MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

PROBLEM SET #4

This problem set is due at the *beginning* of class on Monday 4 March. Only Problem 2 will be formally assessed, but I think you will find Problem 1 intriguing.

Topics: Basic properties of inner-product spaces and Hilbert spaces.

Readings:

- Handout #4: Introduction to Hilbert space.
- Kreyszig, Sections 3.1–3.6 and 3.8 (handout).

1. **The parallelogram law.** As discussed in class, the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 \;=\; 2(\|x\|^2 + \|y\|^2) \qquad \text{for all } x, y \in X$$

is a necessary condition for a normed linear space to be an inner-product space (this is almost trivial). Here I would like you to prove that it is also a sufficient condition, restricting attention for simplicity to the case of a *real* normed linear space. [*Hint:* Use the polarization identity

$$(x,y) \;=\; \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

to define the inner product. To prove the additivity $(x+y,z) = (x,z) + (y,z)$, use the parallelogram identity cleverly. To prove the homogeneity $(\alpha x, y) = \alpha(x,y)$, prove it first for rational α and then argue by continuity.]

2. **An example of the Gram–Schmidt process.** Consider the inner-product space $\mathcal{C}[-1,1]$ of continuous functions on the interval $[-1,1]$, equipped with the inner product

$$(f,g) \;=\; \int_{-1}^{1} f(t)\,\overline{g(t)}\,dt\ .$$

(a) Apply the Gram–Schmidt process to the functions $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$ to find the corresponding orthonormal set.

(b) Apply the Gram–Schmidt process to the functions $f_1(t) = t^2$, $f_2(t) = t$, $f_3(t) = 1$ to find the corresponding orthonormal set. Is the answer different? Should it be?

Remarks. 1. As discussed in Kreyszig, Section 3.7, the orthonormal system obtained by applying the Gram–Schmidt process to $f_n(t) = t^n$ $(n = 0, 1, 2, \ldots)$ coincides (up to normalization) with the **Legendre polynomials**.

2. As proved several weeks ago, the inner-product space being considered in this problem is an *incomplete* inner-product space. Those of you familiar with measure theory will know that its completion can be identified with the space $L^2[-1, 1]$ of Lebesgue-square-integrable functions on $[-1, 1]$.[1]

[1] More precisely, $L^2[-1, 1]$ is the space of equivalence classes, modulo modification on a set of measure zero, of Lebesgue-square-integrable functions on $[-1, 1]$.