MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

PROBLEM SET #5

This problem set is due at the *beginning* of class on Monday 11 March. Only Problems 2, 4 and 7 will be formally assessed, but I think you will find the other problems intriguing as well. Problem 8 is of particular interest, as it illustrates the interconnection between different fields of mathematics (functional analysis and complex analysis).

Topics: Spaces of continuous functions: Urysohn's lemma and the Tietze extension theorem; Dini's theorem; the Stone–Weierstrass theorem; the Arzelà–Ascoli theorem.

Readings:

- Handout #5: Spaces of continuous functions.

1. **Generalization of the Tietze extension theorem.** Prove the following slightly generalized version of the Tietze extension theorem:

   > Let A be a closed subset of a metric space X, and let $f\colon A \to \mathbb{R}$ be a continuous function. Then there exists a continuous function $g\colon X \to \mathbb{R}$ that extends f (i.e. $g \restriction A = f$) and does not take any values that are larger or smaller than all the values taken by f.

   In other words, we now allow unbounded continuous functions f, and we also show that if f is bounded above (resp. below) but does not actually attain this upper (resp. lower) bound, then g can be chosen so that it does not attain this bound either. [*Hint:* Consider $f/(1+|f|)$.]

2. **Continuous functions on noncompact metric spaces.**

   (a) A few weeks ago we proved the (easy) theorem that every real-valued continuous function on a compact metric space X is bounded. Now I would like you to prove the converse: namely, if X is a *noncompact* metric space, then there exists an *unbounded* real-valued continuous function on X. [*Hint:* Use the result of Problem 1.]

   Remark: For general (nonmetrizable) topological spaces, the situation is a bit more complicated: compactness is *not* equivalent to the boundedness of all continuous functions. But a variant of the above result holds: namely, for *normal* topological spaces (those for which the Tietze extension theorem holds), a space is *countably compact* (i.e. every *countable* open covering has a finite subcovering) if and only if every real-valued continuous function is bounded.

(b) In Theorem 5.16 we proved that if X is a compact metric space, then $\mathcal{C}(X)$ is separable. Now I would like you to prove the converse: namely, if X is a *noncompact* metric space, then $\mathcal{C}(X)$ is *nonseparable*. [*Hint:* Imitate the proof that ℓ^∞ is nonseparable, using the Tietze extension theorem.]

3. **Dini's theorem for semicontinuous functions.** Prove that if (f_n) is a decreasing sequence of *upper semicontinuous* real-valued functions on a compact metric space X that converges pointwise to a *lower semicontinuous* function g, then the convergence is uniform.

4. **A Stone–Weierstrass theorem for noncompact metric spaces?** The Stone–Weierstrass theorem applies to $\mathcal{C}(X)$ when X is a *compact* metric space, but what happens if X is noncompact? Here are two examples:

   (a) If $X = \mathbb{R}$ (or any unbounded subset of $\mathbb{R}$), it doesn't make sense to talk about uniform approximation of bounded continuous functions by polynomials, because all nonconstant polynomials are unbounded! But we can still ask about algebras $\mathcal{A}$ of *bounded* continuous functions: Does the Stone–Weierstrass theorem hold for these?

   Give an example of an algebra $\mathcal{A} \subset \mathcal{C}(\mathbb{R})$ that contains the constant functions and separates points of $\mathbb{R}$, and a function $f \in \mathcal{C}(\mathbb{R})$ that is *not* in the sup-norm closure of $\mathcal{A}$. [*Hint:* This is very easy.]

   (b) Next consider a bounded but non-closed subset of $\mathbb{R}$, e.g. $X = (0,1)$. Now it makes sense to ask about uniform approximation of bounded continuous functions by polynomials.

   Give an example of a bounded continuous function f on $(0,1)$ that cannot be uniformly approximated by polynomials. [*Hint:* f cannot have a continuous extension to $[0,1]$: for if it did, then it *would* be uniformly approximable by polynomials.]

