MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

HANDOUT #4: INTRODUCTION TO HILBERT SPACE

Euclidean geometry on $\mathbb{R}^2$ or $\mathbb{R}^3$ or $\mathbb{R}^n$ uses more than the vector-space structure of $\mathbb{R}^n$; it also uses the notion of *angle*. Algebraically this arises by equipping $\mathbb{R}^n$ with an *inner product*

$$(x, y) \;=\; \sum_{i=1}^{n} x_i y_i \; . \tag{4.1}$$

Note that the Euclidean (ℓ^2) norm on $\mathbb{R}^n$ is then given by $\|x\|_2 = \sqrt{(x,x)}$.

By abstracting the key properties of the particular inner product (4.1), we arrive at the general definition of an inner product on a (finite-dimensional or infinite-dimensional) vector space X. Here the cases of real and complex vector spaces have to be treated separately; I shall start with the real case.

The first important property of the inner product (4.1) on $\mathbb{R}^n$ is that it is *bilinear*. Recall that if X, Y, Z are vector spaces (over the same field F of scalars), then a map $T\colon X \times Y \to Z$ is called a **bilinear map** if it is linear in each variable separately whenever the other variable is fixed, i.e.

$$T(\alpha_1 x_1 + \alpha_2 x_2, y) \;=\; \alpha_1 T(x_1, y) + \alpha_2 T(x_2, y) \qquad \text{for all } x_1, x_2 \in X,\ y \in Y \text{ and } \alpha_1, \alpha_2 \in F \tag{4.2a}$$

$$T(x, \beta_1 y_1 + \beta_2 y_2) \;=\; \beta_1 T(x, y_1) + \beta_2 T(x, y_2) \qquad \text{for all } x \in X,\ y_1, y_2 \in Y \text{ and } \beta_1, \beta_2 \in F \tag{4.2b}$$

If the target space Z is the field F of scalars (considered as a one-dimensional vector space), then we say that T is a **bilinear form**.

A second important property of the inner product (4.1) on $\mathbb{R}^n$ is that it is *symmetric*, i.e. $(x, y) = (y, x)$. For a general bilinear map, this condition makes sense only if $Y = X$. The definition is then: a bilinear map $T\colon X \times X \to Z$ is called **symmetric** if $T(x, y) = T(y, x)$ for all $x, y \in X$.

The final important property of the inner product (4.1) on $\mathbb{R}^n$ is that it is *positive-definite*. In general, a bilinear form $T\colon X \times X \to F$ (where $F = \mathbb{R}$ or $\mathbb{C}$) is called **positive** (or **positive-semidefinite**) if $T(x, x) \ge 0$ for all $x \in X$. It is called **positive-definite** if it is positive and, in addition, $T(x, x) = 0$ *only* for $x = 0$.

Definition 4.1 *An* **inner product** *on a* real *vector space X is a symmetric positive-definite bilinear form on X. We usually write the inner product as (x, y) rather than $T(x, y)$.*[1]

An **inner-product space** *(or* **Euclidean space** *or* **prehilbert space***) over the field of real numbers is the pair $(X, (\,\cdot\,,\,\cdot\,))$ where X is a real vector space and $(\,\cdot\,,\,\cdot\,)$ is an inner product on X.*

[1] Some authors use the notation $\langle x, y\rangle$ or $\langle x|y\rangle$.

I stress, as always, that the inner-product space is the *pair* $(X, (\,\cdot\,,\,\cdot\,))$. The same vector space X can be equipped with many different inner products, and these give rise to *different* inner-product spaces. However, we shall often refer informally to "the inner-product space X" whenever it is understood from the context what the inner product is.

Things have to be modified to handle complex vector spaces, and we can see this already in the finite-dimensional case — indeed, already in the one-dimensional case! To start with, the bilinear form (4.1) is *not* positive on $\mathbb{C}^n$, not even when $n = 1$, since the square of a complex number need not be nonnegative (indeed, it need not even be real). Worse yet, *no* bilinear form on a complex vector space can be positive, other than the identically-zero form. To see this, it suffices to note that, by bilinearity, $T(ix, ix) = -T(x, x)$. So bilinearity is certainly *not* the property we want!

