MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

PROBLEM SET #2

This problem set is due at the *beginning* of class on Monday 4 February. I urge you to start on it *early* in the week, as some of the problems are not entirely trivial. Only Problems 1 and 3 will be formally assessed, but I strongly urge you not to neglect the others!

Topics:

- *Completeness.* Completeness of the sequence spaces ℓ^∞, c_0, ℓ^1 and ℓ^2. Incompleteness of the space $C[a,b]$ with the L^1 or L^2 norm. Completion of a metric space.
- *Compactness.* Equivalent versions of compactness for metric spaces. Continuous functions on compact metric spaces. Examples of compactness and noncompactness in infinite-dimensional spaces. Locally compact metric spaces.

Readings:

- Kreyszig, Sections 1.5 and 1.6 (handout).
- Handout #2: Compactness of Metric Spaces.
- Dieudonné, Sections III.16, III.17 and III.18 (handout).

1. **Completeness of the space $\mathcal{B}(A)$ of bounded functions.** Let A be an arbitrary nonempty set, and let $\mathcal{B}(A)$ be the space of bounded real-valued functions on A, equipped with the sup norm. Prove that $\mathcal{B}(A)$ is complete.

2. **Compactness of finite and countably infinite Cartesian products.**

   (a) Let (X_1, d_1), ..., (X_n, d_n) be metric spaces, and let X be the Cartesian-product space $X_1 \times \cdots \times X_n$ [that is, the space consisting of n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in X_i$]. Equip X with either of the equivalent metrics

$$\begin{aligned} \mathbf{d}_1(x,y) &= \sum_{i=1}^{n} d_i(x_i, y_i) \\ \mathbf{d}_\infty(x,y) &= \max_{1 \le i \le n} d_i(x_i, y_i) \end{aligned}$$

   (see Problem 2(a) of Problem Set #1). Prove that if the spaces $X_1, \ldots, X_n$ are all compact, then so is X.

(b) Let (X_1, d_1), (X_2, d_2), ... be an infinite sequence of metric spaces, and let X be the Cartesian-product space $X_1 \times X_2 \times \cdots$ [that is, the space consisting of infinite sequences $x = (x_1, x_2, \ldots)$ with $x_i \in X_i$], equipped with the metric

$$d(x, y) \;=\; \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{d_j(x_j, y_j)}{1 + d_j(x_j, y_j)}$$

(see Problem 2(c) of Problem Set #1). Prove that if the spaces $X_1, X_2, \ldots$ are all compact, then so is X.

3. **Compactness of some sets in ℓ^1 and ℓ^2.** Given a sequence $x = (x_1, x_2, \ldots)$ of real numbers, let us define S_x to be the set consisting of those infinite sequences of real numbers that are "bounded above elementwise by x", i.e.

$$S_x \;=\; \{y \in \mathbb{R}^{\mathbb{N}} \colon\, |y_n| \le |x_n| \text{ for all } n\} \,.$$

In the notes we proved that if $x \in c_0$, then S_x is a compact subset of c_0 (and hence also of ℓ^∞). Here you will prove the analogous results for ℓ^1 and ℓ^2:

(a) If $x \in \ell^1$, then S_x is a compact subset of ℓ^1.

(b) If $x \in \ell^2$, then S_x is a compact subset of ℓ^2.

4. **More on compactness in ℓ^∞ and c_0.** Define S_x as in the preceding problem.

(a) Let $x \in \ell^\infty$. Prove that the following are equivalent:

(i) $x \in c_0$.

(ii) S_x is compact (as a subspace of ℓ^∞).

(iii) S_x is separable (as a subspace of ℓ^∞).

(b) Prove that a closed subset $A \subseteq c_0$ is compact *if and only if* it is contained in the set S_x for some $x \in c_0$. [*Hint:* If A were indeed contained in some set S_x, what would the smallest such x be? Define it and then prove that it lies in c_0 whenever A is compact.]

5. **Upper semicontinuous and lower semicontinuous functions.** Let X be a metric space, let $f\colon X \to \mathbb{R}$ be a real-valued function, and let $x_0 \in X$. We recall that f is continuous at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that

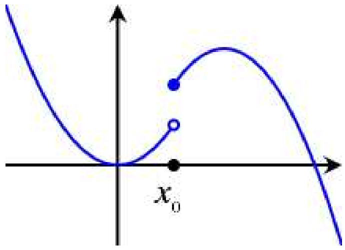
$$f(x_0) - \epsilon \;<\; f(x) \;<\; f(x_0) + \epsilon$$

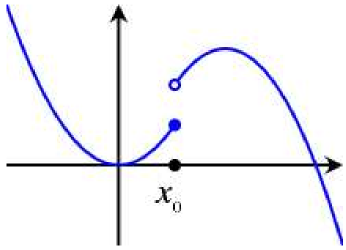
for all $x \in U$. It is sometimes useful to consider these two inequalities separately: let us say that f is

- **upper semicontinuous** at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) < f(x_0) + \epsilon$ for all $x \in U$; and

- **lower semicontinuous** at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) > f(x_0) - \epsilon$ for all $x \in U$.

(Clearly, a function is continuous at x_0 if and only if it is both upper semicontinuous and lower semicontinuous there.) Here are two drawings that can help to remember what upper and lower semicontinuity mean:





The function at the left is upper semicontinuous, while the one at the right is lower semicontinuous; in both cases the solid dot indicates $f(x_0)$.

A function $f\colon X \to \mathbb{R}$ is said to be upper (resp. lower) semicontinuous if it is upper (resp. lower) semicontinuous at every point of X.

Clearly, f is upper semicontinuous if and only if $-f$ is lower semicontinuous, so it suffices to study one of the two concepts; we can then immediately deduce results for the other. So let us focus on lower semicontinuity.

(a) Show that a function $f\colon X \to \mathbb{R}$ is lower semicontinuous if and only if the set $\{x \in X\colon\ f(x) > a\}$ is an open set for every $a \in \mathbb{R}$.

(b) Let $(f_\alpha)_{\alpha \in I}$ be a collection of real-valued functions on X (indexed by some arbitrary index set I), and define f as the pointwise supremum

$$f(x) \;=\; \sup_{\alpha \in I} f_\alpha(x) \;.$$

Let us assume for simplicity that $f(x) < \infty$ for all $x \in X$, so that f is again a real-valued function on X.

Show that if all the functions f_α are lower semicontinuous, then so is f. Show also, by example, that f need not be continuous even if all the functions f_α are continuous and the index set I is countably infinite and the metric space X is compact. (Of course, a *finite* maximum of continuous functions *is* continuous.)

(c) Let f be a lower semicontinuous function on a *compact* metric space X. Show that f is bounded below and attains its minimum. [*Hint:* Use open coverings.]

Remark. The most natural context for studying upper and lower semicontinuous functions is that of functions taking values in the **extended real line** $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.[1] Then the statement of part (b) would be true without the assumption that $f(x) < \infty$ for all $x \in X$.

[1] With e.g. the metric $d(x,y) = |\tanh x - \tanh y|$ where $\tanh(+\infty)$ is defined as $+1$ and $\tanh(-\infty)$ is defined as -1. This metric has the property that $\lim\limits_{n\to\infty} (\pm n) = \pm\infty$.