MATHEMATICS 3103 (Functional Analysis) **YEAR 2012–2013, TERM 2**

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Lectures: Mondays 10–11 and Thursdays 11–1, all in Room 807
Office hours: To be negotiated with you later this week

Course Description and Objectives: Elementary analysis mostly studies real-valued functions on the real line $\mathbb{R}$ or on n-dimensional space $\mathbb{R}^n$. Functional analysis, by contrast, shifts the point of view: we collect all the functions of a given class (for instance, all bounded continuous functions) into a *space of functions*, and we study that space (and operations on it) as an object in its own right. Since spaces of functions are nearly always infinite-dimensional, we are led to study analysis on infinite-dimensional vector spaces, of which the most important cases are Banach spaces and Hilbert spaces. This course provides an introduction to the basic concepts of functional analysis. These concepts are crucial in the modern study of partial differential equations, Fourier analysis, quantum mechanics, probability and many other fields.

Recommended Texts: There is no single book that is perfect for all the topics in this course. You will therefore have to get into the (good) habit of reading the relevant sections in *several* textbooks (each one with its own idiosyncrasies, gems and flaws) and then synthesizing what you have learned. With each topic I will give you a detailed reading list, but here is a preliminary list of useful reference books:

Kolmogorov and Fomin, *Introductory Real Analysis*
Kreyszig, *Introductory Functional Analysis with Applications*
Giles, *Introduction to the Analysis of Metric Spaces*
Giles, *Introduction to the Analysis of Normed Linear Spaces*
Rynne and Youngson, *Linear Functional Analysis*
Saxe, *Beginning Functional Analysis*
Dieudonné, *Foundations of Modern Analysis*
Royden, *Real Analysis*

I will also hand out my own notes when I feel (modestly) that I can explain a topic better than any of the existing texts.

Prerequisites: You are assumed to have a good background in analysis on $\mathbb{R}$ and $\mathbb{R}^n$, as well as the elementary theory of metric spaces, as presented in module 7102 (Real Analysis). A good reference for this material is Rudin, *Principles of Mathematical Analysis*.

I will *not* assume that you know measure theory (which is presented in module 3101), but I will occasionally make remarks intended for those who do know a bit about Lebesgue integration.

Detailed Syllabus:

1. Brief review of metric spaces and their basic properties. Completeness and completion. Compactness and equivalent properties.

2. Normed linear spaces and Banach spaces. Examples: Sequence spaces ℓ^p $(1 \le p \le \infty)$ and c_0; spaces $C(X)$ of bounded continuous functions. Proofs of completeness of these spaces. Special properties of finite-dimensional normed linear spaces.

3. Hilbert spaces. Basic properties. Orthogonal systems and the orthogonalization process. Representation of linear functionals on Hilbert space.

4. Spaces of continuous functions. Tietze extension theorem. Dini, Stone–Weierstrass and Arzelà–Ascoli theorems.

5. Zorn's lemma and the Hahn–Banach theorem.

6. Linear functionals and duality. Dual of ℓ^p is ℓ^q. Second dual and reflexive spaces.

7. Baire category theorem. Uniform boundedness theorem, open mapping theorem, closed graph theorem.

Problem Sets: One cannot learn mathematics solely by watching *someone else* do mathematics (even if that "someone" is a UCL professor). To learn mathematics, you must solve mathematics problems — lots of them — by *yourself*. Therefore, I will assign problem sets every week; they are to be handed in at the following week's problem class. *These problem sets are the most important part of the course.*

It is essential that you do the problem sets faithfully each week; if you put them off, even a little bit, you will have an *extremely* hard time catching up. Give yourself lots of time — mathematics is not a speed race — and do *not* expect to do a whole problem set in one sitting. If you get completely stuck on a problem, go on to another problem, and come back to the first one on a later day — your unconscious mind will be working on it in the meantime! I suggest therefore that you start on the problem set early in the week.

I do not expect you to get everything right on the problem sets the first time around. (Indeed, if you do get everything right, then you should complain to me that the problem sets are not challenging enough!) Rather, the purpose of the problem sets is to give you an opportunity to struggle with the ideas discussed in class by applying them to concrete mathematical problems, and in this way to solidify your understanding of those ideas. Only by such an intellectual struggle can you learn mathematics (or anything else of value, for that matter).

In writing up the problem sets, therefore, you must attempt to explain, as clearly and precisely as you can, the logic behind what you are trying to do: what is the mathematical situation, what are the principles to be applied, how you intend to apply those principles, etc. (Please use full English sentences, and large clearly-labelled drawings.) This explanation is especially important if you are *not* able to complete the problem: you should try to pinpoint, as clearly as possible, at what point you got stuck and why — this will serve as the basis for feedback to you. The coursework grade will be based on the logic and clarity of your explanation.

Only a fraction of each week's problem set will be formally assessed; but I will be happy to discuss with you any of the other problems, during my office hours or at any other mutually convenient time.

Assessment: 90% examination, 10% coursework.