The hint is provided already in the case $n = 1$: the correct measure of the (squared) length of a vector $x \in \mathbb{C}$ is not x^2 but rather $|x|^2$. And recall that $|x|^2 = x\overline{x}$ where $\overline{}$ denotes complex conjugate. So the standard inner product on $\mathbb{C}^n$ is given not by (4.1) but rather by

$$(x, y) \;=\; \sum_{i=1}^{n} x_i \overline{y_i}\,. \tag{4.3}$$

Then we do have $(x, x) \ge 0$ for all $x \in \mathbb{C}^n$, and the Euclidean (ℓ^2) norm on $\mathbb{C}^n$ is given by $\|x\|_2 = \sqrt{(x, x)}$.

So what are the fundamental properties of (4.3)? It is *not* bilinear: rather, it is linear in the first argument and *antilinear* in the second argument. More precisely, a map $T\colon Y \to Z$ from one complex vector space Y to another complex vector space Z is called **antilinear** (or **conjugate-linear**) if

$$T(\alpha_1 y_1 + \alpha_2 y_2) \;=\; \overline{\alpha_1} T(y_1) + \overline{\alpha_2} T(y_2) \tag{4.4}$$

for all $y_1, y_2 \in Y$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. If X, Y, Z are complex vector spaces, then a map $T\colon X \times Y \to Z$ is called a **sesquilinear map**[2] if it is linear in the first argument and antilinear in the second argument, i.e.

$$T(\alpha_1 x_1 + \alpha_2 x_2, y) \;=\; \alpha_1 T(x_1, y) + \alpha_2 T(x_2, y) \qquad \text{for all } x_1, x_2 \in X,\ y \in Y \text{ and } \alpha_1, \alpha_2 \in \mathbb{C} \tag{4.5a}$$

$$T(x, \beta_1 y_1 + \beta_2 y_2) \;=\; \overline{\beta_1} T(x, y_1) + \overline{\beta_2} T(x, y_2) \qquad \text{for all } x \in X,\ y_1, y_2 \in Y \text{ and } \beta_1, \beta_2 \in \mathbb{C} \tag{4.5b}$$

If the target space Z is $\mathbb{C}$ (considered as a one-dimensional vector space), then we say that T is a **sesquilinear form**.

Consider now a sesquilinear form in the case $Y = X$, i.e. $T\colon X \times X \to \mathbb{C}$. Such a map *cannot* be symmetric (unless it is identically zero), because linearity in the first variable clashes with antilinearity in the second. (Can you give a precise proof that symmetric sesquilinear form must be identically zero?) Rather, the property we want is *hermiticity*: a sesquilinear form $T\colon X \times X \to \mathbb{C}$ is called **hermitian** if $T(x, y) = \overline{T(y, x)}$ for all $x, y \in X$.

Finally, positive-semidefiniteness and positive-definiteness are defined as before.

We then have:

[2] From the Latin prefix "sesqui" meaning "one-and-a-half".

Definition 4.2 *An* **inner product** *on a* complex *vector space X is a hermitian positive-definite sesquilinear form on X. We usually write the inner product as (x, y) or $\langle x, y\rangle$ or $\langle x|y\rangle$ rather than $T(x, y)$.*

An **inner-product space** *(or* **Euclidean space** *or* **prehilbert space***) over the field of complex numbers is the pair $(X, (\,\cdot\,,\,\cdot\,))$ where X is a complex vector space and $(\,\cdot\,,\,\cdot\,)$ is an inner product on X.*

Warning: I (following majority practice) have defined our sesquilinear forms (and hence our inner products) to be linear in the first variable and antilinear in the second. However, some authors (mostly mathematical physicists, especially those working in quantum mechanics) define their sesquilinear forms and inner products to be antilinear in the first variable and linear in the second. When reading a book or article, it is important to determine which convention the author is using!

The basic theory of inner-product spaces and Hilbert spaces is fairly standard, and is well covered in nearly all of the textbooks I have suggested. In particular, I will distribute to you copies of Kreyszig, Sections 3.1–3.6 and 3.8, to which I have little to add.