5. **A variant of the Stone–Weierstrass theorem.** Let $\mathcal{A}$ be an algebra of real-valued continuous functions on a compact metric space X, and suppose that $\mathcal{A}$ separates the points of X but does not necessarily contain the constant functions. Show that either $\overline{\mathcal{A}} = \mathcal{C}(X)$ or else there exists a (unique) point $p \in X$ such that $\overline{\mathcal{A}} = \{f \in \mathcal{C}(X)\colon f(p) = 0\}$.

6. **Equicontinuity versus uniform equicontinuity.** We know that any continuous mapping from a *compact* metric space X to a metric space Y is in fact uniformly continuous. I would like you now to prove a generalization of this: any equicontinuous family of mappings from a *compact* metric space X to a metric space Y is in fact uniformly equicontinuous.

7. **Fredholm integral operators.** If X and Y are Banach spaces, a linear operator $T\colon X \to Y$ is called **compact** if the image of every bounded set in X is relatively compact in Y. (Recall that a subset of a metric space is called *relatively compact* if its closure is compact.) Equivalently, T is compact if the image of the (open or closed) unit ball in X is relatively compact in Y (why is this equivalent?).

   In Problem 6 of Problem Set #3, you studied the Fredholm integral operator $T\colon \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ defined by

$$(Tf)(s) \;=\; \int_a^b K(s,t)\, f(t)\, dt$$

   where $K\colon [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function, and you proved that T is a *bounded* linear operator. Now I would like you to prove that T is in fact a *compact* linear operator.

8. **Application of equicontinuity to complex analysis.** Analytic functions of a complex variable are much more "rigid" than functions (even C^∞ functions) of a real variable: for instance, knowing an analytic function on a small neighborhood tells you the function everywhere (**analytic continuation**); and knowing a bound on the function on a simple closed curve allows you to bound the function and all its derivatives inside the curve (**Cauchy integral formula**). These principles combined with equicontinuity arguments lead to some surprising and powerful results. Here are two examples:

   (a) Let $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ be the open unit disc in the complex plane. The **Hardy space** $H^\infty(\mathbb{D})$ consists of the bounded analytic functions on $\mathbb{D}$, equipped with the sup norm $\|f\|_{H^\infty} = \sup\limits_{|z|<1} |f(z)|$.

   For any complex number λ with $|\lambda| \le 1$, we can define a linear operator $T_\lambda\colon H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ by

$$(T_\lambda f)(z) \;=\; f(\lambda z)\,.$$

   Clearly T_λ is a bounded linear operator of norm 1 (why?).

   Prove that if $|\lambda| < 1$, then T_λ is compact. (See the preceding problem for the definition of a compact linear operator.)

   [*Hint:* Use the Cauchy integral formula to bound the derivative of f, then use the Arzelà–Ascoli theorem.]

   (b) Prove **Montel's theorem**: Let D be a domain (i.e. connected open set) in the complex plane, and let (f_n) be a sequence of analytic functions on D that is uniformly bounded (i.e. there exists $M < \infty$ such that $\|f_n\|_\infty \le M$ for all n). Then there exists a subsequence (f_{n_i}) that converges, uniformly on compact subsets of D, to an analytic function g. [*Hint:* Do it first when D is an open disc. Use the Cauchy integral formula to bound the derivative of f, then use the Arzelà–Ascoli theorem and a diagonal argument.]

Remark. This result, which was proven by French mathematician Paul Montel in 1907, plays a central role in complex analysis. But it is only the first step, and a vastly stronger result, proven by Montel in 1912, turns out to be true: namely, fix two distinct points $a, b \in \mathbb{C}$, and consider a sequence (f_n) of analytic functions on D that do not take the value a or b. Then there exists a subsequence (f_{n_i}) that converges, uniformly on compact subsets of D, either to an analytic function g or to infinity. This is a truly amazing result (avoiding two points is a hell of a lot weaker than avoiding the exterior of a disc!), and its proof is quite a bit more difficult than that of Montel's 1907 theorem. A nice book on this subject is Joel L. Schiff, *Normal Families*. Montel's 1912 theorem plays a key role in the study of *holomorphic dynamics*: for an accessible introduction to this fascinating area combining complex analysis and dynamical systems, see Alan F. Beardon, *Iteration of Rational Functions*.