UNIVERSITY COLLEGE LONDON
Department of Mathematics

MATH0054 - Analytical Dynamics
2023 Main Summer Assessment Period

Duration: 3 hours (+ 30min upload window)

<u>INSTRUCTIONS:</u> (READ CAREFULLY!)

- **All questions should be attempted. Marks obtained in all solutions will count.**
- You may write your answers electronically or by hand. If you write your answer by hand, either take a picture with your phone or scan them. *You will be able to submit pdf or jpg files.*
  Tip: create a separate file for each question and name it using the question's number, then you can upload each file independently for each section. This will make the upload process a lot quicker.
- **You are responsible for your images being legible. If they are not clearly legible, you will receive no credit for your answer.**
- **Organize your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- You may consult any books of your choice, any handouts and problem-set solutions distributed in class, your own notes and problem sets, and any non-interactive publicly available Internet sources (e.g. Wikipedia or the lecture notes of professors at other universities).
- **The work you submit must be entirely your own.** You must NOT copy material from anyone, or discuss the questions with anyone, copy material from sources (unless you are quoting it with attribution), make any posts on social media or e-mail, consult in any way with any other person, or engage in any other form ofacademic misconduct.

Plagiarism, cheating, and collusion can have serious consequences, up to and including loss of the degree.

If you have any issues or questions, contact math.ugexams@ucl.ac.uk.

Good luck!

Question 1. (25 marks)

A person is stranded on a large frictionless frozen pond. She has on her back a device that shoots liquid backwards at a speed u relative to her, and at a rate α (mass per unit time). Her initial mass, including the device and the liquid, is M. She heads in a straight line towards the shore. There is an air-resistance force $-\gamma v$.

(a) Find the equation of motion.

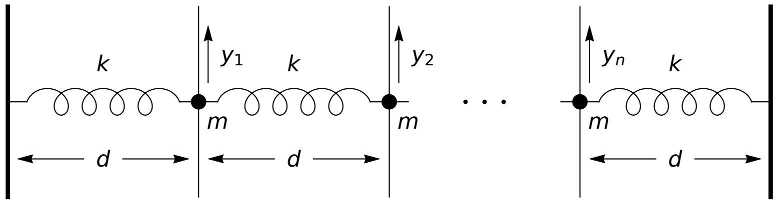
(15 marks)

(b) Find her position $x(t)$, assuming that at time 0 the position is x_0 and the velocity is v_0.

(10 marks)

Question 2. (25 marks)

n beads, each of mass m, are threaded on frictionless vertical rods, separated by a distance d, as shown in the figure below. The beads are connected by $n+1$ springs, each of equilibrium length d' and spring constant k. Let y_i be the vertical displacement of the ith bead. There is *no* gravitational field.



(a) Find the exact potential energy $U(y_1, \dots, y_n)$.

(5 marks)

(b) Under what conditions is $y_1 = \ldots = y_n = 0$ a *stable* equilibrium?

(3 marks)

(c) Find the linearised equations of motion. (You may use either Newtonian or Lagrangian methods.)

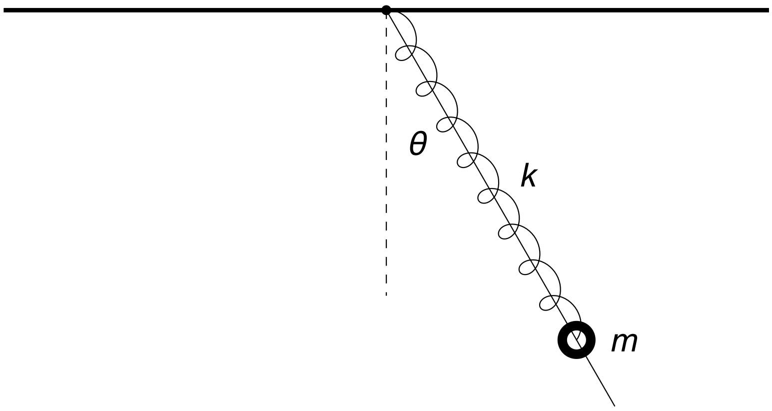
(6 marks)

(d) In the case $n = 3$ with $d' < d$, find the normal modes (that is, the eigenfrequencies and the eigenvectors).

(11 marks)

Question 3. (25 marks)

A spring (of spring constant k and equilibrium length d) is threaded around a massless rigid rod. At the end of the spring there is attached a ring of mass m, which can slide frictionlessly up and down the rod along with the spring. The top of the rod is attached to the ceiling, and the whole apparatus is allowed to swing like a pendulum. There is a uniform downward gravitational field g.



(a) Choose generalised coordinates for the ring, and find the Lagrangian in terms of them.

(6 marks)

(b) Find the Lagrange equations of motion.

(6 marks)

(c) Are there any conserved quantities? If so, say what they are and give formulas for them.

(5 marks)

(d) Find the equilibrium position.

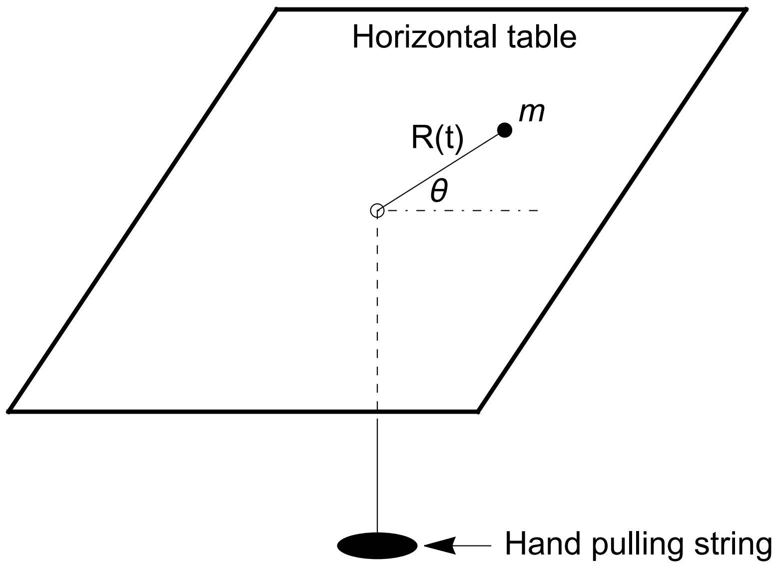
(4 marks)

(e) Find the normal modes of oscillation around the equilibrium position, and the corresponding eigenfrequencies.

(4 marks)

Question 4. (25 marks)

A particle of mass m moves on a smooth horizontal table. It is connected to a massless inextensible string that passes through a small hole in the table, and the string is pulled from below in such a way that the particle's distance from the hole is a specified function $R(t)$. Use polar coordinates (r, θ) with the origin located at the hole.



(a) Using θ as the generalised coordinate, find the kinetic energy, the potential energy, and the Lagrangian.

(8 marks)

(b) Show that θ is a cyclic coordinate, and find the corresponding conjugate momentum p_θ. What is the physical meaning of p_θ? Is p_θ conserved? Why or why not?

(4 marks)

(c) Find the Hamiltonian and the Hamilton equations of motion.

(7 marks)

(d) Compare the Hamiltonian and the total energy. Is the Hamiltonian conserved? Is the total energy conserved? Justify your answers, and explain physically.

(6 marks)