MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

HANDOUT #0: REVIEW OF SET THEORY

The basic theory of finite and infinite sets — and in particular the distinction between "countably infinite" and "uncountably infinite" sets — is an essential prerequisite for Functional Analysis and indeed for nearly all higher mathematics. It is well covered in Sections 1.1 and 1.2 of Kolmogorov–Fomin, and in a more conversational style in Vilenkin's book *Stories about Sets*. You should study this material without delay!

Here I will merely summarize the principal definitions and, without proof, the principal results of this theory (see Kolmogorov–Fomin for the proofs).

Two sets A and B are said to **have the same cardinality** (or be **equivalent** or **equipotent** or **equipollent** or **equinumerous**), written $A \sim B$, if there exists a bijection from A to B. I stress the word "there exists": we don't demand that every attempted bijection be successful (or for instance that every injection of A into B be a surjection, or vice versa), but only that there *exist* a bijection.

A set is said to be **finite** if it is equivalent to the set $\{1, 2, \ldots, n\}$ for some integer $n \geq 0$. (Note that the empty set corresponds to the case $n = 0$.) A set is said to be **infinite** if it is not finite.

A set is said to be **countably infinite** if it is equivalent to the set $\mathbb{N} = \{1, 2, \ldots\}$ of natural numbers. (Such a set is obviously infinite — why?) An infinite set that is not countably infinite is said to be **uncountably infinite**. A set is said to be **countable** if it is either finite or countably infinite.[1] We then have the following fundamental facts (see Kolmogorov–Fomin for proofs):

Theorem 0.1 *Every infinite set has a countably infinite subset.*

Theorem 0.2 *Every infinite set is equivalent to one of its proper subsets.*

(Obviously such a thing *cannot* happen for a finite set — why?)

Theorem 0.3

(a) *Every subset of a countably infinite set is countable (i.e. either finite or countably infinite).*

(b) *A finite or countably infinite union of countably infinite sets is countably infinite.*

(c) *A* finite *Cartesian product of countably infinite sets is countably infinite.*

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[1]**Warning:** Kolmogorov–Fomin (or perhaps their translator) are a bit sloppy about distinguishing between "countably infinite" and "countable". But one can usually easily reconstruct from the context what is meant.

Note also that some authors (e.g. Dieudonné) use the terms **denumerable**, **nondenumerable** and **at most denumerable** for what I have called "countably infinite", "uncountably infinite" and "countable".

It follows from (c) and (a) [or alternatively from (b)] that the set $\mathbb{Q}$ of rational numbers is countably infinite (why?).

Warning: A *countably infinite* Cartesian product of countably infinite sets is *not* in general countably infinite! Indeed, even $S = \prod_{n=1}^{\infty} \{0,1\}$ — that is, a countably infinite Cartesian product of copies of the two-element set $\{0,1\}$ — is uncountably infinite: this is an immediate corollary of Theorem 0.5 below (why? what is the relation between subsets of $\{0,1\}$ and infinite sequences of 0's and 1's?).

Theorem 0.4 (uncountability of the reals) *The set of real numbers in the interval* $[0,1]$ *is uncountably infinite.*

Theorem 0.5 (Cantor's theorem) *For any set A, the set $\mathcal{P}(A)$ of all subsets of A is* not *equivalent to A; it has* strictly larger *cardinality.*

Theorem 0.6 (Cantor–Bernstein–Schröder theorem) *Let A, B be two sets. Suppose that A contains a subset A_1 that is equivalent to B, and that B contains a subset B_1 that is equivalent to A. Then A and B are equivalent.*

This can also be rephrased as: Suppose that there exist injective functions $f\colon A \to B$ and $g\colon B \to A$. Then there exists a bijective function $h\colon A \to B$.