $b_{x,\gamma}(y, \eta) \in C_c^\infty(T^*M)$ have

$$\operatorname{supp} b_{x,\gamma} \subset \{(y, \eta) \mid (y, \eta) = G_t(x_0, \xi) \text{ for some } (x_0, \xi) \in S^*_{x_0}M(3\gamma) \text{ with } r(x, x_0) < \gamma, |t| \leq 3\delta\}$$

with

$$b_{x,\gamma} \equiv 1 \text{ on } \{(y, \eta) \mid (y, \eta) = G_t(x_0, \xi) \text{ for some } (x_0, \xi) \in S^*_{x_0}M(2\gamma) \text{ with } r(x, x_0) < 2\gamma, |t| \leq 4\delta\}.$$

Then, by wavefront calculus, it follows that

$$u_h(x) = \int_M \bar{V}(x, y, h)\, b_{x,\gamma}(y, hD_y) u_h(y) dy + O_\gamma(h^\infty), \tag{7.2}$$

where,

$$\bar{V}(x, y, h) := \int_{\mathbb{R}} \hat{\rho}(t) \big(e^{it[-h^2\Delta-1]/h} \chi(x, hD)\big)(t, x, y)\, dt.$$

By a standard stationary phase argument [Sog93, Chapter 5],

$$\bar{V}(x, y, h) = h^{\frac{1-n}{2}} e^{\pm ir(x,y)/h} a_\pm(x, y, h)\, \hat{\rho}(r(x, y)) + O_\gamma(h^\infty), \tag{7.3}$$

where $a_\pm(x, y, h) \in S^0(1)$.

Then, in view of (7.3) and (7.2),

(7.4)

$$u_h(x) = (2\pi h)^{\frac{1-n}{2}} \sum_\pm \int_{\delta<|y-x|<2\delta} e^{\pm ir(x,y)/h} a_\pm(x, y, h) \hat{\rho}(r(x, y))\, b_{x,\gamma}(y, hD_y) u_h(y) dy + O_\gamma(h^\infty).$$

Let χ_j, be as in (5.4) with $T(\xi_j, r_j)$ satisfying (5.7) and $\sum r_j^{n-1} < \varepsilon$. Define $\psi = 1 - \sum_j \chi_j$. Then

$$u_h(x) = \sum_\pm I_\pm + II_\pm + O_\gamma(h^\infty)$$

where
(7.5)

$$I_\pm = (2\pi h)^{\frac{1-n}{2}} \int_{\delta<|y-x|<2\delta} e^{\pm ir(x,y)/h} a_\pm(x,y,h)\hat{\rho}(r(x,y))\,\psi(y,hD_y)b_{x,\gamma}(y,hD_y)u_h(y)dy$$

$$II_\pm = \sum_j (2\pi h)^{\frac{1-n}{2}} \int_{\delta<|y-x|<2\delta} e^{\pm ir(x,y)/h} a_\pm(x,y,h)\hat{\rho}(r(x,y))\,\chi_j(y,hD_y)b_{x,\gamma}(y,hD_y)u_h(y)dy$$

An application of Cauchy-Schwarz to I gives

$$\limsup_{h\to 0} h^{\frac{n-1}{2}}|I_\pm| \leq C \limsup_{h\to 0} \|\psi(y,hD_y)b_{x,\gamma}(y,hD_y)u_h\|_{L^2} \tag{7.6}$$

But,

$$\lim_{\gamma\to 0}\lim_{h\to 0}\|\psi(y,hD_y)b_{x,\gamma}(y,hD_y)u\|^2_{L^2} = \lim_{\gamma\to 0}\int_{S^*M}|\psi|^2|b_{x,\gamma}(y,\xi)|^2 d\mu \leq C\rho(\operatorname{supp}\psi) \leq C\varepsilon$$

On the other hand, by propagation of singularities, for each χ_j in II, we may insert $\varphi_j \in C_c^\infty(M)$ localized to

$$\pi(T(\xi_j,r_j)\cap\{\delta < r(x,x_0) < 2\delta\})$$

where $\pi : T^*M \to M$ is projection to the base. In particular, replacing $\chi_j(y,hD_y)$ by $\varphi_j(y)\chi_j(y,hD_y)$ and applying Cauchy-Schwarz to each term of II, we have

$$\limsup_{h\to 0} h^{\frac{n-1}{2}}|II_\pm| \leq C\sum_j \|\varphi_j\|_{L^2} \limsup_{h\to 0}\|\chi_j b_{x,\gamma}(y,hD_y)u_h\|_{L^2} \tag{7.7}$$

Now, since φ_j is supported on a tube of radius r_j, $\|\varphi_j\|_{L^2} \leq C r_j^{(n-1)/2}$. Furthermore,

$$\lim_{\gamma\to 0}\lim_{h\to 0}\|\chi_j(y,hD_y)b_{x,\gamma}(y,hD_y)u\|^2_{L^2} = \lim_{\gamma\to 0}\int_{S^*M}\chi_j^2|b_{x,\gamma}(y,\xi)|^2 d\mu \leq C\int_{\Lambda_x}\chi_j^2 d\mu$$

Thus, applying Cauchy-Schwarz once again to the sum in (7.7),

$$\limsup_{h\to 0} h^{\frac{n-1}{2}}|II_\pm| \leq C\left(\sum_j r_j^{n-1}\right)^{1/2}\left(\int\sum_j \chi_j^2 d\mu\right)^{1/2} \leq C\varepsilon^{1/2}.$$

Sending $\varepsilon \to 0$ proves the theorem. □

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