*Answer **all** questions.*

1. A rocket moves vertically in the Earth's gravitational field (near the Earth's surface) by expelling downwards a mass α per unit time of hot gas, at a speed u relative to the rocket. (Here α and u are both constants.) The rocket is also subject to an air-resistance force $-\gamma v$. Suppose that at time $t = 0$ the rocket (together with its fuel) has mass M and velocity $v_0 = 0$.

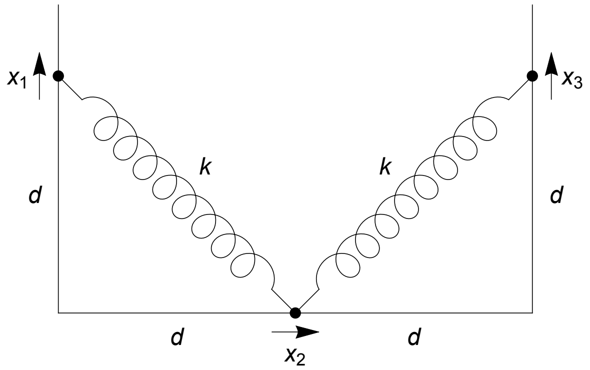
(a) Find the equation of motion of the rocket.

(b) Find the rocket's velocity as a function of time in the interval $0 \le t \le M/\alpha$. (You may assume that $0 < \gamma < \alpha$.) What is the velocity at the moment the fuel runs out?

(c) What inequality must be satisfied by the product αu in order for the rocket to get off the launching pad?

(25 marks)

2. Three beads, each of mass m, are threaded onto a rigid framework of frictionless rods, as shown in the diagram below. Beads 1 and 3 are free to move vertically, while bead 2 is free to move horizontally. The positions of beads 1,2,3 are thus $(0, d + x_1)$, $(d + x_2, 0)$ and $(2d, d + x_3)$, respectively. Each of the springs has equilibrium length $\sqrt{2}\, d$ and spring constant k. There is *no* gravitational field.



(a) Derive the linearized equations of motion. (You may use either Newtonian or Lagrangian methods.)

(b) Find the frequencies of the normal modes.

(c) Find the eigenvectors corresponding to each of the normal modes.

(25 marks)

3. Consider the differential equation

$$\ddot{x} + \omega_0^2 x = \epsilon x^3 \dot{x}^2$$

with initial condition $x(0) = A$, $\dot{x}(0) = 0$, using perturbation theory in the small parameter ϵ.

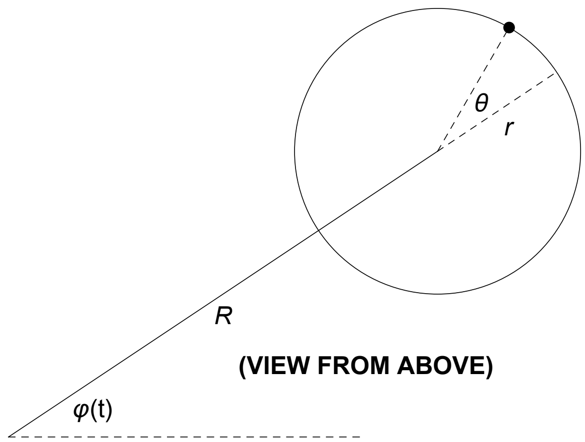
(a) Find the solution $x(t)$ through order ϵ^1. [*Hint:* $\cos^3\psi = \frac{1}{4}\cos 3\psi + \frac{3}{4}\cos\psi$ and $\cos^5\psi = \frac{1}{16}\cos 5\psi + \frac{5}{16}\cos 3\psi + \frac{5}{8}\cos\psi$.]

(b) Explain what a "secular term" is, and say which term in your answer from part (a) is a secular term.

(c) Use the Lindstedt renormalization procedure to compute the frequency of oscillation ω through order ϵ^1.

(25 marks)

4. A circular hoop of radius r is connected at its center to a rigid rod of length R; the whole apparatus is made to rotate (in a horizontal plane) around the origin with a specified angle $\varphi(t)$, as shown in the diagram below. A bead of mass m then slides frictionlessly on the hoop. Let θ be the angle of the bead relative to the rod.



(a) Using θ as the generalized coordinate, find the Lagrangian and the equation of motion.

(b) For the case $\varphi(t) = \omega t$, find the frequency of small oscillations around $\theta = 0$.

(c) Find the Hamiltonian.

(d) For the case $\varphi(t) = \omega t$: Does the Hamiltonian equal the total energy? Is the Hamiltonian conserved? Is the total energy conserved? Make sure to explain each answer.

(25 marks)